SEMIGROUPS OF QUASINORMAL OPERATORS

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Strongly continuous semi-groups \{Q_t\} of quasinormal operators on Hilbert space are characterized as follows: there exist Hilbert spaces \(L\) and \(K\), a strongly continuous normal semi-group \(\{N_t\}\) on \(L\) and a strongly continuous self-adjoint semi-group \(\{h(t)\}\) on \(K\) such that \(\{Q_t\}\) is unitarily equivalent to \(\{N_t\} \oplus \{h(t)\} L_t^\ast\) on \(L \oplus L^2(K)\), where \(\{L_t\}\) is the forward translation semi-group on \(L^\infty(K)\) and \((h(t)f)(x) = h(t)f(x)\) a.e. for each \(f\) in \(L^2(K)\).

1. Preliminaries. In this paper we characterize one parameter strongly continuous semi-groups of quasinormal operators. The major result, found in Theorem 6, bears a marked resemblance to the characterization of quasinormal operators given by Brown in [2]. He showed that an operator \(A\) is quasinormal (\(A\) commutes with \(A^*A\)) if and only if there exist Hilbert spaces \(L\) and \(K\), a normal operator \(N\) on \(L\) and a positive operator \(P\) on \(K\) such that \(A\) is unitarily equivalent to \(N \oplus S^P\) on \(L \oplus L^\infty(K)\) where \(S\) is the unilateral shift on \(L^\infty(K)\) and \((Px)_k = Px_k\) whenever \(\{x_k\} \in L^2(K)\).

We shall use the following notation and conventions. \(H\) is a separable Hilbert space and \(B(H)\) is the space of continuous linear operators on \(H\). \(L^\infty(H)\) is the Hilbert space of all sequences \(\{x_n\}\) where \(x_n \in H\) and \(\Sigma \|x_n\|^2 < \infty\). In particular, \(L^2 = L^2(\mathbb{C})\), where \(\mathbb{C}\) is the set of complex numbers. \(\mathbb{R}_+\) denotes the set of non-negative real numbers. \(L^2(H)\) will stand for the Hilbert space of (equivalence classes) of weakly measurable functions from \(\mathbb{R}_+\) into \(H\) such that

\[
\int_0^\infty \|f(x)\|^2 dx < \infty. \quad \text{In particular, } L^2 = L^2(\mathbb{C}).
\]

An operator \(A\) on \(H\) is self-adjoint if \(A = A^*\), normal if \(AA^* = A^*A\), subnormal if \(A\) is the restriction of a normal operator to an invariant subspace, an isometry if \(A^*A = I\) where \(I\) is the identity operator on \(H\), a partial isometry if \((A^*A)^2 = A^*A\), and unitary if \(A\) is a normal isometry.

We use [3] as a general reference on semi-groups of operators. The set \(\{S_t\} = \{S_t : t \in \mathbb{R}_+\}\) is a semi-group of elements of \(B(H)\) if \(S_{t+r} = S_t S_r\) for all \(t\) and \(r\) in \(\mathbb{R}_+\) and \(S_0 = I\). We say that \(\{S_t\}\) has a certain property (for example, is quasinormal) if each of the operators \(S_t\) has that property. A semi-group \(\{S_t\}\) is strongly con-
tinuous if \( \lim_{t \to 0} \| S_t f - f \| = 0 \) for each \( f \) in \( \mathcal{H} \) and uniformly continuous if \( \lim_{t \to 0} \| S_t - I \| = 0 \). The generator of a strongly continuous semi-group \( \{ S_t \} \) is the (not necessarily bounded) linear transformation \( S \) defined by \( Sf = \lim_{t \to 0} (S_t f - f) / t \), whenever this limit exists in the strong topology.

One semi-group which will play a prominent part in the development of ideas is the forward translation semi-group \( \{ L_t \} \) defined for each \( f \) in \( L^2(\mathbb{R}) \) by \( (L_t f)(x) = f(x - t) \) if \( x \geq t \) and zero otherwise. It is well-known that \( \{ L_t \} \) is a strongly continuous semi-group and the infinitesimal generator of \( \{ L_t \} \) is defined by \( f \mapsto -f' \) for all \( f \) in \( L^2(\mathbb{R}) \) for which \( f \) is absolutely continuous, \( f' \in L^2(\mathbb{R}) \) and \( f(0) = 0 \). We shall denote this unbounded operator by \(-D\). The semi-group of adjoints \( \{ L_t^* \} \) is the backward translation semi-group and for each \( f \) in \( L^2(\mathbb{R}) \), \( (L_t^* f)(x) = f(x + t) \). The generator of \( \{ L_t^* \} \) is defined by \( f \mapsto f' \) for all \( f \) in \( L^2(\mathbb{R}) \) for which \( f \) is absolutely continuous and \( f' \in L^2(\mathbb{R}) \).

