TOTALLY BOUNDED GROUP TOPOLOGIES AND CLOSED SUBGROUPS

S. JANAKIRAMAN AND T. SOUNDARARAJAN
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Let \((G, J)\) be an infinite compact totally disconnected 
abelian group. Finer totally bounded group topologies \(J'\) 
such that every \(J'\)-closed subgroup is \(J\)-closed are studied. 
Necessary and sufficient conditions for the existence of 
such a \(J' \neq J\) are given.

Introduction. Throughout this paper all topologies are Hausdorff 
topological group topologies and all the groups are written in the 
additive notation.

A topological group \(G\) is called totally bounded if for every 
identity neighborhood \(V\) there is a finite subset \(F\) of \(G\) with 
\(G = F + V\). This is tantamount to saying that \(G\) is embedded algebraically 
and topologically into its Bohr compactification under the 
natural map \(G \to \alpha G\). We recall that for abelian \(G\) we have \(\alpha G = ((G^\sim)_d)^\sim\) and that \((G^\sim)_d = (\alpha G)^\sim\).

Now let \(G\) be a compact abelian group with topology \(J\) and let 
\(G'\) be the same underlying group with a possibly finer totally bounded 
topology \(J'\). Then \(G^\sim \subseteq (G')^\sim = (G'^\sim)_d \subseteq (G_d)^\sim\); and conversely, any 
group \(H\) of (not necessarily continuous) characters of \(G\) with \(G^\sim \subseteq H \subseteq (G_d)^\sim\) induces on \(G\) a coarsest topology \(J'\) making all characters 
of \(H\) continuous, and then the group \(G'\) with the topology \(J'\) is 
totally bounded such that \((\alpha G')^\sim = G'^\sim = H\). Thus there is a lattice 
isomorphism between the lattice of totally bounded topologies \(J'\) on 
\(G\) refining \(J\) and the lattice of subgroups of \((G_d)^\sim\) containing \(G^\sim\). 
(These nice results are proved by W. W. Comfort and K. A. Ross 
in [1].) Furthermore, the diagram

\[
\begin{array}{ccc}
G' & \longrightarrow & \alpha G' \\
e \downarrow & & \downarrow \alpha e \\
G & \longrightarrow & \alpha G = G
\end{array}
\]

shows that \(\alpha G'\) is algebraically the direct sum of the image of \(G'\) 
in \(\alpha G'\) and of \(\ker \alpha e\).

The problem we are interested in studying is the following:

\((P)\) Determine all those totally bounded topologies \(J'\) containing 
\(J\) such that every \(J'\)-closed subgroup of \(G'\) is a \(J\)-closed subgroup 
of \(G\).

In view of the isomorphism of lattices mentioned before, this
is tantamount to the following problem:

\((P')\) Determine all those intermediate groups \(H\) with \(G^\sim \subseteq H \subseteq (G_d)^\sim\) for which the associated topology \(J' = J_H\) has the same closed subgroups as \(J = J_\emptyset\).

In a totally bounded group the smallest closed subgroup containing a subset \(S\) is its bipolar \(S^{\pm 1}\); hence a subgroup \(S\) is closed if and only if it agrees with its bipolar if and only if it is the intersection of a collection of kernels of continuous characters. As a consequence, the \(J_\mu\)-closed subgroups are precisely the intersections of families of groups \(\ker f\) with \(f \in H\). Consequently problem \((P')\) is equivalent to \((P'')\): Determine all those groups \(H\) with \(G^\sim \subseteq H \subseteq (G_d)^\sim\) such that \(\ker f\) is \(J = J_\emptyset\) — closed for all \(f \in H\).

