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**TAUBERIAN THEOREMS BETWEEN THE LOGARITHMIC
AND ABEL-TYPE SUMMABILITY METHODS**

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TAUBERIAN THEOREMS BETWEEN THE LOGARITHMIC AND ABEL-TYPE SUMMABILITY METHODS

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The object of this paper is to show that if a series is summable by the logarithmic method L , then the series is also summable by the Abel method A_λ , provided a tauberian condition of the "slowly decreasing" type is satisfied.

1. Introduction. Suppose throughout that $\{s_n\}$ is a sequence of numbers, λ real is real, $\varepsilon_0^\lambda = 1$, $\varepsilon_n^\lambda = \binom{n + \lambda}{n}$ for $n = 1, 2, 3, \dots$, and

$$v_n^\lambda = \frac{\varepsilon_n^\lambda \Gamma(\lambda + 1)}{(n + 1)^\lambda} \quad \text{for } n = 0, 1, 2, \dots$$

We are concerned with the methods of summability A_λ introduced and studied by Borwein [1] and the logarithmic method L . They are defined as follows. Let

$$(1) \quad \sigma_\lambda(y) = (1 + y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1 + y} \right)^n, \quad \text{and}$$

$$(2) \quad L(y) = \frac{1}{\log(1 + y)} \sum_{n=0}^{\infty} \frac{s_n}{n + 1} \left(\frac{y}{1 + y} \right)^{n+1}.$$

If $\sigma_\lambda(y)$ converges for $y > 0$ and tends to s as $y \rightarrow \infty$, then we say that the sequence $\{s_n\}$ is A_λ -convergent to s and write $s_n \rightarrow s(A_\lambda)$. The method A_0 is the ordinary Abel method.

If $L(y)$ converges for $y > 0$ and tends to s as $y \rightarrow \infty$, then we say that $\{s_n\}$ is L -convergent to s and write $s_n \rightarrow s(L)$.

Evidently, $s_n \rightarrow s(L)$ if and only if

$$- \frac{1}{\log(1 - x)} \sum_{n=0}^{\infty} \frac{s_n}{n + 1} x^{n+1}$$

converges for $0 < x < 1$ and tends to s as $x \rightarrow 1^-$.

LEMMA 1. A_λ is regular for $\lambda > -1$. [That is, $s_n \rightarrow s$ implies $s_n \rightarrow s(A_\lambda)$].

LEMMA 2. L is regular.

LEMMA 3. $A_{\lambda+\varepsilon} \subset A_\lambda$ for $\lambda > -1$, and $\varepsilon > 0$. [That is, $s_n \rightarrow s(A_{\lambda+\varepsilon})$ implies $s_n \rightarrow s(A_\lambda)$ and there exists a sequence $\{s_n\}$, depending on λ and ε , such that $\{s_n\}$ is A_λ -convergent but not $A_{\lambda+\varepsilon}$ -convergent.]

LEMMA 4. $A_\lambda \subset L$ for $\lambda > -1$.

Lemmas 1 and 3 were established by Borwein in [1]. Lemma 4 was proved by Borwein in [2] as a particular case of a more general inclusion theorem on methods of summability based on power series. Lemma 2 is a standard result found, for example, in [4].

2. The main theorem. Suppose that Φ is a nonnegative, continuous, strictly increasing function on $[a, \infty)$, for some a , such that $\Phi(t) \rightarrow \infty$ as $t \rightarrow \infty$.

The real-valued function f is said to be *slowly decreasing with respect to Φ* if $\liminf \{f(y) - f(x)\} \geq 0$ whenever $y \geq x \rightarrow \infty$ and $\Phi(y) - \Phi(x) \rightarrow 0$.

THEOREM 1. For $\lambda > -1$, if $s_n \rightarrow s(L)$ and $\sigma_\lambda(t)$ is slowly decreasing with respect to $\log \log t$, then $s_n \rightarrow s(A_\lambda)$.

In connection with the methods A_λ , we proved the following lemma in [3].

LEMMA 5. For $\lambda > -1$ and $\varepsilon > 0$, if $s_n \rightarrow s(A_\lambda)$ and $\sigma_{\lambda+\varepsilon}(t)$ is slowly decreasing with respect to $\log t$, then $s_n \rightarrow s(A_{\lambda+\varepsilon})$.