The isometric semi-groups \( \{ U_t \} \) are obviously quasinormal. In [5] Cooper characterizes them as follows: a strongly continuous semi-group \( \{ U_t \} \) is isometric if and only if there exist Hilbert spaces \( \mathcal{L} \) and \( \mathcal{H} \) and a unitary semi-group \( \{ W_t \} \) on \( \mathcal{L} \) such that \( \{ U_t \} \) is unitarily equivalent to \( \{ W_t \} \oplus \{ L_t \} \) on \( \mathcal{L} \oplus L^2(\mathbb{R}) \). In §2 we show that \( \{ Q_t \} \) can be factored into an isometric semi-group and a self-adjoint semi-group, each of which is strongly continuous and which commute with one another. This reduces the general problem of characterizing quasinormal semi-groups to that of characterizing those semi-groups of the form \( \{ H_t L_t \} \) where \( \{ H_t \} \) is a self-adjoint semi-group commuting with \( \{ L_t \} \). In §3 we complete the characterization.

In §4 we investigate the properties of the infinitesimal generator of a quasinormal semi-group and give an explicit representation for it in terms of the characterization of the semi-group.

2. Factoring semi-groups. Let \( \phi \) be a continuous, almost every where nonzero function from \( \mathbb{R}_+ \) into \( \mathbb{C} \) and define \( (S_t f)(x) = (\phi(x)/\phi(x - t))f(x - t) \) if \( x \geq t \) and zero otherwise for \( f \) in \( L^2(\mathbb{R}) \). Under suitable boundedness conditions on \( \phi \), \( \{ S_t \} \) is a strongly continuous translation semi-group in \( \mathcal{B}(L^2) \) [7, p. 334] and is called a weighted translation semi-group. Such a semi-group is quasinormal exactly when \( \phi \) is a multiple of an exponential: \( \phi(x) = Me^{\alpha x} \) [7, p. 340–341]. A straightforward computation shows that \( \{ S_t^* S_t \} \) is a semi-group exactly when \( \phi(x + t + s)\phi(x) = \phi(x + t)\phi(x + s) \) for all \( x, t, s \), or equivalently, when \( \phi \) is a multiple of an exponential. Therefore \( \{ S_t \} \) is quasinormal exactly when \( \{ S_t^* S_t \} \) is a semi-group. In Lemma 1 we show that this equivalence always occurs.
LEMMA 1. Let \( \{Q_t\} \) be a strongly continuous semi-group of operators. \( \{Q_t\} \) is quasinormal if and only if \( \{Q^*_tQ_t\} \) is a semi-group. Moreover in this case \( \{Q^*_tQ_t\} \) is strongly continuous and \( Q_r \) commutes with \( \{Q^*_tQ_t\} \) for each \( r \) and \( t \).

Proof. Assume first that \( \{Q_t\} \) is quasinormal. Every quasinormal operator is subnormal [9] and every strongly continuous semi-group of subnormal operators has a normal extension as a semi-group [10]. That is, there exists a strongly continuous normal semi-group \( \{N_t\} \) of operators on a Hilbert space \( \mathcal{H} \), containing \( \mathcal{H} \), with \( N_t\mathcal{H} = Q_t \). Since \( Q_t \) is quasinormal, then \( \mathcal{H} \) is invariant under \( N_t^*N_t \) [4] and since \( \{N_t\} \) is a strongly continuous normal semi-group, it follows that \( \{N_t^*N_t\} \) is a strongly continuous semi-group and \( N_t \) commutes with \( N_t^*N_t \) for each \( r \) and \( t \). Consequently, \( \{Q_t^*Q_t\} \) inherits the same properties.

On the other hand if we assume that \( \{Q_t^*Q_t\} \) is a semi-group, then for each \( t \) and each nonnegative integer \( n \), \( (Q_t^*)^n(Q_t)^n = Q_{nt}^*Q_{nt} = (Q_t^*Q_t)^n \), which is sufficient to imply that each \( Q_t \) is quasinormal [6].

By the polar decomposition of an operator \( A \) we mean the unique representation \( A = UP \) where \( P \) is the unique square root of \( A^*A \) and \( U \) is a partial isometry such that \( \ker U = \ker P = \ker A \). A necessary and sufficient condition that \( A \) be quasinormal is that \( U \) and \( P \) commute [2]. It is not difficult to show that when \( A \) is quasinormal, the polar decomposition of \( A^* \) is \( U^*P^* \). The continuous analogues of these assertions are found in the following theorem.

THEOREM 2. For each \( t \) in \( \mathbb{R}_+ \) let \( U_tP_t \) be the polar decomposition of \( Q_t \). Then \( \{Q_t\} \) is a strongly continuous quasinormal semi-group if and only if

(i) \( \{P_t\} \) is a strongly continuous self-adjoint semi-group,
(ii) \( \{U_t\} \) is a strongly continuous isometric semi-group, and
(iii) \( P_r \) commutes with \( U_t \) for each \( r \) and \( t \).