In this paper we consider only the case \((G, J)\) is totally disconnected \(ie G^\sim\) is a torsion group [5, p. 385]. We show that if \(H\) is a subgroup “admissible” in the sense of problem \((P'')\) then \(G^\sim\) is the torsion subgroup of \(H\) [Lemma 1.3]. In particular \(\ker \alpha e\) is always connected in this case. Next for any \(f \in (G_d)^\sim\) whose \(\ker f\) is \(J\)-closed, \(G^\sim + \langle f \rangle\) is admissible [Lemma 2.3]. We then prove that \((G, J)\) has an admissible \(H \supseteq G^\sim\) if and only if \(G\) has a direct factor which is \(p\)-adic integer group \(A_p\) or an infinite product of cyclic groups of prime power order for infinitely many different primes. [Theorem 2.5].

It is also shown that if there are admissible groups properly containing \(G^\sim\) then there is no largest admissible \(H\) [Theorem 2.10].

That one can never expect pseudocompact \(J' \neq J\) (whether or not \(J\) is totally disconnected) and existence of maximal admissible subgroups \(H\) is dealt with in a paper by W. W. Comfort and the second author [3].

The authors conclude the paper with a few remarks on the nonabelian case and a remark on Galois theory (in § 3).

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1. Preliminaries. Throughout this paper \((G, J)\) denotes an infinite compact totally disconnected abelian group, \(G^\sim\) is the group of all continuous characters of \((G, J)\), \(G_d\) is the group \(G\) endowed with the discrete topology and \((G_d)^\sim\) is the group of all characters on \(G\).

**Definition 1.1.** A subgroup \(H\) of \((G_d)^\sim\) is said to be admissible
if $H$ contains $G^\wedge$ and ker $f$ is a $J$-closed subgroup of $G$ for all $f \in H$.

**Proposition 1.2.** If $G$ is of finite exponent then the only admissible subgroup is $G^\wedge$.

**Proof.** Let $f \in (G_d)^\wedge$ with ker $f$ a $J$-closed subgroup of $G$. Since $G$ is of finite exponent say $m$, $mx = 0$ for all $x \in G$ yields $f(G)$ is of finite exponent in $T = \mathbb{R}/\mathbb{Z}$. Hence $f(G)$ is a finite subgroup of $T$. Thus ker $f$ is of finite index in $G$. Already it is $J$-closed. Hence ker $f$ is $J$-open and so we get $f$ is continuous and hence $f \in G^\wedge$. The proposition now easily follows.

**Lemma 1.3.** Let $H$ be an admissible subgroup. Then $G^\wedge$ is the torsion subgroup of $H$.

**Proof.** It is enough to show that if $f \in H$ is of finite order then $f \in G^\wedge$. Let $mf = 0$. Then $mf(G) = 0$. Hence $f(G)$ is of finite exponent in $T$ and so $f(G)$ is a finite group. Hence ker $f$ is of finite index in $G$. It is $J$-closed implies now that it is $J$-open. Hence we get $f$ is continuous and $f \in G^\wedge$.

2. In this section we prove the main theorem.

**Lemma 2.1.** Let $(A, \tau)$ be an abelian totally bounded topological group and $B$ a closed subgroup of $(A, \tau)$. If $B^\perp$ is the set of all continuous homomorphisms of $(A, \tau)$ into $T$ which map $B$ to 0, then $B = \bigcap_{f \in B^\perp} \ker f$.

**Proof.** Let $\alpha A$ be the compact topological group in which $(A, \tau)$ is densely embedded. Then $\alpha A$ is also abelian. Let $\overline{B}$ be the closure of $B$ in $\alpha A$. We have $\overline{B} \cap A = B$. By Pontrjagin-van Kampen duality theory we have $\overline{B} = \bigcap_{f \in B^\perp} \ker f$. By taking the restrictions of the $f \in B^\perp$ to $A$, the lemma follows.

**Lemma 2.2.** Let $A$ be an abelian group and $f, g$ be two homomorphisms of $A$ into $T$ such that $g$ is of finite order. Let $n$ be any integer. Then $\ker (g + nf)$ contains $(\ker f \cap \ker g)$ as a subgroup of finite index.