3. Methods of summability based on power series. Suppose that $p_n \geq 0$, $q_n \geq 0$, $\sum_{v=n}^{\infty} p_v > 0$, and $\sum_{v=n}^{\infty} q_v > 0$ for $n = 0, 1, 2, \dots$. Set

$$p(x) = \sum_{n=0}^{\infty} p_n x^n, \quad \text{and}$$

$$q(x) = \sum_{n=0}^{\infty} q_n x^n.$$

Let ρ_p and ρ_q denote their respective radii of convergence. We also write

$$p_s(x) = \frac{1}{p(x)} \sum_{n=0}^{\infty} p_n s_n x^n$$

$$q_s(x) = \frac{1}{q(x)} \sum_{n=0}^{\infty} q_n s_n x^n.$$

The power series method P is defined as follows. If $\rho_p > 0$, $\sum_{n=0}^{\infty} p_n s_n x^n$ converges for $0 < x < \rho_p$ and $\lim_{x \rightarrow \rho_p^-} p_s(x) = s$, then we write $s_n \rightarrow s(P)$.

The method Q is defined similarly.

Borwein has proved [2] the following lemma.

LEMMA 6. (i) *If $0 < \rho_p < \infty$, then a necessary and sufficient condition for P to be regular is that $\sum_{n=0}^{\infty} p_n(\rho_p)^n = \infty$.*

(ii) *If $\rho_p = \infty$ then P is regular.*

Suppose that $\chi(t)$ is a function of bounded variation on $[0, 1]$, and $\chi^*(t)$ is its associated normalized function. That is,

$$\chi^*(t) = \begin{cases} 0 & t = 0 \\ \frac{1}{2}\{\chi(t+) + \chi(t-)\} - \chi(0) & 0 < t < 1 \\ \chi(1) - \chi(0) & t = 1. \end{cases}$$

A sequence $\{\mu_n\}$ is called an m -sequence if, for some χ ,

$$\mu_n = \int_0^1 t^n d\chi(t) \quad \text{for } n = 0, 1, 2, \dots$$

If, in addition,

$$\mu_n \geq \delta \int_0^1 t^n |d\chi^*(t)| \quad \text{for } 0 < \delta \leq 1 \quad \text{and}$$

$n = N, N + 1, \dots$, then $\{\mu_n\}$ is called an \bar{m} -sequence.

LEMMA 7. *If $p_n = \mu_n q_n (n = N, N + 1, \dots)$, $\{\mu_n\}$ is an \bar{m} -sequence, $\rho_p = \rho_q > 0$, and P is regular, then $Q \subseteq P$. (That is, $s_n \rightarrow s(Q)$ implies $s_n \rightarrow s(P)$.)*

This result is due to Borwein (see [2], Theorem A').

We require the following two lemmas.

LEMMA 8. *An m -sequence which converges to a positive limit is an \bar{m} -sequence.*

LEMMA 9. *The sequences $\{v_n^\lambda\}$ and $\{1/v_n^\lambda\}$ are \bar{m} -sequences for $\lambda > -1$.*

The proof of Lemma 8 is straightforward and Lemma 9 was established in [4], Theorem 211.

The next result is used in the proof of Theorem 1.

THEOREM 2. *Let Q be a regular power series method and suppose that $\{\mu_n\}$ is an \bar{m} -sequence such that $\mu_n \rightarrow a > 0$. Then $\mu_n s_n \rightarrow as(Q)$*

whenever $s_n \rightarrow s(Q)$.

Proof. Suppose that $s_n \rightarrow s(Q)$. Set $p_n = \mu_n q_n$ for $n = 0, 1, 2, \dots$. Since $\mu_n \geq 0$ and $\mu_n \rightarrow a$ it is easy to verify that $\rho_p = \rho_q$. If $\rho_p = \infty$, then P is regular by Lemma 6(ii). Otherwise, since $p_n \sim a q_n$, P is regular by Lemma 6(i).