Proof. Obviously, if conditions (i), (ii), and (iii) are true, then \( \{Q_t\} \) is a quasinormal semi-group. Moreover, in this case \( \{Q_t\} \) is the product of strongly continuous semi-groups and is, itself, strongly continuous.

Assume now that \( \{Q_t\} \) is a strongly continuous quasinormal semi-group. \( P_t \) is the positive square root of \( Q_t^*Q_t \). Therefore since \( P_t^2 \) and \( P_t^2 \) commute, so do \( P_t \) and \( P_r \) for all \( t \) and \( r \). This implies that \( (P_{t+r})^2 = (P_tP_r)^2 \). Since the positive square roots are unique, \( P_{t+r} = P_tP_r \) and \( \{P_t\} \) is a semi-group of self-adjoint operators. Moreover, since \( P_t - I = (P_t + I)^{-1}(P_t^2 - I) \) and \( \{P_t\} \) is strongly continuous by Lemma 1, then so is \( \{P_t\} \). (We use here the fact that
\[(P_t + I)^{-1} \leq 1 \text{ since } P_t \text{ is positive.}\]

To show that \(U_t\) is an isometry, we only need show that \(\ker P_t = \{0\}\). But if \(P_t f = 0\), then \(P_t U_t f = 0\) since \(P_t\) is positive. Thus by induction there is a sequence \(t_n \to 0\) such that \(P_{t_n} f = 0\). Using the strong continuity of \(\{P_t\}\) we arrive at \(f = 0\).

Since \(\ker P_t = \{0\}\), any operator commuting with \(Q_t\) and \(P_t\) also commutes with \(U_t\). Also, \(Q_r\) commutes with \(P_t\) for each \(r\) and \(t\) by Lemma 1. Therefore since each of \(\{P_t\}\) and \(\{Q_t\}\) is commutative, \(U_r\) commutes with \(P_t\) and \(U_t\) for each \(r\) and \(t\). Also \(U_t U_s P_{t+s} = U_t P_t U_s P_s = Q_t Q_s = Q_{t+s} = U_{t+s} P_{t+s}\) so that \(U_t U_s = U_{t+s}\) on the range of \(P_{t+s}\) which is a dense subset of \(\mathcal{H}\). We have shown that \(\{U_t\}\) is an isometric semi-group.

To show that \(\{U_t\}\) is strongly continuous we argue as follows: For \(f\) and \(g\) in \(\mathcal{H}\)

\[
\|f - U_t f, g\| = | \langle f - Q_t f, g \rangle + \langle P_t f - f, U_t^* g \rangle | \\
\leq (\|f - Q_t f\| + \|P_t f - f\|) \|g\|,
\]

and consequently

\[
\|f - U_t f\| \leq \|f - Q_t f\| + \|P_t f - f\|.
\]

Strong continuity of \(\{Q_t\}\) and \(\{P_t\}\) now implies strong continuity of \(\{U_t\}\).

**Remark 1.** We note that \(\{Q_t\}\) is normal if and only if \(\{U_t\}\) is unitary. This follows from Theorem 2(ii) and the fact that a quasinormal operator is normal if and only if the partial isometry in the polar decomposition of \(Q\) is normal.

In view of the nice behavior of the sets \(\{U_t\}\) and \(\{P_t\}\) when \(\{Q_t\}\) is quasinormal, we shall write \(\{Q_t\} = \{U_t\}\{P_t\}\) and call \(\{U_t\}\) the isometric factor of \(\{Q_t\}\) and \(\{P_t\}\) the positive factor.

3. A characterization of quasinormal semi-groups.

**Theorem 3.** Let \(\{Q_t\}\) be a strongly continuous quasinormal semi-group. There exist Hilbert spaces \(L\) and \(\mathcal{H}\), a strongly continuous normal semi-group \(\{N_t\}\) on \(L\) and a strongly continuous self-adjoint semi-group \(\{H_t\}\) on \(L(\mathcal{H})\) commuting with \(\{L_t\}\), such that \(\{Q_t\}\) is unitarily equivalent to \(\{N_t\} \oplus \{H_t L_t\}\) on \(L \oplus L'(\mathcal{H})\). Conversely, any semi-group constructed in this fashion is a strongly continuous quasinormal semi-group.

**Proof.** The converse is immediate since \(\{N_t\}\) is trivially quasi-
normal and \( \{H_t L_t\} \) is a strongly continuous quasinormal semi-group by Theorem 2.

