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TAUBERIAN THEOREMS BETWEEN THE LOGARITHMIC AND ABEL-TYPE SUMMABILITY METHODS

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The object of this paper is to show that if a series is summable by the logarithmic method L, then the series is also summable by the Abel method A_λ, provided a tauberian condition of the "slowly decreasing" type is satisfied.

1. Introduction. Suppose throughout that $\{s_n\}$ is a sequence of numbers, λ real is real, $\varepsilon_0^\lambda = 1$, $\varepsilon_n^\lambda = \binom{n+\lambda}{n}$ for $n = 1, 2, 3, \cdots$, and

$$v_n^\lambda = \frac{\varepsilon_n^\lambda \Gamma(\lambda+1)}{(n+1)^\lambda} \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

We are concerned with the methods of summability A_λ introduced and studied by Borwein [1] and the logarithmic method L. They are defined as follows. Let

$$(1) \qquad \sigma_\lambda(y) = (1+y)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{y}{1+y}\right)^n , \quad \text{and}$$

$$(2) \qquad L(y) = \frac{1}{\log(1+y)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} \left(\frac{y}{1+y}\right)^{n+1} .$$

If $\sigma_\lambda(y)$ converges for $y > 0$ and tends to s as $y \to \infty$, then we say that the sequence $\{s_n\}$ is A_λ-convergent to s and write $s_n \to s(A_\lambda)$. The method A_0 is the ordinary Abel method.

If $L(y)$ converges for $y > 0$ and tends to s as $y \to \infty$, then we say that $\{s_n\}$ is L-convergent to s and write $s_n \to s(L)$.

Evidently, $s_n \to s(L)$ if and only if

$$-\frac{1}{\log(1-x)} \sum_{n=0}^{\infty} \frac{s_n}{n+1} x^{n+1}$$

converges for $0 < x < 1$ and tends to s as $x \to 1^-$.

LEMMA 1. *A_λ is regular for $\lambda > -1$. [That is, $s_n \to s$ implies $s_n \to s(A_\lambda)$].*

LEMMA 2. *L is regular.*

LEMMA 3. *$A_{\lambda+\varepsilon} \subset A_\lambda$ for $\lambda > -1$, and $\varepsilon > 0$. [That is, $s_n \to s(A_{\lambda+\varepsilon})$ implies $s_n \to s(A_\lambda)$ and there exists a sequence $\{s_n\}$, depending on λ and ε, such that $\{s_n\}$ is A_λ-convergent but not $A_{\lambda+\varepsilon}$-convergent.]*

LEMMA 4. *$A_\lambda \subset L$ for $\lambda > -1$.*

Lemmas 1 and 3 were established by Borwein in [1]. Lemma 4 was proved by Borwein in [2] as a particular case of a more general inclusion theorem on methods of summability based on power series. Lemma 2 is a standard result found, for example, in [4].

2. The main theorem. Suppose that Φ is a nonnegative, continuous, strictly increasing function on $[a, \infty)$, for some a, such that $\Phi(t) \to \infty$ as $t \to \infty$.

The real-valued function f is said to be *slowly decreasing with respect to* Φ if $\liminf\{f(y) - f(x)\} \geqq 0$ whenever $y \geqq x \to \infty$ and $\Phi(y) - \Phi(x) \to 0$.

THEOREM 1. *For $\lambda > -1$, if $s_n \to s(L)$ and $\sigma_\lambda(t)$ is slowly decreasing with respect to* $\log\log t$, *then $s_n \to s(A_\lambda)$.*

In connection with the methods A_λ, we proved the following lemma in [3].

LEMMA 5. *For $\lambda > -1$ and $\varepsilon > 0$, if $s_n \to s(A_\lambda)$ and $\sigma_{\lambda+\varepsilon}(t)$ is slowly decreasing with respect to* $\log t$, *then $s_n \to s(A_{\lambda+\varepsilon})$.*