Therefore, by Lemma 7, $s_n \rightarrow s(P)$. That is,

$$(3) \quad \frac{1}{p(x)} \sum_{n=0}^{\infty} s_n \mu_n q_n x^n \longrightarrow s \quad \text{as } x \longrightarrow \rho_p^-.$$

In addition, since Q is regular,

$$(4) \quad \frac{p(x)}{q(x)} = \frac{1}{q(x)} \sum_{n=0}^{\infty} \mu_n q_n x^n \longrightarrow a \quad \text{as } x \longrightarrow \rho_q^-.$$

Application of Q to $\{\mu_n s_n\}$ yields

$$\begin{aligned} & \frac{1}{q(x)} \sum_{n=0}^{\infty} \mu_n s_n q_n x^n \\ &= \frac{p(x)}{q(x)} \frac{1}{p(x)} \sum_{n=0}^{\infty} s_n \mu_n q_n x^n \\ & \longrightarrow as \quad \text{as } x \longrightarrow \rho_q^- = \rho_p^- \text{ by (3) and (4).} \end{aligned}$$

This completes the proof.

COROLLARY TO THEOREM 2. $s_n \rightarrow s(L)$ if and only if $v_n^2 s_n \rightarrow s(L)$.

This is immediate in view of Lemmas 8 and 9, and the fact that $v_n^2 \rightarrow 1$ as $n \rightarrow \infty$.

4. An integral transformation. The integral transformation $J_\lambda(w)$ of the function $f(t)$, for $\lambda > -1$ and $w > 0$, is defined as follows.

$$(5) \quad J_\lambda(w) = \frac{1}{\log(1+w)} \int_0^w (1+t)^{\lambda-1} \left(\log \frac{w(1+t)}{t(1+w)} \right)^\lambda f(t) dt.$$

THEOREM 3. If $\lambda > -1$ and $f(t) = \sigma_\lambda(t)$ is convergent for all $t > 0$, then $J_\lambda(w) \rightarrow s$ as $w \rightarrow \infty$ if and only if $s_n \rightarrow s(L)$.

Proof. Setting $u = (t(1+w))/(w(1+t))$ in $J_\lambda(w)$ gives

$$\begin{aligned} & J_\lambda(w) \\ &= \frac{1}{\log(1+w)} \int_0^w (1+t)^{\lambda-1} \left(\log \frac{w(1+t)}{t(1+w)} \right)^\lambda (1+t)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{t}{1+t} \right)^n dt \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\log(1+w)} \int_0^1 \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{w}{1+w}\right)^{n+1} u^n \left(\log \frac{1}{u}\right)^\lambda du \\
 &= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{w}{1+w}\right)^{n+1} \int_0^1 u^n \left(\log \frac{1}{u}\right)^\lambda du \\
 &= \frac{\Gamma(\lambda+1)}{\log(1+w)} \sum_{n=0}^{\infty} \frac{\varepsilon_n^\lambda}{(n+1)^{\lambda+1}} s_n \left(\frac{w}{1+w}\right)^{n+1} \\
 &= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \frac{v_n^\lambda s_n}{n+1} \left(\frac{w}{1+w}\right)^{n+1}.
 \end{aligned}$$

The convergence, for $t > 0$, of the series defining $\sigma_\lambda(t)$ implies its absolute convergence. This justifies the integration term by term and, in view of the corollary to Theorem 2, the proof is complete.

5. Additional lemmas.

LEMMA 10. For $\lambda > -1$, $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n$ is absolutely convergent for $|x| < 1$ if and only if $\sum_{n=0}^{\infty} (s_n/(n+1))x^n$ is absolutely convergent for $|x| < 1$.

We omit the simple proof.

LEMMA 11. For $0 < t < w$,

$$\log \frac{w(1+t)}{t(1+w)} > \frac{w-t}{w(1+t)}.$$

Proof. For $x > 1$,

$$\log x = \log x - \log 1 = \frac{x-1}{\theta} > \frac{x-1}{x}$$

where $1 < \theta < x$. The result follows by observing that, for $0 < t < w$, $x = (w(1+t))/(t(1+w)) > 1$.

LEMMA 12. For fixed $\gamma > 1$ and $\lambda > -1$,

$$\begin{aligned}
 I(x) &= \int_0^x (1+t)^{\lambda-1} \left(\left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)} \right)^\lambda - \left(\log \frac{x(1+t)}{t(1+x)} \right)^\lambda \right) dt \\
 &= O(1).
 \end{aligned}$$

Proof. Suppose $\lambda \geq 1$. Then, for $x \geq 1$,

$$\begin{aligned}
 |I(x)| &= I(x) \\
 &\leq \lambda \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)} \int_0^x (1+t)^{\lambda-1} \left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)} \right)^{\lambda-1} dt
 \end{aligned}$$

$$\begin{aligned} &\leq \lambda \log \frac{x^r(1+x)}{x(1+x^r)} \left(\int_0^1 + \int_1^x \right) (1+t)^{\lambda-1} \left(\log \frac{1+t}{t} \right)^{\lambda-1} dt \\ &= I_1(x) + I_2(x) . \end{aligned}$$