Assume that \( \{Q_t\} \) is a strongly continuous quasinormal semi-group. By Theorem 2 \( \{Q_t\} = \{P_t\} \{U_t\} \) where \( \{P_t\} \) is self-adjoint and commutes with the isometric semi-group \( \{U_t\} \). Cooper's theorem \([5]\) tells us that \( \{U_t\} \) is unitarily equivalent to \( \{W_t\} \oplus \{V_t\} \) where \( \{W_t\} \) is unitary and defined on \( \mathcal{L} \) and \( \mathcal{L} \) is the range of the projection \( \lim_{t \to \infty} U_t U_t^* \). Moreover \( \{V_t\} \) is unitarily equivalent to the forward translation semi-group \( \{L_t\} \) on \( \mathcal{L}^2(\mathcal{H}) \) for some Hilbert space \( \mathcal{H} \).

Since by Theorem 2 \( P_r \) commutes with \( U_t \) for each \( r \) and \( t \), then \( \mathcal{L} \) reduces \( \{P_t\} \). Thus we have \( \{P_t\} \) unitarily equivalent to \( \{K_t\} \oplus \{H_t\} \) where \( \{K_t\} \) is self-adjoint and commutes with \( \{W_t\} \) on \( \mathcal{L} \) and \( \{H_t\} \) is self-adjoint and commutes with \( \{L_t\} \) on \( \mathcal{L}^2(\mathcal{H}) \). Thus \( \{Q_t\} \) is unitarily equivalent to \( \{K_t W_t\} \oplus \{H_t L_t\} \) on \( \mathcal{L} \oplus \mathcal{L}^2(\mathcal{H}) \), and \( \{K_t W_t\} \) is normal since \( \{W_t\} \) is unitary and commutes with \( \{K_t\} \).

The semi-group \( \{H_t L_t\} \) is completely nonnormal in the sense that there exists no subspace which reduces \( \{H_t L_t\} \) and on which \( \{H_t L_t\} \) is normal. The last step in characterizing quasinormal semi-groups is to characterize the self-adjoint semi-groups commuting with \( \{L_t\} \) on \( \mathcal{L}^2(\mathcal{H}) \).

Each \( h \) in \( \mathcal{B}(\mathcal{H}) \) induces an operator \( \bar{h} \) in \( \mathcal{B}(\mathcal{L}^2(\mathcal{H})) \) by \( (\bar{h} f)(x) = h f(x) \) a.e. whenever \( f \in \mathcal{L}^2(\mathcal{H}) \). Each such induced operator \( \bar{h} \) commutes with \( \{L_t\} \) and if \( \{h(t)\} \) is a (self-adjoint) semi-group in \( \mathcal{B}(\mathcal{H}) \), then \( \{\bar{h}(t)\} \) is a (self-adjoint) semi-group in \( \mathcal{B}(\mathcal{L}^2(\mathcal{H})) \). (We shall show in Theorem 5 that the strong continuity of either implies strong continuity of the other.) All of this leads to the following: \( \{\bar{h}(t)\} \) is a strongly continuous self-adjoint semi-group, commuting with \( \{L_t\} \) whenever \( \{h(t)\} \) is a strongly continuous self-adjoint semi-group on \( \mathcal{H} \). In Theorem 5 we shall show that this is the only way to construct a positive factor for a quasinormal semi-group with isometric factor \( \{L_t\} \). The key to this result lies in the following lemma concerning the commutant of \( \{L_t\} \).

The commutant of a collection \( \mathcal{A} \) of operators on \( \mathcal{H} \) is the algebra \( \mathcal{A}' = \{T: T \in \mathcal{B}(\mathcal{H}) \text{ and } TA = AT \text{ for all } A \text{ in } \mathcal{A}\} \).

**Lemma 4.** Let \( \{L_t\} \) be the forward translation semi-group on \( \mathcal{L}^2(\mathcal{H}) \). Then \( \{L_t^*\}' \cap \{L_t\}' = \{\bar{h}: h \in \mathcal{B}(\mathcal{H})\} \).

**Proof.** We have already observed that each \( \bar{h} \) is in \( \{L_t^*\}' \). Since \( (L_t^* f)(x) = f(x + t) \), a quick check shows that each \( \bar{h} \) is also in \( \{L_t^*\}' \).