3. Methods of summability based on power series. Suppose that $p_n \geqq 0$, $q_n \geqq 0$, $\sum_{v=n}^{\infty} p_v > 0$, and $\sum_{v=n}^{\infty} q_v > 0$ for $n = 0, 1, 2, \cdots$. Set

$$p(x) = \sum_{n=0}^{\infty} p_n x^n\,, \quad \text{and}$$

$$q(x) = \sum_{n=0}^{\infty} q_n x^n\,.$$

Let ρ_p and ρ_q denote their respective radii of convergence. We also write

$$p_s(x) = \frac{1}{p(x)} \sum_{n=0}^{\infty} p_n s_n x^n$$

$$q_s(x) = \frac{1}{q(x)} \sum_{n=0}^{\infty} q_n s_n x^n\,.$$

The power series method P is defined as follows. If $\rho_p > 0$, $\sum_{n=0}^{\infty} p_n s_n x^n$ converges for $0 < x < \rho_p$ and $\lim_{x \to \rho_p -} p_s(x) = s$, then we write $s_n \to s(P)$.

The method Q is defined similarly.

Borwein has proved [2] the following lemma.

LEMMA 6. (i) *If* $0 < \rho_p < \infty$, *then a necessary and sufficient condition for* P *to be regular is that* $\sum_{n=0}^{\infty} p_n(\rho_p)^n = \infty$.
(ii) *If* $\rho_p = \infty$ *then* P *is regular.*

Suppose that $\chi(t)$ is a function of bounded variation on $[0, 1]$, and $\chi^*(t)$ is its associated normalized function. That is,

$$\chi^*(t) = \begin{cases} 0 & t = 0 \\ \frac{1}{2}\{\chi(t+) + \chi(t-)\} - \chi(0) & 0 < t < 1 \\ \chi(1) - \chi(0) & t = 1 . \end{cases}$$

A sequence $\{\mu_n\}$ is called an *m-sequence* if, for some χ,

$$\mu_n = \int_0^1 t^n d\chi(t) \quad \text{for} \quad n = 0, 1, 2, \cdots .$$

If, in addition,

$$\mu_n \geqq \delta \int_0^1 t^n |d\chi^*(t)| \quad \text{for} \quad 0 < \delta \leqq 1 \quad \text{and}$$

$n = N, N+1, \cdots$, then $\{\mu_n\}$ is called an *$\bar{m}$-sequence.*

LEMMA 7. *If* $p_n = \mu_n q_n (n = N, N+1, \cdots)$, $\{\mu_n\}$ *is an $\bar{m}$-sequence,* $\rho_p = \rho_q > 0$, *and* P *is regular, then* $Q \subseteqq P$. *(That is,* $s_n \to s(Q)$ *implies* $s_n \to s(P)$.)

This result is due to Borwein (see [2], Theorem A′).
We require the following two lemmas.

LEMMA 8. *An m-sequence which converges to a positive limit is an $\bar{m}$-sequence.*

LEMMA 9. *The sequences* $\{v_n^\lambda\}$ *and* $\{1/v_n^\lambda\}$ *are $\bar{m}$-sequences for* $\lambda > -1$.

The proof of Lemma 8 is straightforward and Lemma 9 was established in [4], Theorem 211.

The next result is used in the proof of Theorem 1.

THEOREM 2. *Let* Q *be a regular power series method and suppose that* $\{\mu_n\}$ *is an $\bar{m}$-sequence such that* $\mu_n \to a > 0$. *Then* $\mu_n s_n \to as(Q)$

whenever $s_n \to s(Q)$.

Proof. Suppose that $s_n \to s(Q)$. Set $p_n = \mu_n q_n$ for $n = 0, 1, 2, \cdots$. Since $\mu_n \geqq 0$ and $\mu_n \to a$ it is easy to verify that $\rho_p = \rho_q$. If $\rho_p = \infty$, then P is regular by Lemma 6(ii). Otherwise, since $p_n \sim a q_n$, P is regular by Lemma 6(i).

Therefore, by Lemma 7, $s_n \to s(P)$. That is,

$$(3) \qquad \frac{1}{p(x)} \sum_{n=0}^{\infty} s_n \mu_n q_n x^n \longrightarrow s \quad \text{as} \quad x \longrightarrow \rho_P^- .$$

In addition, since Q is regular,

$$(4) \qquad \frac{p(x)}{q(x)} = \frac{1}{q(x)} \sum_{n=0}^{\infty} \mu_n q_n x^n \longrightarrow a \quad \text{as} \quad x \longrightarrow \rho_q^- .$$

Application of Q to $\{\mu_n s_n\}$ *yields*

$$\begin{aligned} &\frac{1}{q(x)} \sum_{n=0}^{\infty} \mu_n s_n q_n x^n \\ &\quad = \frac{p(x)}{q(x)} \frac{1}{p(x)} \sum_{n=0}^{\infty} s_n \mu_n q_n x^n \\ &\quad \longrightarrow as \quad \text{as} \quad x \longrightarrow \rho_q^- = \rho_p^- \text{ by (3) and (4)} . \end{aligned}$$

This completes the proof.