Now,

$$\int_0^1 (1+t)^{\lambda-1} \left(\log \frac{1+t}{t} \right)^{\lambda-1} dt < \infty .$$

Hence,

$$I_1(x) = O(1) .$$

Also,

$$\begin{aligned} I_2(x) &= O(1) \log \frac{x^r(1+x)}{x(1+x^r)} \int_1^x dt \\ &= O(1)x \log \frac{1+x}{x} = O(1) . \end{aligned}$$

Suppose $0 < \lambda < 1$. By Lemma 11 we have,

$$\begin{aligned} |I(x)| &= I(x) \\ &\leq \lambda \log \frac{x^r(1+x)}{x(1+x^r)} \int_0^x (1+t)^{\lambda-1} \left(\log \frac{x(1+t)}{t(1+x)} \right)^{\lambda-1} dt \\ &< \lambda \frac{M}{x} \int_0^x (1+t)^{\lambda-1} \left(\frac{x-t}{x(1+t)} \right)^{\lambda-1} dt \end{aligned}$$

since $x \log (x^r(1+x))/(x(1+x^r)) \leq M$.

Therefore

$$I(x) \leq \lambda \frac{M}{x^\lambda} \int_0^x (x-t)^{\lambda-1} dt = M .$$

Suppose $-1 < \lambda < 0$. Then

$$\begin{aligned} |I(x)| &= -I(x) \\ &= \left(\int_0^{x/2} + \int_{x/2}^x \right) (1+t)^{\lambda-1} \left(\left(\log \frac{x(1+t)}{t(1+x)} \right)^\lambda - \left(\log \frac{x^r(1+t)}{t(1+x^r)} \right)^\lambda \right) dt \\ &= I_1(x) + I_2(x) . \end{aligned}$$

Using Lemma 11 and the fact that

$$\left| x \log \frac{x(1+x^r)}{(1+x)x^r} \right| \leq M$$

we have

$$\begin{aligned}
 0 \leq I_1(x) &\leq \lambda \left(\log \frac{x(1+x^\gamma)}{x^\gamma(1+x)} \right) \int_0^{x/2} (1+t)^{\lambda-1} \left(\log \frac{x(1+t)}{t(1+x)} \right)^{\lambda-1} dt \\
 &\leq -\frac{\lambda M}{x} \int_0^{x/2} (1+t)^{\lambda-1} \left(\frac{x-t}{x(1+t)} \right)^{\lambda-1} dt \\
 &= M((1/2)^\lambda - 1).
 \end{aligned}$$

For $I_2(x)$, since $1+t > x/2$,

$$\begin{aligned}
 0 \leq I_2(x) &\leq \int_{x/2}^x (1+t)^{\lambda-1} \left(\log \frac{x(1+t)}{t(1+x)} \right)^\lambda dt \\
 &\leq \int_{x/2}^x (1+t)^{\lambda-1} \left(\frac{x-t}{x(1+t)} \right)^\lambda dt \\
 &= \frac{1}{x^\lambda} \int_{x/2}^x (x-t)^\lambda \frac{dt}{1+t} \\
 &\leq \frac{2}{x^{\lambda+1}} \int_{x/2}^x (x-t)^\lambda dt \\
 &= \frac{1}{(\lambda+1)2^\lambda}.
 \end{aligned}$$

Hence, $I(x) = O(1)$ in this case.

Finally, since the case $\lambda = 0$ is trivial, the lemma is established.