Now assume that \( H \) commutes with \( \{L_t\} \) and \( \{L_t^*\} \). Without loss
of generality we may assume that \( H \) is self-adjoint since each of 
\( \text{Re} \ H \) and \( \text{Im} \ H \) commutes with \( \{ L_t \} \) and \( \{ L_t^* \} \). Let \( \{ e_n : n \in \mathbb{N} \} \) be a 
complete orthonormal basis of the separable Hilbert space \( \mathcal{H} \) and 
identify \( \mathcal{L}^2(\mathcal{H}) \) with \( \mathcal{L}^2(\mathcal{E}) \) in the usual fashion [8, p. 32]. 
The coordinate functions of each element \( f \) of \( \mathcal{L}^2(\mathcal{H}) \) are defined 
by \( f_n(x) = \langle f(x), e_n \rangle \) and the matrix \( [T_{nm}] \) of an operator \( T \) on 
\( \mathcal{L}^2(\mathcal{H}) \) is defined by \( T_{nm}f = (T(fe_m))_n \) whenever \( f \in \mathcal{L}^2 \). 
(\( fe_m \) is the element of \( \mathcal{L}^2(\mathcal{H}) \) whose value at \( x \) is \( f(x)e_m \) a.e.) 
Straight-forward computations show the following:

1. \([L_t]_{nm} \) is diagonal and \( (L_t)_{nn} = Lf \) the forward translation 
by \( t \) on \( \mathcal{L}^2 = \mathcal{L}^2(\mathcal{E}) \);
2. \( H_{nm}^* = H_{mn} \) for each \( n \) and \( m \) since \( H \) is self-adjoint;
3. \( H_{nm} \) commutes with \( L_t^{(0)} \) for each \( n \) and \( m \) since \( H \) commutes 
with \( L_t \) and the matrix of \( L_t \) is diagonal.

But the forward translation semi-group on \( \mathcal{L}^2 \) is irreducible [1, p. 76]. Thus the self-adjoint operators on \( \mathcal{L}^2 \) commuting with \( \{ L_t^{(0)} \} \) are the scalar multiples of the identity operator \( I \) on \( \mathcal{L}^2 \). It now 
follows from (2) and (3) that \( \text{Re} \ H_{nm} \), \( \text{Im} \ H_{nm} \) and consequently \( H_{nm} \) 
are scalar multiples of \( I \). Let \( H_{nm} = h_{nm}I \). For each \( f \) in \( \mathcal{L}^2(\mathcal{H}) \) and each \( n \)

\[
(1) \quad (Hf)_n = \sum_{m \in \mathcal{E}} h_{nm}f_m = \sum_{m \in \mathcal{E}} h_{nm}f_m.
\]

Let \( k \in \mathcal{H} \) and define \( f(x) = k \) for \( x \) in \([0,1] \) and 0 elsewhere. Then 
\( (Hf)_n(x) = \sum_{m \in \mathcal{E}} h_{nm}k_m \) for \( x \) in \([0,1] \) and 0 elsewhere. Also \( \| f \| = \| k \| \) and 
\( \sum_{n \in \mathcal{E}} | \sum_{m \in \mathcal{E}} h_{nm}k_m |^2 = \sum_{n \in \mathcal{E}} \int_0^1 |(Hf)_n(x)|^2 dx = \| Hf \|^2 \). Thus the 
matrix \( [h_{nm}] \) defines a (bounded) operator \( h \) on \( \mathcal{H} \). Finally, we see 
from equation (1) that for each \( f \) in \( \mathcal{L}^2(\mathcal{H}) \), \( (Hf)(x) = hf(x) \) a.e. 
so that \( H = \overline{h} \).

**Lemma 4** is the continuous analogue of the fact that \( \{ A \}' \cap \{ A^* \}' = \{ \overline{m} : m \in \mathcal{H} \} \) when \( A \) is the unilateral shift on \( \mathcal{H}^2(\mathcal{H}) \) [8, §4]. 
The connection between the unilateral shift on \( \mathcal{H}^2(\mathcal{H}) \) and the 
forward translation semi-group on \( \mathcal{L}^2(\mathcal{H}) \) is discussed in [11, 
p. 29–31].

**Theorem 5.** The strongly continuous self-adjoint semi-groups 
on \( \mathcal{L}^2(\mathcal{H}) \), commuting \( \{ L_t \} \), are induced by the strongly continuous 
self-adjoint semi-groups on \( \mathcal{H} \).

**Proof.** First let \( \{ h(t) \} \) be a strongly continuous self-adjoint 
semi-group on \( \mathcal{H} \). We have already noted that \( \{ h(t) \} \) is a self-
adjoint semi-group on \( \mathcal{L}^2(\mathcal{H}) \), commuting with \( \{ L_t \} \). We need to 
show that \( \{ h(t) \} \) is strongly continuous. Let \( f \) be an element of 
\( \mathcal{L}^2(\mathcal{H}) \). Then for each \( x \), \( \lim_{t \to 0} h(t)f(x) = f(x) \), since \( \{ h(t) \} \) is
strongly continuous on $\mathcal{H}$. Moreover $\{h(t)\}$ is bounded on finite intervals $[3, \Gamma]$. Hence for $t$ in $[0, 1]$ \( \|h(t)f(x)\| \leq M\|f(x)\| \) and consequently by the Lebesgue Dominated Convergence Theorem, \( \|h(t)f - f\| \to 0 \), showing that $\{h(t)\}$ is strongly continuous.