COROLLARY TO THEOREM 2. $s_n \to s(L)$ *if and only if* $v_n^\lambda s_n \to s(L)$.

This is immediate in view of Lemmas 8 and 9, and the fact that $v_n^\lambda \to 1$ as $n \to \infty$.

4. An integral transformation. The integral transformation $J_\lambda(w)$ of the function $f(t)$, for $\lambda > -1$ and $w > 0$, is defined as follows.

$$(5) \qquad J_\lambda(w) = \frac{1}{\log(1 + w)} \int_0^w (1 + t)^{\lambda - 1} \left(\log \frac{w(1 + t)}{t(1 + w)}\right)^\lambda f(t) dt .$$

THEOREM 3. *If* $\lambda > -1$ *and* $f(t) = \sigma_\lambda(t)$ *is convergent for all* $t > 0$, *then* $J_\lambda(w) \to s$ *as* $w \to \infty$ *if and only if* $s_n \to s(L)$.

Proof. Setting $u = (t(1 + w))/(w(1 + t))$ in $J_\lambda(w)$ gives

$$\begin{aligned} &J_\lambda(w) \\ &\quad = \frac{1}{\log(1+w)} \int_0^w (1 + t)^{\lambda - 1} \left(\log \frac{w(1+t)}{t(1+w)}\right)^\lambda (1 + t)^{-\lambda - 1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{t}{1+t}\right)^n dt \end{aligned}$$

$$= \frac{1}{\log(1+w)} \int_0^1 \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{w}{1+w}\right)^{n+1} u^n \left(\log \frac{1}{u}\right)^\lambda du$$

$$= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n \left(\frac{w}{1+w}\right)^{n+1} \int_0^1 u^n \left(\log \frac{1}{u}\right)^\lambda du$$

$$= \frac{\Gamma(\lambda+1)}{\log(1+w)} \sum_{n=0}^{\infty} \frac{\varepsilon_n^\lambda}{(n+1)^{\lambda+1}} s_n \left(\frac{w}{1+w}\right)^{n+1}$$

$$= \frac{1}{\log(1+w)} \sum_{n=0}^{\infty} \frac{v_n^\lambda s_n}{n+1} \left(\frac{w}{1+w}\right)^{n+1}.$$

The convergence, for $t > 0$, of the series defining $\sigma_\lambda(t)$ implies its absolute convergence. This justifies the integration term by term and, in view of the corollary to Theorem 2, the proof is complete.

5. Additional lemmas.

Lemma 10. *For* $\lambda > -1$, $\sum_{n=0}^{\infty} \varepsilon_n^\lambda s_n x^n$ *is absolutely convergent for* $|x| < 1$ *if and only if* $\sum_{n=0}^{\infty} (s_n/(n+1))x^n$ *is absolutely convergent for* $|x| < 1$.

We omit the simple proof.

Lemma 11. *For* $0 < t < w$,

$$\log \frac{w(1+t)}{t(1+w)} > \frac{w-t}{w(1+t)}.$$

Proof. For $x > 1$,

$$\log x = \log x - \log 1 = \frac{x-1}{\theta} > \frac{x-1}{x}$$

where $1 < \theta < x$. The result follows by observing that, for $0 < t < w$, $x = (w(1+t))/(t(1+w)) > 1$.