LEMMA 13. For $\gamma > 1$, and $\lambda > -1$,

$$\begin{aligned}
 &\int_x^{x^\lambda} (1+t)^{\lambda-1} \left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)} \right)^\lambda dt \\
 &= (\gamma-1) \log(1+x) + o(\log(1+x)).
 \end{aligned}$$

Proof. Set $\{s_n\} = \{1\}$. Then $\sigma_\lambda(t) = 1$ and, by Theorem 3, putting $f(t) = \sigma_\lambda(t)$ in (5) gives

$$J_\lambda(x) = 1 + o(1) \quad \text{as } x \longrightarrow \infty.$$

Now by Lemma 12,

$$\begin{aligned}
 &\int_x^{x^\lambda} (1+t)^{\lambda-1} \left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)} \right)^\lambda dt \\
 &= \left(\int_0^{x^\lambda} - \int_0^x \right) (1+t)^{\lambda-1} \left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)} \right)^\lambda dt \\
 &= \log(1+x^\gamma) + o(\log(1+x^\gamma)) - \log(1+x) + o(\log(1+x)) \\
 &\quad + o(1) \\
 &= (\gamma-1) \log(1+x) + o(\log(1+x)).
 \end{aligned}$$

This establishes the lemma.

6. A general tauberian result.

THEOREM 4. *Suppose that the following conditions hold:*

(6) $K(w, t)$ is defined, real-valued, and nonnegative for $w > 0, t \geq 0$; moreover, $\int_0^\infty K(w, t)dt$ exists in the sense of Lebesgue for each $w > 0$,

(7) $\int_0^\infty K(w, t)dt \longrightarrow 1$ as $w \longrightarrow \infty$,

(8) f is real-valued and continuous on $(0, \infty)$,

(9) $F(w) = \int_0^\infty K(w, t)f(t)dt$ exists in the Cauchy-Lebesgue sense for each $w > 0$,

(10) $\liminf\{f(y) - f(x)\} \geq -\mu$ for some fixed finite nonnegative μ , whenever $y \geq x \rightarrow \infty$ and $\Phi(y) - \Phi(x) \rightarrow 0$,

(11) $\Phi(x) - \Phi(x-1) \longrightarrow 0$ as $x \longrightarrow \infty$,

(12) $\int_0^x K(w, t)dt \longrightarrow 0$ whenever $w > x \longrightarrow \infty$ and $\Phi(w) - \Phi(x) \longrightarrow \infty$,

(13) $\int_x^\infty K(w, t)(\Phi(t) - \Phi(x))dt \longrightarrow 0$ whenever $x > w \longrightarrow \infty$ and $\Phi(x) - \Phi(w) \longrightarrow \infty$, and

(14) $F(w) = O(1)$ for $w > 0$.

Then $f(t) = O(1)$ for $t > 0$.

This result was established in [5]. A version of this theorem with (10) replaced by the stronger condition that f be slowly decreasing with respect to Φ can be found in [3]. The proofs are very similar.

7. A theorem on boundedness. In this section we deduce a weakened form of Theorem 1 from the general tauberian result of § 6.

THEOREM 5. *If $\lambda > -1, \infty > \mu \geq 0, s_n \rightarrow s(L)$, and $\liminf\{\sigma_\lambda(y) - \sigma_\lambda(x)\} \geq -\mu$ whenever $y \geq x \rightarrow \infty$ and $\Phi(y) - \Phi(x) \rightarrow 0$, then $\sigma_\lambda(t) = O(1)$.*

Proof. Set

$$K(w, t) = \begin{cases} \frac{1}{\log(1+w)}(1+t)^{\lambda-1} \left(\log \frac{w(1+t)}{t(1+w)} \right)^2 & 0 < t < w \\ 0 & \text{otherwise,} \end{cases}$$

$$\Phi(t) = \begin{cases} t/e^e & 0 \leq t < e^e \\ \log \log t & e^e \leq t, \end{cases}$$

and

$$f(t) = \sigma_\lambda(t).$$

First, note that if $\{s_n\} = \{1\}$, then $s_n \rightarrow 1(L)$ and $\sigma_\lambda(t) = 1$. Hence, by Theorem 3 with $f(t) = \sigma_\lambda(t) = 1$ in (5), we have

$$\begin{aligned} \int_0^\infty K(w, t) dt &= \frac{1}{\log(1+w)} \int_0^w (1+t)^{\lambda-1} \left(\log \frac{w(1+t)}{t(1+w)} \right)^2 dt \\ &= J_\lambda(w) \longrightarrow 1 \quad \text{as } w \longrightarrow \infty. \end{aligned}$$

This establishes (6) and (7).

Conditions (8), (9), (10) and (14) hold by hypotheses, and (11) clearly holds.