Secondly, assume that $\{H_t\}$ is a strongly continuous self-adjoint semi-group, commuting with $\{L_t\}$ on $\mathcal{L}_2(\mathcal{K})$. By Lemma 4, $H_t = h(t)$ for some $h(t)$ in $\mathcal{B}(\mathcal{H})$. To verify that $\{h(t)\}$ has the desired properties we proceed as follows: Let $k \in \mathcal{H}$ and define $f$ by $f(x) = k$ if $x \in [0, 1]$ and 0 otherwise. Then $f \in \mathcal{L}_2(\mathcal{K})$ and

\[
\begin{align*}
(1) & \quad h(t + s)k = (H_{t+s}f)(x) = (H_tH_s f)(x) = h(t)(H_s f)(x) = h(t)h(s)k, \\
(2) & \quad \langle H_t f, f \rangle = \int_0^\infty \langle h(t)f(x), f(x) \rangle dx = \langle h(t)k, k \rangle, \\
(3) & \quad \|H_t f - f\|^2 = \int_0^\infty \|h(t)f(x) - f(x)\|^2 dx = \|h(t)k - k\|^2.
\end{align*}
\]

Thus $\{h(t)\}$ is (1) a semi-group, (2) self-adjoint, and (3) strongly continuous.

We combine the results of Theorems 3 and 5 to arrive at the continuous analogue of Brown's characterization of quasinormal operators.

Theorem 6. $\{Q_t\}$ is a strongly continuous quasinormal semi-group if and only if there exist Hilbert spaces $\mathcal{L}$ and $\mathcal{K}$, a strongly continuous normal semi-group $\{N_t\}$ on $\mathcal{L}$ and a strongly continuous self-adjoint semi-group $\{h(t)\}$ on $\mathcal{K}$ such that $\{Q_t\}$ is unitarily equivalent to $\{N_t \oplus (h(t)L_t)\}$ on $\mathcal{L} \oplus \mathcal{L}_2(\mathcal{K})$.

Corollary 7. Let $\mathcal{K}$ and $\{h(t)\}$ be as in Theorem 6. If $\mathcal{K}$ is finite $n$-dimensional, then there exist real numbers $a_1, \ldots, a_n$ such that $(h(t)L_t)$ is unitarily equivalent to $e^{a_1 t}L_1^{(0)} \oplus \cdots \oplus e^{a_n t}L_n^{(0)}$, where $\{L_i^{(0)}\}$ is the forward translation semi-group on $\mathcal{L}_2(\mathcal{L})$.

Proof. Since $\mathcal{K}$ is finite dimensional, the generator $h$ of $\{h(t)\}$ is bounded, and since $h$ is self-adjoint, $h$ is diagonal. Let $\{e_k\}$ be a basis of $\mathcal{K}$ such that the matrix of $h$ is diagonal with diagonal elements $a_1, \ldots, a_n$. Then $\{h(t)\}$ is diagonal with diagonal elements $e^{i a_1}, \ldots, e^{i a_n}$. Recall from the proof of Lemma 4 that $[(L_t)_{nm}]$ is diagonal and $(L_t)_{kk} = L_t^{(0)}$. Thus the matrix of $(h(t)L_t)_{kk}$ is diagonal with $((h(t)L_t)_{kk})_{kk} = (e^{i a_k})L_t^{(0)}$, as desired.

We see now that the quasinormal weighted translation semi-groups introduced at the beginning of § 2 were quite typical. By Corollary 7 each quasinormal semi-group is a finite direct sum of quasinormal weighted translation semi-groups whenever the auxiliary space $\mathcal{K}$ is finite dimensional. We can go a little farther: if $\{h(t)\}$ is uniformly continuous and if the infinitesimal generator of $\{h(t)\}$ is a diagonal
operator on $\mathcal{H}$, then the proof of Corollary 7 is valid whether $\mathcal{H}$ is finite or infinite dimensional. Consequently we can conclude that \( \{h(t)L_t\} \) is unitarily equivalent to a direct sum of quasinormal semi-groups of the form \( \{e^{itL_t}\} \). However, if $\mathcal{H}$ is infinite dimensional and we choose a self-adjoint operator $h$ on $\mathcal{H}$ with no point spectrum, then the induced operator $\overline{h}$ on $L^2(\mathcal{H})$ also fails to have point spectrum and consequently \( \{e^{itL_t}\} \) is not unitarily equivalent to a direct sum of quasinormal weighted translation semi-groups.