Lemma 12. *For fixed* $\gamma > 1$ *and* $\lambda > -1$,

$$I(x) = \int_0^x (1+t)^{\lambda-1}\left(\left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda - \left(\log \frac{x(1+t)}{t(1+x)}\right)^\lambda\right)dt$$
$$= O(1).$$

Proof. Suppose $\lambda \geqq 1$. Then, for $x \geqq 1$,

$$|I(x)| = I(x)$$
$$\leqq \lambda \log \frac{x^\gamma(1+x)}{x(1+x^\gamma)} \int_0^x (1+t)^{\lambda-1}\left(\log \frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^{\lambda-1} dt$$

$$\leqq \lambda \log \frac{x^r(1+x)}{x(1+x^r)}\left(\int_0^1 + \int_1^x\right)(1+t)^{\lambda-1}\left(\log\frac{1+t}{t}\right)^{\lambda-1}dt$$
$$= I_1(x) + I_2(x) .$$

Now,

$$\int_0^1 (1+t)^{\lambda-1}\left(\log\frac{1+t}{t}\right)^{\lambda-1}dt < \infty .$$

Hence,

$$I_1(x) = O(1) .$$

Also,

$$I_2(x) = O(1) \log \frac{x^r(1+x)}{x(1+x^r)} \int_1^x dt$$
$$= O(1) x \log \frac{1+x}{x} = O(1) .$$

Suppose $0 < \lambda < 1$. By Lemma 11 we have,

$$|I(x)| = I(x)$$
$$\leqq \lambda \log \frac{x^r(1+x)}{x(1+x^r)} \int_0^x (1+t)^{\lambda-1}\left(\log\frac{x(1+t)}{t(1+x)}\right)^{\lambda-1}dt$$
$$< \lambda\frac{M}{x}\int_0^x (1+t)^{\lambda-1}\left(\frac{x-t}{x(1+t)}\right)^{\lambda-1}dt$$

since $x \log (x^r(1+x))/(x(1+x^r)) \leqq M$.

Therefore

$$I(x) \leqq \lambda\frac{M}{x^\lambda}\int_0^x (x-t)^{\lambda-1}dt = M .$$

Suppose $-1 < \lambda < 0$. Then

$$|I(x)| = -I(x)$$
$$= \left(\int_0^{x/2} + \int_{x/2}^x\right)(1+t)^{\lambda-1}\left(\left(\log\frac{x(1+t)}{t(1+x)}\right)^\lambda - \left(\log\frac{x^r(1+t)}{t(1+x^\lambda)}\right)^\lambda\right)dt$$
$$= I_1(x) + I_2(x) .$$

Using Lemma 11 and the fact that

$$\left| x \log \frac{x(1+x^r)}{(1+x)x^r} \right| \leqq M$$

we have

$$0 \leqq I_1(x) \leqq \lambda\left(\log \frac{x(1+x^\gamma)}{x^\gamma(1+x)}\right)\int_0^{x/2}(1+t)^{\lambda-1}\left(\log \frac{x(1+t)}{t(1+x)}\right)^{\lambda-1}dt$$

$$\leqq -\frac{\lambda M}{x}\int_0^{x/2}(1+t)^{\lambda-1}\left(\frac{x-t}{x(1+t)}\right)^{\lambda-1}dt$$

$$= M((1/2)^\lambda - 1)\ .$$

For $I_2(x)$, since $1 + t > x/2$,

$$0 \leqq I_2(x) \leqq \int_{x/2}^{x}(1+t)^{\lambda-1}\left(\log \frac{x(1+t)}{t(1+x)}\right)^{\lambda}dt$$

$$\leqq \int_{x/2}^{x}(1+t)^{\lambda-1}\left(\frac{x-t}{x(1+t)}\right)^{\lambda}dt$$

$$= \frac{1}{x^\lambda}\int_{x/2}^{x}(x-t)^\lambda \frac{dt}{1+t}$$

$$\leqq \frac{2}{x^{\lambda+1}}\int_{x/2}^{x}(x-t)^\lambda dt$$

$$= \frac{1}{(\lambda+1)2^\lambda}\ .$$

Hence, $I(x) = O(1)$ in this case.

Finally, since the case $\lambda = 0$ is trivial, the lemma is established.