Furthermore, condition (13) is immediate since $K(w, t) = 0$ whenever $t \geq w$. It remains to show (12). Suppose $-1 < \lambda < 0$. Then, by Lemma 11, we have

$$\begin{aligned} \int_0^x K(w, t) dt &= \frac{1}{\log(1+w)} \int_0^x (1+t)^{\lambda-1} \left(\log \frac{w(1+t)}{t(1+w)} \right)^2 dt \\ &\leq \frac{1}{\log(1+w)} \int_0^x (1+t)^{\lambda-1} \left(\frac{w-t}{w(1+t)} \right)^2 dt \\ &= \frac{1}{\log(1+w)} \int_0^x (1-t/w)^\lambda \frac{dt}{1+t} \\ &\leq \frac{(1-x/w)^\lambda}{\log(1+w)} \int_0^x \frac{dt}{1+t} \\ &= (1-x/w)^\lambda \frac{\log(1+x)}{\log(1+w)} = o(1) \end{aligned}$$

as $w > x \rightarrow \infty$ and $\log \log w - \log \log x \rightarrow \infty$, since the latter implies $\log x/\log w \rightarrow 0$ and $x/w \rightarrow 0$.

Suppose $\lambda \geq 0$ and $x > 1$. Then

$$\begin{aligned}
\log(1+w) \int_0^x K(w, t) dt &= \int_0^x (1+t)^{\lambda-1} \left(\log \frac{w(1+t)}{t(1+w)} \right)^\lambda dt \\
&\leq \left(\int_0^1 + \int_1^x \right) (1+t)^{\lambda-1} \left(\log \frac{1+t}{t} \right)^\lambda dt \\
&= I_1 + I_2.
\end{aligned}$$

Setting $u = 1/t$ in I_1 gives

$$\begin{aligned}
I_1 &= \int_1^\infty (1+1/u)^{\lambda-1} (\log(1+u))^\lambda \frac{du}{u^2} \\
&= O(1).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
I_2 &= O(1) \int_1^x (1+t)^{-1} dt \\
&= O(1) \log(1+x) - O(1).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_0^x K(w, t) dt \\
&= \frac{1}{\log(1+w)} \{I_1 + I_2\} \\
&= o(1) + O(1) \frac{\log(1+x)}{\log(1+w)} = o(1)
\end{aligned}$$

as $w > x \rightarrow \infty$ and $\log \log w - \log \log x \rightarrow \infty$.

This completes the proof.

8. Proof of Theorem 1. Assign $\varepsilon > 0$. Since $\sigma_\lambda(t)$ is slowly decreasing with respect to $\Phi(t) = \log \log t$, there exist positive numbers X and δ such that $\sigma_\lambda(y) - \sigma_\lambda(x) > -\varepsilon$ whenever $y > x > X$ and $\log \log y - \log \log x < \delta$; or equivalently, writing $\delta = \log \gamma$

$$(15) \quad \sigma_\lambda(x) - \varepsilon < \sigma_\lambda(y) \quad \text{whenever} \quad X < x < y < x^\gamma.$$

Suppose, without loss of generality, that $s = 0$. Then $J_\lambda(w) \rightarrow 0$ as $w \rightarrow \infty$.

Relation (15) implies, for $x > X$, that

$$\begin{aligned}
I_1 &= \int_x^{x^\lambda} (1+t)^{\lambda-1} \left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)} \right)^\lambda (\sigma_\lambda(x) - \varepsilon) dt \\
&\leq \int_x^{x^\gamma} (1+t)^{\lambda-1} \left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)} \right)^\lambda \sigma_\lambda(t) dt \\
&= I_2.
\end{aligned}$$

Now, by Theorem 5 and Lemma 12,

$$\begin{aligned} I_2 &= \left(\int_0^{x^r} - \int_0^x \right) (1+t)^{\lambda-1} \left(\log \frac{x^r(1+t)}{t(1+x^r)} \right)^\lambda \sigma_\lambda(t) dt \\ &= \log(1+x^r) J_\lambda(x^r) - \log(1+x) J_\lambda(x) + O(1) \\ &= o(\log(1+x^r)) + o(\log(1+x)) \\ &= o(\log(1+x)). \end{aligned}$$

By Lemma 13,

$$\begin{aligned} I_1 &= (\sigma_\lambda(x) - \varepsilon) \int_x^{x^r} (1+t)^{\lambda-1} \left(\log \frac{x^r(1+t)}{t(1+x^r)} \right)^\lambda dt \\ &= (\sigma_\lambda(x) - \varepsilon) ((\gamma - 1) \log(1+x) + o(\log(1+x))). \end{aligned}$$

But $I_1 \leq I_2$ implies

$$\sigma_\lambda(x) - \varepsilon \leq \frac{o(1)}{(\gamma - 1) + o(1)}.$$

Therefore,

$$(16) \quad \limsup_{x \rightarrow \infty} \sigma_\lambda(x) \leq \varepsilon.$$

In a similar fashion, we can show that

$$(17) \quad -\varepsilon \leq \liminf_{x \rightarrow \infty} \sigma_\lambda(x).$$

Combining (16) and (17) completes the proof of theorem.