4. The generator of a quasinormal semi-group. Recall that the (infinitesimal) generator of a strongly continuous semi-group \( \{S_t\} \) is the operator $S$ (not necessarily bounded) defined by $Sf = \lim_{t \to 0} (S_t f - f)/t$, whenever this limit exists in the strong topology. We shall denote the domain of $S$ by $\mathcal{D}(S)$. In general if \( \{S_t\} \) is the product of two strongly continuous semi-groups \( \{R_t\} \) and \( \{T_t\} \), the most one can show is that $R + T \subset S$ in the sense that $\mathcal{D}(R) \cap \mathcal{D}(T) \subset \mathcal{D}(S)$ and that $R + T = S$ on $\mathcal{D}(R) \cap \mathcal{D}(T)$. However quite a bit more can be said about the generators of a quasinormal semi-group and its isometric and positive factors.

**Theorem 8.** Let \( \{Q_t\} = \{U_t\}\{P_t\} \) be a strongly continuous quasinormal semi-group and let $Q$, $U$, and $P$ be the generators of \( \{Q_t\} \), \( \{U_t\} \) and \( \{P_t\} \), respectively. Then

(i) $\mathcal{D}(Q) \subset \mathcal{D}(Q^*)$

(ii) $\mathcal{D}(Q) = \mathcal{D}(P) \cap \mathcal{D}(U)$

(iii) $Q = P + U$ and $Q^* = P - U$ on $\mathcal{D}(Q)$ and

(iv) $Q^*(\mathcal{D}(Q^*)) \subset \mathcal{D}(Q)$ and $QQ^* = Q^*Q$ on $\mathcal{D}(Q^*)$.

**Proof.** Assertion (i) follows from the fact that $\|Q_t^*f - f\| \leq \|Q_t f - f\|$ for all $f$ and $t$. Moreover $Q_t f = \lim_{t \to 0} (Q_t^*f - f)/t$ on $\mathcal{D}(Q)$.

To prove (ii) and (iii) we first prove that $\mathcal{D}(Q) \subset \mathcal{D}(P)$ and $P = (1/2)(Q + Q^*)$ on $\mathcal{D}(Q)$. For each $f$ in $\mathcal{H}$ and each $t > 0$, $P_t f - f = (P_t + I)^{-1}[Q_t^*(Q_t f - f) + (Q_t^* f - f)]$. But as $t \to 0$, $(P_t + I)^{-1}$ converges strongly to $(1/2)I$, $Q_t^*$ converges strongly to $I$, and if $f \in \mathcal{D}(Q)$, $(Q_t f - f)/t$ converges to $Qf$ and $(Q_t^* f - f)/t$ converges to $Q^*f$. Therefore $\lim_{t \to 0} (P_t f - f)/t = (1/2)(Qf + Q^*f)$, so that $f \in \mathcal{D}(P)$ and $Pf = (1/2)(Qf + Q^*f)$.

Now observe that for each $f$ and $t$

(2) $Q_t f - f = U_t(P_t f - f) + (U_t f - f)$.

Equation (2) immediately implies that $\mathcal{D}(P) \cap \mathcal{D}(U) \subset \mathcal{D}(Q)$ and $\mathcal{D}(Q) \cap \mathcal{D}(P) \subset \mathcal{D}(U)$. We have already shown $\mathcal{D}(Q) \subset \mathcal{D}(P)$. 

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These three set inclusions yield \( \mathcal{D}(Q) = \mathcal{D}(P) \cap \mathcal{D}(U) \). Therefore, equation (2) can be used to conclude that \( Qf = Pf + Uf \) for all \( f \) in \( \mathcal{D}(Q) \). Finally since \( Pf = (1/2)(Qf + Q^*f) \) for all \( f \) in \( \mathcal{D}(Q) \), we also have \( Q^*f = Pf - Uf \) for all \( f \) in \( \mathcal{D}(Q) \).

Note now that if \( f \in \mathcal{D}(Q^2) \), then by definition \( f \in \mathcal{D}(Q) \) and \( Qf \in \mathcal{D}(Q) \). But then \( f \in \mathcal{D}(P) \) and \( Qf \in \mathcal{D}(P) \) by (ii). Consequently \( (Pf - f)t \to Pf \) and since \( P \) commutes with \( Q \), \( Q(Pf - f)t \to PQf \).

Every generator is closed [3, p. 10] so that \( Pf \in \mathcal{D}(Q) \) and \( QPf = PQf \). Similarly \( Vf \in \mathcal{D}(Q) \) and \( QVf = VQf \). Finally, since \( Q^*f = Pf - Vf \), we know that \( Q^*f \in \mathcal{D}(Q) \). Moreover \( QQ^*f = Q(Pf - Vf) = PQf - VQf = Q^*Qf \) by (iii) since \( Qf \in \mathcal{D}(Q) \).

The fourth conclusion in Theorem 3 indicates that the generator \( Q \) behaves very much like a normal operator. In general it is not true that \( Q^*(\mathcal{D}(Q)) \subset \mathcal{D}(Q) \) (for example, if \( Q = -D \), the generator of the forward translation semi-group on \( L^2 \)). Thus the assertion \( QQ^* = Q^*Q \) on \( \mathcal{D}(Q) \) is not meaningful. We also note that the first conclusion of Theorem 3 cannot in general be strengthened.