**Proof.** Let $g$ be of order $m$. Then for each $x$ in $A$, $g(mx) = mg(x) = (mg)(x) = 0$. Hence $g(A)$ is of finite exponent and so is a finite subgroup of $T$. Consequently ker $f \cap \ker g$ is of finite index in ker $f$. Easily ker $f \cap \ker g$ is a subgroup of ker $(g + nf)$. Let $S = \ker (g + nf)$. Then for every $x \in S$ we have $0 = (g + nf)(mx) = \ldots$
Let $B$ be the finite subgroup of order $mn$ in $T$. Then clearly $S \subseteq f^{-1}(B)$. Also $\ker f$ is of finite index in $f^{-1}(B)$. Already $\ker f \cap \ker g$ is of finite index in $\ker f$. Hence $\ker f \cap \ker g$ is of finite index in $f^{-1}(B)$. Since $(\ker f \cap \ker g) \subseteq S \subseteq f^{-1}(B)$, the lemma follows.

**Lemma 2.3.** For any $f \in (G^\wedge \backslash G^\wedge$ such that $\ker f$ is a $J$-closed subgroup of $G$, $G^\wedge + \langle f \rangle$ is an admissible subgroup.

**Proof.** It is enough to show that $\ker h$ is a $J$-closed subgroup for all $h \in G^\wedge + \langle f \rangle$. Now let $h \in G^\wedge + \langle f \rangle$. Then $h = g + nf$ with $g \in G^\wedge$ and $n$ an integer. As $G^\wedge$ is a torsion abelian group $g$ is of finite order. Hence by Lemma 2.2, $\ker g \cap \ker f$ is of finite index in $\ker h$. Now $\ker g$ is $J$-closed since $g \in G^\wedge$ and $\ker f$ is $J$-closed by hypothesis. Hence $(\ker f \cap \ker g)$ is a $J$-closed subgroup and $\ker h$ is a finite union of cosets of $(\ker f \cap \ker g)$. Hence $\ker h$ is $J$-closed.

**Proposition 2.4.** Let $(G, J)$ be one of the following two groups;

1. $\Lambda_p$, the topological group of all the $p$-adic integers with the usual topology, $p$ a prime.
2. $\Pi_{p \in I} Z(p^{n_p})$, the product of cyclic groups of prime power order $p^{n_p}$, with the product topology, where $I$ is an infinite set of primes. (We shall denote this compact group by $C(p, n_i).$)

Then there is an admissible subgroup $H \neq G^\wedge$.

**Proof.** (1) Algebraically, $\Lambda_p$ is a torsion free abelian group of cardinality $c$. Now $T = \sum Z(p^\omega) \oplus R$ algebraically where the sum is extended over all primes [4, p. 105]. $R$ being a torsion free divisible group of cardinality $c$, we can find an algebraic monomorphism $f: \Lambda_p \to T$. Clearly $\ker f = 0$ is a $J$-closed subgroup. Also $f \in \Lambda^\wedge$ since $mf = 0$ will imply $f(mx) = 0$ for all $x \in \Lambda_p$ contradicting that $\ker f = 0$. Now Lemma 2.3 completes the proof.

(2) For this case we use a product decomposition. Algebraically $T = \prod Z(p^\omega)$ (see [4, p. 105]) the product extending over all primes. Again we have an algebraic monomorphism $f: G \to T$; with $\ker f = 0$, a $J$-closed subgroup. Since $I$ is infinite $G$ has elements of infinite order. Hence $mf = 0$ will yield $f(mx) = 0$ and contradict $\ker f = 0$. Hence $f \notin G^\wedge$. Now Lemma 2.3 completes the proof.

**Theorem 2.5.** Let $(G, J)$ be an infinite compact totally disconnected abelian topological group. Then the following statements are equivalent.

1. There exists a totally bounded group topology $J'$ containing
TOTALLY BOUNDED GROUP TOPOLOGIES AND CLOSED SUBGROUPS

127

(2) $G$ has an infinite monothetic factor group.

(3) $G$ has a direct factor $M$ which is either a $p$-adic group $\Delta_p$ or a group $C(p^\alpha, n_i)$.