9. A counterexample. In this section we give an example which shows that Theorem 1 would be false if $\log \log t$ were replaced by $\log t$. That is, a more delicate tauberian condition on $\sigma_\lambda(t)$ is required than what is obtained by using the standard definition of slowly decreasing.

LEMMA 14. *If $f(x)$ is absolutely continuous on $[0, T]$ for each $T > 0$ and $f'(x) > -M/x$ for all $x > 0$, then $f(x)$ is slowly decreasing with respect to $\log x$.*

Proof. Assign $\varepsilon > 0$. Then if $y > x > 0$

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t) dt \\ &> -M \int_x^y \frac{1}{t} dt \\ &= -M(\log y - \log x) > -\varepsilon \end{aligned}$$

whenever $\log y - \log x < \varepsilon/M$. This completes the proof.

THEOREM 6. *There exists a sequence $\{s_n\}$ such that $s_n \rightarrow s(L)$ and, for every $\lambda > -1$, $\sigma_\lambda(t)$ is slowly decreasing with respect to $\log t$, but $\{s_n\}$ is not A_λ -convergent.*

Proof. Let $\{s_n\}$ be the real part of the sequence $\{\varepsilon_n^i\}$. For any $\lambda > -1$, $\sigma_\lambda(t)$ exists for $t > 0$, and we have

$$\varepsilon_n^i = \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} \frac{\varepsilon_n^{\lambda+i}}{\varepsilon_n^\lambda} + o(1).$$

Therefore, $\sigma_\lambda(t)$ is the real part of

$$\begin{aligned} (1+t)^{-\lambda-1} \sum_{n=0}^{\infty} \frac{\Gamma(\lambda+i+1)}{\Gamma(\lambda+1)\Gamma(i+1)} \varepsilon_n^{\lambda+i} \left(\frac{t}{1+t}\right)^n &+ (1+t)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda o(1) \left(\frac{t}{1+t}\right)^n \\ &= \frac{\Gamma(\lambda+i+1)}{\Gamma(\lambda+1)\Gamma(i+1)} (1+t)^i + o(1). \end{aligned}$$

The first term above has a derivative which is $O(1/t)$ and, hence, the real part of the first term has a derivative which is $O(1/t)$. The second term is $o(1)$ since A_λ is regular. Hence, the real part of this term is slowly decreasing with respect to any ϕ . Therefore, by Lemma 14, $\sigma_\lambda(t)$ is slowly decreasing with respect to $\log t$.

Next, it is clear that $\{s_n\}$ is not A_λ -convergent.

However,

$$\begin{aligned} J_0(w) &= \frac{1}{\log(1+w)} \int_0^w (1+t)^{-1} \sigma_0(t) dt \\ &= \frac{1}{\log(1+w)} \int_0^w \frac{\cos \log(1+t)}{1+t} dt \\ &= \frac{\sin \log(1+w)}{\log(1+w)} \longrightarrow 0 \quad \text{as } w \longrightarrow \infty. \end{aligned}$$

Hence, by Theorem 3, $s_n \rightarrow O(L)$. This completes the proof.

REFERENCES

1. D. Borwein, *On a scale of Abel-type summability methods*, Proc. Cambridge Phil. Soc., **53** (1957), 318-322.
2. ———, *On methods of summability based on power series*, Proc. Royal Soc. Edinburgh, **64** (1957), 342-349.
3. D. Borwein and B. Watson, *Tauberian theorems on a scale of Abel-type summability methods*, Journal Fur Die Reine Und Angewandte Mathematik, **298** (1978), 1-7.
4. G. H. Hardy, *Divergent Series*, Oxford, 1949.
5. B. Watson, *Tauberian theorems on a scale of Abel-type summability methods*, Ph. D. Thesis, The University of Western Ontario, 1974.

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