LEMMA 13. *For* $\gamma > 1$, *and* $\lambda > -1$,

$$\int_x^{x^\lambda}(1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^{\lambda}dt$$

$$= (\gamma - 1)\log(1+x) + o(\log(1+x))\ .$$

Proof. Set $\{s_n\} = \{1\}$. Then $\sigma_\lambda(t) = 1$ and, by Theorem 3, putting $f(t) = \sigma_\lambda(t)$ in (5) gives

$$J_\lambda(x) = 1 + o(1) \quad \text{as} \quad x \longrightarrow \infty\ .$$

Now by Lemma 12,

$$\int_x^{x^\lambda}(1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^{\lambda}dt$$

$$= \left(\int_0^{x^\lambda} - \int_0^{x}\right)(1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^{\lambda}dt$$

$$= \log(1+x^\gamma) + o(\log(1+x^\gamma)) - \log(1+x) + o(\log(1+x)) + o(1)$$

$$= (\gamma - 1)\log(1+x) + o(\log(1+x))\ .$$

This establishes the lemma.

6. A general tauberian result.

THEOREM 4. *Suppose that the following conditions hold*:

(6) $K(w, t)$ *is defined, real-valued, and nonnegative for* $w > 0, t \geqq 0$; *moreover,* $\int_0^\infty K(w, t)dt$ *exists in the sense of Lebesgue for each* $w > 0$,

$$\int_0^\infty K(w, t)dt \longrightarrow 1 \quad as \quad w \longrightarrow \infty\,, \tag{7}$$

(8) f *is real-valued and continuous on* $(0, \infty)$,

(9) $F(w) = \int_0^\infty K(w, t)f(t)dt$ *exists in the Cauchy-Lebesgue sense for each* $w > 0$,

(10) $\liminf\{f(y) - f(x)\} \geqq -\mu$ *for some fixed finite nonnegative* μ, *whenever* $y \geqq x \to \infty$ *and* $\Phi(y) - \Phi(x) \to 0$,

$$\Phi(x) - \Phi(x-1) \longrightarrow 0 \quad as \quad x \longrightarrow \infty\,, \tag{11}$$

$$\int_0^x K(w, t)dt \longrightarrow 0 \quad whenever \quad w > x \longrightarrow \infty \quad and \quad \Phi(w) - \Phi(x) \longrightarrow \infty\,, \tag{12}$$

$$\int_x^\infty K(w, t)(\Phi(t) - \Phi(x))dt \longrightarrow 0 \quad whenever \quad x > w \longrightarrow \infty \quad and \quad \Phi(x) - \Phi(w) \longrightarrow \infty\,, \quad and \tag{13}$$

$$F(w) = O(1) \quad for \quad w > 0\,. \tag{14}$$

Then $f(t) = O(1)$ *for* $t > 0$.

This result was established in [5]. A version of this theorem with (10) replaced by the stronger condition that f be slowly decreasing with respect to Φ can be found in [3]. The proofs are very similar.

7. A theorem on boundedness. In this section we deduce a weakened form of Theorem 1 from the general tauberian result of §6.

THEOREM 5. *If* $\lambda > -1$, $\infty > \mu \geqq 0$, $s_n \to s(L)$, *and* $\liminf\{\sigma_\lambda(y) - \sigma_\lambda(x)\} \geqq -\mu$ *whenever* $y \geqq x \to \infty$ *and* $\Phi(y) - \Phi(x) \to 0$, *then* $\sigma_\lambda(t) = O(1)$.

Proof. Set

$$K(w, t) = \begin{cases} \dfrac{1}{\log(1+w)}(1+t)^{\lambda-1}\left(\log\dfrac{w(1+t)}{t(1+w)}\right)^{\lambda} & o < t < w \\ 0 & \text{otherwise}, \end{cases}$$

$$\Phi(t) = \begin{cases} t/e^e & 0 \leqq t < e^e \\ \log\log t & e^e \leqq t, \end{cases}$$

and

$$f(t) = \sigma_\lambda(t) .$$

First, note that if $\{s_n\} = \{1\}$, then $s_n \to 1(L)$ and $\sigma_\lambda(t) = 1$. Hence, by Theorem 3 with $f(t) = \sigma_\lambda(t) = 1$ in (5), we have

$$\begin{aligned}\int_0^\infty & K(w, t)dt \\ &= \frac{1}{\log(1+w)}\int_0^w (1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^{\lambda} dt \\ &= J_\lambda(w) \longrightarrow 1 \quad \text{as} \quad w \longrightarrow \infty .\end{aligned}$$

This establishes (6) and (7).