(4) $G$ has an infinite procyclic direct factor.

Proof. (1) $\Rightarrow$ (2). Suppose there exists a totally bounded group topology $J'$ on $G$ containing $J$ properly such that every $J'$-closed subgroup is $J$-closed. Let $H = \{f: f$ is a continuous homomorphism of $(G, J')$ into $T\}$. Hence there is an $f \in H \setminus G^\wedge$. Clearly then $f \in (G_d)^\wedge \setminus G^\wedge$. Now $f \in H$ implies that $\ker f$ is $J'$-closed and so is $J$-closed by hypothesis. Thus $f$ is a discontinuous character for $(G, J)$ with $\ker f$ being $J$-closed. Let $\bar{G} = G/\ker f$ and $\bar{J}$ be the quotient topology on $\bar{G}$ obtained from $J$. We now have a monomorphism $\bar{f}: \bar{G} \to T$ i.e., $(\bar{G})_d$ can be injected into $T$. Hence by [5, p. 407] the torsion free rank of $(\bar{G})_d$ is at most $c$ and the $p$-rank of the torsion subgroup of $(\bar{G})_d$ is at most 1 for all $p$. Also $(\bar{G}, \bar{J})$ is an infinite compact totally disconnected abelian group (since $f \notin G^\wedge$, ker$f$ is not $J$-open and hence $\bar{G}$ cannot be finite). Since $\bar{G}^\wedge$ is a torsion abelian group let $\bar{G}^\wedge = \sum G_{\bar{p}}^\wedge$, $G_{\bar{p}}^\wedge$ being the $p$-primary part of $\bar{G}^\wedge$. If for some $p$, $G_{\bar{p}}^\wedge$ contains $\mathbb{Z}(p^\omega)$ then we get $(\bar{G}, \bar{J})$ has a factor $\Delta_p$ and hence $(G, J)$ has a factor group $\Delta_p$ which is an infinite compact monothetic group and we are done. Otherwise $\bar{G}_{\bar{p}}^\wedge$ is a reduced group for each $\bar{p}$. We claim now $\bar{G}_{\bar{p}}^\wedge$ is cyclic of prime power order. Otherwise by [4, p. 117] we can have $\bar{G}_{\bar{p}}^\wedge = \mathbb{Z}(p^r) \oplus \mathbb{Z}(p^s) \oplus B$ and consequently by duality $(\bar{G}, \bar{J})$ will have a direct factor $Z(p^r) \oplus Z(p^s)$ contradicting that $p$-rank of $(\bar{G})_d$ is at most 1. Thus each $\bar{G}_{\bar{p}}^\wedge$ is cyclic of prime power order and so $\bar{G}^\wedge$ is isomorphic to a subgroup of $T$. (See [5, p. 407].) Hence $(\bar{G}, \bar{J})$ is monothetic and (2) follows.

(2) $\Rightarrow$ (3). Let $(\bar{G}, \bar{J})$ be an infinite monothetic factor of $(G, J)$. Then $\bar{G}^\wedge$ is a torsion subgroup of $T$ (see [5, p. 385]).

We consider now $G^\wedge$. If $G^\wedge$ contains a $\mathbb{Z}(p^\omega)$ for some $p$ then $(G, J)$ will have a direct factor $\Delta_p$ and we are done. Otherwise $G^\wedge$ is a reduced group. Now $\bar{G}^\wedge$ is a subgroup of $G^\wedge$. Since $\bar{G}^\wedge$ is infinite and a subgroup of $T$ and $\bar{G}^\wedge$ also has to be reduced we get $\bar{G}_{\bar{p}}^\wedge \neq 0$ for infinitely many $p$. Hence $G_{\bar{p}}^\wedge \neq 0$ for infinitely many $p$. Now applying [4, p. 117] to each of these $G_{\bar{p}}$ we easily get $\sum \mathbb{Z}(p_i^\omega)$ is a direct summand of $G^\wedge$. Hence $(G, J)$ has a direct factor $C(p_i, n_i)$. Hence (3) follows.