Although we have not been able to verify it we conjecture that if \( Q \) is the generator of a strongly continuous semi-group \( \{Q_t\} \) and \( Q \) satisfies conditions (i)-(iv) of Theorem 8, then \( \{Q_t\} \) is quasinormal.

**Remark 2.** Since a generator is closed and densely defined [3, p. 10], it is bounded if and only if it is everywhere defined. It follows now from Theorem 8(ii) that \( Q \) is bounded if and only if both \( U \) and \( P \) are bounded. But this is equivalent to \( \{Q_t\} \) being uniformly continuous [3, p. 13] and normal, the normality resulting from each of the quasinormal operators \( Q_t \) being invertible (and hence normal) when \( Q \) is bounded.

It is well-known that the generator of a normal semi-group \( \{N_t\} \) is normal. Applying Theorem 8 we note that the generator of \( \{N_t\} \) is the sum of the generators of the unitary factor \( \{W_t\} \) and the positive factor \( \{K_t\} \) of \( \{N_t\} \). The generator of \( \{W_t\} \) is \( iT \), where \( T \) is self-adjoint [8, p. 93] and the generator of \( \{K_t\} \) is self-adjoint. To complete our analysis of the generator of a quasinormal semi-group we need to determine the generator of \( \{h(t)L_t\} \), the completely nonnormal part of \( \{Q_t\} \).

**Corollary 9.** Let \( \{h(t)\} \) be a strongly continuous self-adjoint semi-group on \( \mathcal{H} \) with generator \( h \). The generator of \( \{h(t)L_t\} \) is \( h + (-D) \), where \( -D \) is the generator of \( \{L_t\} \) on \( \mathcal{L}^2(\mathcal{H}) \) and \( h \) is defined by \( (hf)(x) = hf(x) \) for all \( f \) in \( \mathcal{L}^2(\mathcal{H}) \) such that \( f(x) \in \mathcal{D}(h) \).
Proof. By Theorem 8 we know that the generator of \( \{ h(t)L_t \} \) is \( H + (-D) \), where \( H \) is the generator of \( \{ h(t) \} \). We need to show that \( \mathcal{D}(H) = \mathcal{D}(\overline{h}) \) if and only if \( (Hf)(x) = hf(x) \) a.e.

First let \( f \in \mathcal{D}(\overline{h}) \). Then \( \lim_{t \to 0} \frac{h(t)f(x) - f(x)}{t} = hf(x) \) a.e. and \( hf(\cdot) \in L^2(\mathcal{X}) \). But \( \| (h(t)f(x) - f(x))/t \| \leq \sup_{\|x\| < 1} \| h(t) \| \| f(x) \| \) \([3, p. 88]\) for all \( t \) in \([0, 1]\) and once again the Lebesgue Dominated Convergence Theorem applies. The result is that \( (h(t)f - f)/t \to hf \) in the \( L^2(\mathcal{X}) \) norm. Consequently \( f \in \mathcal{D}(H) \) and \( Hf = hf \).

Now let \( f \in \mathcal{D}(H) \). By \([3, p. 10]\) \( \overline{h(t)}f - f = \int_0^t h(s)Hf ds \). Consequently, for almost all \( x \), \( h(t)f(x) - f(x) = \int_0^t h(s)(Hf)(x) ds \). But since \( \{ h(s) \} \) is strongly continuous, \( \lim_{t \to 0} 1/t \int_0^t h(s) ds = h(0)k = k \) for all \( k \) in \( \mathcal{X} \). Therefore \( \lim_{t \to 0} (h(t)f(x) - f(x))/t = (Hf)(x) \) for almost all \( x \). But then \( f(x) \in \mathcal{D}(h) \) a.e. and \( hf(x) = (Hf)(x) \). Thus \( f \in \mathcal{D}(\overline{h}) \) and \( \overline{h}f = Hf \), completing the proof.

Using Corollary 9 it is now easy to construct a quasinormal semi-group such that neither the isometric nor the positive factor is uniformly continuous. We let \( \{ L_t \} \) on \( L^2(\mathcal{X}) \) be the isometric factor. The Hille-Yosida theorem \([3, p. 36]\) guarantees that the unbounded diagonal operator with diagonal \( (-1, -2, \ldots, -n, \ldots) \) is the generator of a strongly continuous semi-group \( \{ h(t) \} \) on \( L^2(\mathcal{X}) \). The induced semi-group \( \{ L(t) \} \) on \( L^2(\mathcal{X}) \) is self-adjoint and strongly, but not uniformly, continuous. Thus neither factor of \( \{ h(t)L_t \} \) is uniformly continuous.

REFERENCES

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