Conditions (8), (9), (10) and (14) hold by hypotheses, and (11) clearly holds.

Furthermore, condition (13) is immediate since $K(w, t) = 0$ whenever $t \geqq w$. It remains to show (12). Suppose $-1 < \lambda < 0$. Then, by Lemma 11, we have

$$\begin{aligned}\int_0^x & K(w, t)dt \\ &= \frac{1}{\log(1+w)}\int_0^x (1+t)^{\lambda-1}\left(\log\frac{w(1+t)}{t(1+w)}\right)^{\lambda} dt \\ &\leqq \frac{1}{\log(1+w)}\int_0^x (1+t)^{\lambda-1}\left(\frac{w-t}{w(1+t)}\right)^{\lambda} dt \\ &= \frac{1}{\log(1+w)}\int_0^x (1-t/w)^{\lambda}\frac{dt}{1+t} \\ &\leqq \frac{(1-x/w)^{\lambda}}{\log(1+w)}\int_0^x \frac{dt}{1+t} \\ &= (1-x/w)^{\lambda}\frac{\log(1+x)}{\log(1+w)} = o(1)\end{aligned}$$

as $w > x \to \infty$ and $\log\log w - \log\log x \to \infty$, since the latter implies $\log x/\log w \to 0$ and $x/w \to 0$.

Suppose $\lambda \geqq 0$ and $x > 1$. Then

$$\begin{aligned}\log(1+w)\int_0^x K(w,t)dt &= \int_0^x (1+t)^{\lambda-1}\Big(\log\frac{w(1+t)}{t(1+w)}\Big)^\lambda dt\\ &\leqq \Big(\int_0^1+\int_1^x\Big)(1+t)^{\lambda-1}\Big(\log\frac{1+t}{t}\Big)^\lambda dt\\ &= I_1+I_2\,.\end{aligned}$$

Setting $u = 1/t$ in I_1 gives

$$\begin{aligned}I_1 &= \int_1^\infty (1+1/u)^{\lambda-1}(\log(1+u))^\lambda \frac{du}{u^2}\\ &= O(1)\,.\end{aligned}$$

Furthermore,

$$\begin{aligned}I_2 &= O(1)\int_1^x (1+t)^{-1}dt\\ &= O(1)\log(1+x) - O(1)\,.\end{aligned}$$

Therefore,

$$\begin{aligned}&\int_0^x K(w,t)dt\\ &\quad= \frac{1}{\log(1+w)}\{I_1+I_2\}\\ &\quad= o(1) + O(1)\frac{\log(1+x)}{\log(1+w)} = o(1)\end{aligned}$$

as $w > x \to \infty$ and $\log\log w - \log\log x \to \infty$.

This completes the proof.

8. Proof of Theorem 1. Assign $\varepsilon > 0$. Since $\sigma_\lambda(t)$ is slowly decreasing with respect to $\Phi(t) = \log\log t$, there exist positive numbers X and δ such that $\sigma_\lambda(y) - \sigma_\lambda(x) > -\varepsilon$ whenever $y > x > X$ and $\log\log y - \log\log x < \delta$; or equivalently, writing $\delta = \log\gamma$

$$\sigma_\lambda(x) - \varepsilon < \sigma_\lambda(y) \quad \text{whenever} \quad X < x < y < x^\gamma\,. \tag{15}$$

Suppose, without loss of generality, that $s = 0$. Then $J_\lambda(w) \to 0$ as $w \to \infty$.

Relation (15) implies, for $x > X$, that

$$\begin{aligned}I_1 &= \int_x^{x^\lambda}(1+t)^{\lambda-1}\Big(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\Big)^\lambda(\sigma_\lambda(x)-\varepsilon)dt\\ &\leqq \int_x^{x^\gamma}(1+t)^{\lambda-1}\Big(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\Big)^\lambda\sigma_\lambda(t)dt\\ &= I_2\,.\end{aligned}$$

Now, by Theorem 5 and Lemma 12,

$$\begin{aligned} I_2 &= \left(\int_0^{x^\gamma} - \int_0^x\right)(1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda \sigma_\lambda(t)dt \\ &= \log(1+x^\gamma)J_\lambda(x^\gamma) - \log(1+x)J_\lambda(x) + O(1) \\ &= o(\log(1+x^\gamma)) + o(\log(1+x)) \\ &= o(\log(1+x)) . \end{aligned}$$