(3) $\Rightarrow$ (1). Case (i): Let $(G, J) = N \oplus M$, $N$ a $J$-closed subgroup, $M$ is topologically isomorphic to $\Delta_p$ and the sum direct. Then by the proof of (1) in 2.4 we get easily a homomorphism $f: G \to T$ such
that \( \ker f = N \) and \( f \) is injective on \( M \). Surely order of \( f \) is infinite and hence \( f \notin G^\sim \). Also \( \ker f \) is \( J \)-closed. Hence Lemma 2.3 shows that \( G^\sim + \langle f \rangle \) is admissible. Thus a \( J' \) exists by the equivalence in the introduction.

**Case (ii):** Let \((G, J) = N \oplus M, N \) a \( J \)-closed subgroup, \( M \) is isomorphic to \( C(p_i, n_i) \) and the sum direct. Then by the proof (2) in 2.4, we get easily a homomorphism \( f: G \to T \) such that \( \ker f = N \) and \( f \) is injective on \( M \). Since \( C(p_i, n_i) \) has torsion free elements and \( f \) is injective on \( M \), order of \( f \) has to be infinite and so \( f \notin G^\sim \). Also \( \ker f = N \) is \( J \)-closed. Thus Lemma 2.3 yields \( G^\sim + \langle f \rangle \) is admissible. Hence \( J' \)-exists by the equivalence in the introduction. Thus (1) follows.

\( (3) \Rightarrow (4) \). Easy.

\( (4) \Rightarrow (2) \). Let \( P \) be an infinite procyclic direct factor of \( (G, J) \). Then by duality \( P^\sim \) is a torsion group which is a direct limit of finite cyclic groups. By [4, p. 58] \( \hat{\phi} \) is locally cyclic. Hence for each prime \( p \) the \( p \)-rank of \( P^\sim \) is atmost one. So each \( P^\sim_s \) is isomorphic to a \( \mathbb{Z}(p^s) \), \( s = 0, 1, \ldots, \infty \). This yields that \( P^\sim \) is isomorphic to a subgroup of \( T \) and hence \( P \) is monothetic. Thus (2) holds.

Now Theorem 2.5 follows.

We now proceed to discuss the existence of a largest admissible subgroup.

**Lemma 2.6.** There exists a largest admissible subgroup \( L \) if and only if the set of all \( f \in (G_d)^\sim \) such that \( \ker f \) is \( J \)-closed form a group. In this case \( L \) consists precisely of these.

**Proof.** Let \( L \) be a largest admissible subgroup. Let \( f, g \in (G_d)^\sim \) such that \( \ker f, \ker g \) are \( J \)-closed. Then by Lemma 2.3 \( G^\sim + \langle f \rangle \) and \( G^\sim + \langle g \rangle \) are admissible subgroups; they will both be subgroups of \( L \) and hence, \( f, g, f - g \in L \). Clearly then all such \( f \)'s will form a group.

Conversely let \( L = \{ f \in (G_d)^\sim | \ker f \) is \( J \)-closed\} form a group. Then clearly \( L \) is admissible, and by definition any other admissible group should be a subgroup of \( L \). Hence the lemma is proved.

**Proposition 2.7.** In \( ((\Delta_p)_d)^\sim \) there is no largest admissible subgroup.

**Proof.** Since \( \Delta_p \) is a torsion free abelian group of cardinal \( c \) it has a maximal independent set \( B \) of cardinal \( c \). Hence \( \Delta_p/\langle B \rangle \) is a torsion abelian group.
Now \( T = \sum Z(p^\infty) \oplus R \) (see [4, p. 105]). We can write \( R = B_1 + B_2 \) such that \( B_1 \cap B_2 = Q \), \( B_1, B_2 \) each isomorphic to \( \Sigma Q \), \( c \) copies. Now easily we can get embeddings \( h_1, h_2 \) of \( A_\alpha \) into \( \mathbb{R} \), such that \( h_1(A_\alpha) \subset B_1 \), \( h_2(A_\alpha) \subset B_2 \) and \( h_1(1) = h_2(1) = 1 \in Q = B_1 \cap B_2 \), \( h_1, h_2 \) being obtained by mapping \( B \) to the corresponding independent sets. It is easy to see that \( \ker (h_1 - h_2) \) is a countable subgroup, of \( A_\alpha(= \{n/m; (p, m) = 1\}) \). Clearly \( \ker (h_1 - h_2) \) is not \( J \)-closed. Now Lemma 2.6 completes the proof.

**Proposition 2.8.** There is no largest admissible subgroup in \((C(p, n_\alpha))_\alpha\).

**Proof.** We note \( C(p, n_\alpha) = \prod Z(p^{n_\alpha}) \) algebraically and also that \( T = \prod Z(p^\infty) \), \( p \) varies over all primes [4, p. 105]. Hence there is an embedding \( i: C(p, n_\alpha) \rightarrow T \), with \( \ker (i) = 0 \); which is \( J \)-closed. Since \( T = \sum Z(p^\infty) \oplus R \), there is an automorphism \( g: T \rightarrow T \) such that \( g(x) = x \) for elements of finite order and \( g(x) = \sqrt{2}x \) for \( x \) in \( R \). Then \( g+i \) gives another embedding of \( C(p, n_\alpha) \). Now \( \ker (i - g \circ i) \) is a countable subgroup namely \( \sum Z(p^{n_\alpha}) \). Thus we get two embeddings \( f_\alpha, g_\alpha: C(p, n_\alpha) \rightarrow T \) such that \( \ker (f_\alpha - g_\alpha) \) is countable and hence not \( J \)-closed. So Lemma 2.6 completes the proof.

**Definition 2.9.** We say a topology \( J' \) is admissible if it satisfies the condition of \((P)\).

**Theorem 2.10.** The following are equivalent:

1. \( G \) has a largest admissible topology \( J_L \),
2. \( G \) has no admissible topology \( J' \neq J \),
3. \( J \) is the largest admissible topology.

**Proof.** (1) \( \Rightarrow \) (2). Suppose \( G \) has an admissible topology \( J' \neq J \). Then \( G \) has a topological decomposition \( G = A \oplus B \), \( A \) a closed subgroup and \( B \) is isomorphic \( A_\alpha \) or \( C(p, n_\alpha) \). Then by Propositions 2.7 and 2.8, we have two embeddings of \( f, g: B \rightarrow T \) such that \( \ker (f - g) \) is countable and not \( J \)-closed. Hence we easily get two homomorphisms \( F_\alpha, G_\alpha: G \rightarrow T \) such that \( \ker F_\alpha = \ker G_\alpha = A \) is \( J \)-closed but \( \ker (F_\alpha - G_\alpha) \) is not \( J \)-closed. This contradicts Lemma 2.6. Hence (2) follows. (2) \( \Rightarrow \) (3) is easy as also (3) \( \Rightarrow \) (1).

**Proposition 2.11.** On \( A_\alpha \), there is an admissible topology \( J' \) having \( |(A_\alpha, J')^\wedge| = c \).