By Lemma 13,

$$\begin{aligned} I_1 &= (\sigma_\lambda(x) - \varepsilon)\int_x^{x^\gamma}(1+t)^{\lambda-1}\left(\log\frac{x^\gamma(1+t)}{t(1+x^\gamma)}\right)^\lambda dt \\ &= (\sigma_\lambda(x) - \varepsilon)((\gamma - 1)\log(1+x) + o(\log(1+x))) . \end{aligned}$$

But $I_1 \leqq I_2$ implies

$$\sigma_\lambda(x) - \varepsilon \leqq \frac{o(1)}{(\gamma-1) + o(1)} .$$

Therefore,

$$\limsup_{x\to\infty} \sigma_\lambda(x) \leqq \varepsilon . \tag{16}$$

In a similar fashion, we can show that

$$-\varepsilon \leqq \liminf_{x\to\infty} \sigma_\lambda(x) . \tag{17}$$

Combining (16) and (17) completes the proof of theorem.

9. A counterexample. In this section we give an example which shows that Theorem 1 would be false if $\log\log t$ were replaced by $\log t$. That is, a more delicate tauberian condition on $\sigma_\lambda(t)$ is required than what is obtained by using the standard definition of slowly decreasing.

LEMMA 14. *If $f(x)$ is absolutely continuous on $[0, T]$ for each $T > 0$ and $f'(x) > -M/x$ for all $x > 0$, then $f(x)$ is slowly decreasing with respect to* $\log x$.

Proof. Assign $\varepsilon > 0$. Then if $y > x > 0$

$$\begin{aligned} f(y) - f(x) &= \int_x^y f'(t)dt \\ &> -M\int_x^y \frac{1}{t}dt \\ &= -M(\log y - \log x) > -\varepsilon \end{aligned}$$

whenever $\log y - \log x < \varepsilon/M$. This completes the proof.

THEOREM 6. *There exists a sequence* $\{s_n\}$ *such that* $s_n \to s(L)$ *and, for every* $\lambda > -1$, $\sigma_\lambda(t)$ *is slowly decreasing with respect to* $\log t$, *but* $\{s_n\}$ *is not* A_λ*-convergent.*

Proof. Let $\{s_n\}$ be the real part of the sequence $\{\varepsilon_n^i\}$. For any $\lambda > -1$, $\sigma_\lambda(t)$ exists for $t > 0$, and we have

$$\varepsilon_n^i = \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} \frac{\varepsilon_n^{\lambda+1}}{\varepsilon_n^\lambda} + o(1) .$$

Therefore, $\sigma_\lambda(t)$ is the real part of

$$(1 + t)^{-\lambda-1} \sum_{n=0}^{\infty} \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)} \varepsilon_n^{\lambda+i}\left(\frac{t}{1 + t}\right)^n + (1 + t)^{-\lambda-1} \sum_{n=0}^{\infty} \varepsilon_n^\lambda o(1)\left(\frac{t}{1 + t}\right)^n$$
$$= \frac{\Gamma(\lambda + i + 1)}{\Gamma(\lambda + 1)\Gamma(i + 1)}(1 + t)^i + o(1) .$$

The first term above has a derivative which is $O(1/t)$ and, hence, the real part of the first term has a derivative which is $O(1/t)$. The second term is $o(1)$ since A_λ is regular. Hence, the real part of this term is slowly decreasing with respect to any Φ. Therefore, by Lemma 14, $\sigma_\lambda(t)$ is slowly decreasing with respect to $\log t$.

Next, it is clear that $\{s_n\}$ is not A_λ-convergent.

However,

$$\begin{aligned} J_0(w) &= \frac{1}{\log(1 + w)} \int_0^w (1 + t)^{-1}\sigma_0(t)dt \\ &= \frac{1}{\log(1 + w)} \int_0^w \frac{\cos\log(1 + t)}{1 + t} dt \\ &= \frac{\sin\log(1 + w)}{\log(1 + w)} \longrightarrow 0 \quad \text{as} \quad w \longrightarrow \infty . \end{aligned}$$

Hence, by Theorem 3, $s_n \to O(L)$. This completes the proof.

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