**Proof.** We note \( T = \sum Z(p^\infty) \oplus R \) and \( R = \sum Q \), \( c \) copies. Now we can write \( R = \sum B_\alpha, \alpha \in I; |I| = c \) and each \( B_\alpha \) is a torsion free
divisible abelian group of cardinality $c$. This is possible as $c.c = c$. For each $\alpha \in I$, we can have an embedding $h_\alpha : \Lambda_p \to B_\alpha$. Correspondingly we get embeddings $g_\alpha : \Lambda_p \to T$ such that for each $x \neq 0$, the $g_\alpha(x)$ are independent. Let now $H$ be the subgroup of $(\langle \Lambda_p \rangle_\alpha)^\wedge$ generated by $\Lambda_p^\wedge$ and all these $g_\alpha$. Surely $|H| = c$. Let $J'$ be the totally bounded group topology determined by $H$. $J'$ is finer than $J$, the usual topology. We claim $J'$ is admissible. We have only to show that $\ker(h)$ is $J$-closed for each $h \in H$, since $H = (\Lambda_p, J')^\wedge$, (see [5]). Now $h = f + \sum_i n_i g_{\alpha_i}, f \in \Lambda_p^\wedge, n_i$ are integers $k$ finite. If all the $n_i$ are 0, then there is nothing to prove. Let some $n_i \neq 0$. Since $f$ is of finite order by Lemma 2.2, we have only to prove $\ker(f \cap \ker(\sum_i n_i g_{\alpha_i})$ is $J$-closed. We claim $\ker(\sum_i n_i g_{\alpha_i}) = 0$. Let if possible $x \neq 0$ be in the kernel. $\sum_i n_i g_{\alpha_i}(x) = 0$ implies $\sum_i g_{\alpha_i}(n_i x) = 0$ and by independence $g_{\alpha_i}(n_i x) = 0$ for each $i$ and each $g_{\alpha_i}$ being an embedding we get $n_i x = 0$ for each $i$, so $x = 0$. Thus $\ker(f \cap \ker(\sum_i n_i g_{\alpha_i}) = 0, a J$-closed subgroup. Hence the results follows.

3. We now assume $(G, \tau)$ is a noncommutative compact totally disconnected group and make a few remarks on totally bounded group topologies $\tau'$ containing $\tau$ and such that each $\tau'$ closed subgroup is $\tau$-closed. We shall again call such a $\tau'$ an admissible topology.

REMARK 3.1. If $G$ is of finite exponent then $\tau' = \tau$.

Proof. Let $\alpha G'$ be a compact topological group in which $(G, \tau')$ is embedded as a dense subgroup. From hypothesis it now follows easily that for each $x \in \alpha G'$, $mx = 0$ (since $x$ is limit of a net, from $(G, \tau)$). Now $\bigcap_{n=1}^\infty n(\alpha G') = 0$, since $m \alpha G' = 0$. Hence by a theorem of Mycielski [8], $\alpha G'$ is totally disconnected and hence by [7, p. 56] has a basis of open subgroups of finite index at 0.

Hence $(G, \tau')$ has a basis $\{G'_{\alpha}\}$ of open subgroups of finite index at 0. Each of these $G'_{\alpha}$ is now $\tau$-closed and hence $\tau$-open. Hence we get $\tau$ is finer than $\tau'$. Since $\tau$ is compact and $\tau'$ is Hausdorff we get $\tau = \tau'$.

REMARK 3.2. Let $(K, \tau)$ be a compact group of finite exponent. Then $K \times \Lambda_p$ has an admissible topology different from the product topology.

Proof. Let $mx = 0$ for each $x \in K$. Let $J_1$ be an admissible topology on $\Lambda_p$; $J_1 \neq$ the usual topology $J$ of $\Lambda_p$. Let $J'$ be the product of $\tau$ and $J_1$ on $K \times \Lambda_p$. Since $(K \times \Lambda_p, J') \subset (K, \tau) \times (\alpha \Lambda_p, \alpha J_1)$, where $\alpha \Lambda_p$ is the compact group in which $\Lambda_p$ is densely embedded,
we get $J'$ is totally bounded. Also $J'$ is finer than the product topology $\tau \times J$. We have only to show that any $J'$-closed subgroup $S$ is $\tau \times J$ closed. If $S \subset (K \times O)$ then we easily get the result. Suppose $S \not\subset K \times O$. If $(x, y) \in S \setminus (K \times O)$ then $m(x, y) = (mx, my) = (0, my) \in 0 \times A_p$. Let $S \cap (0 \times A_p) = M \neq (0, 0)$ and $S \cap (K \times O) = M$. $M$ is a $J'$-closed subgroup of $A_p$ and hence $J$-closed. So $M = 0 \times p^n A_p$ for some $n$. $M_i$ is $J'$-closed and hence $J$-closed since $K \times O$ is $J'$-closed and $J' = \tau \times J$ on $K \times O$. Now $M_i \times M$ is $J$-closed and $\subset S$. We claim $M_i \times M$ is of finite index in $S$. Let $p_i : S \to A_p$ be the projection. Then $p_i(S) \supset M$. Hence $M$ is of finite index in $p_i(S)$ (since $M$ is of finite index $p^n$ in $A_p$ itself). Let $p_i(S) = M \cup (a_2 + M) \cup \cdots \cup (a_k + M)$ where $(y_i, a_i) \in S$, $i = 1, 2, \cdots, k$. We claim now $S = U_i((y_i, a_i) + (M_i \times M))$. Let $(x, y) \in S$. Then $p_i(x, y) = y = a_i + t$ for some $i$ and $t \in M$. Also $(0, t) \in M_i \times M \subset S$. Hence we can assume $p_i(x, y) = a_i$. Also $p_i(y_i, a_i) = a_i$. Hence $(-y_i, a_i) + (x, y) = (-y_i + x, 0) \in M_i \subset M_i \times M$. Hence $(x, y) \in (y_i, a_i) + (M_i \times M)$. Hence $S$ is a finite union of cosets of $M_i \times M$ and so we get $S$ is $\tau \times J$-closed. That $\tau \times J_1$ is an admissible topology follows now easily.

Remark 3.3. If $E$ is an infinite algebraic separable normal extension of a field $F$ and $G$ is the Galois group of $E$ over $F$ then W. Krull [6] has shown that one can introduce a topology $\tau$ on $G$ (the Krull topology) such that there is a $1 - 1$ Galois correspondence between all intermediate fields of $E$ over $F$ and all $\tau$-closed subgroups of $G$. Furthermore $(G, \tau)$ is a compact totally disconnected group. It might be of some interest that if $\tau'$ is any other admissible topology on $G$ then again there is a $1 - 1$ Galois correspondence between all intermediate fields of $E$ over $F$ and all $\tau'$-closed subgroups of $G$.

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Madurai Kamaraj University
Madurai 625 021 India
Natália Bebiano, On the evaluation of permanents ........................................... 1
David Borwein and Bruce Brigham Watson, Tauberian theorems between
the logarithmic and Abel-type summability methods ..................................... 11
Leo George Chouinard, II, Hermite semigroup rings ...................................... 25
Kun-Jen Chung, Remarks on nonlinear contractions ....................................... 41
Lawrence Jay Corwin, Representations of division algebras over local
fields. II .................................................................................................................. 49
Mahlon M. Day, Left thick to left lumpy—a guided tour .................................. 71
M. Edelstein and Mo Tak Kiang, On ultimately nonexpansive
semigroups ............................................................................................................ 93
Mary Rodriguez Embry, Semigroups of quasinormal operators ....................... 103
William Goldman and Morris William Hirsch, Polynomial forms on
affine manifolds .................................................................................................... 115
S. Janakiraman and T. Soundararajan, Totally bounded group topologies
and closed subgroups .......................................................................................... 123
John Rowlay Martin, Lex Gerard Oversteegen and Edward D.
Tymchatyn, Fixed point set of products and cones ........................................... 133
Jan van Mill, A homogeneous Eberlein compact space which is not
metrizable ............................................................................................................... 141
Steven Paul Plotnick, Embedding homology 3-spheres in $S^5$ ......................... 147
Norbert Riedel, Classification of the $C^*$-algebras associated with minimal
rotations ............................................................................................................... 153
Benedict Seifert, Combinatorial and geometric properties of weight systems
of irreducible finite-dimensional representations of simple split Lie
algebras over fields of 0 characteristic ............................................................... 163
James E. Simpson, Dilations on locally convex spaces ..................................... 185
Paolo M. Soardi, Schauder bases and fixed points of nonexpansive
mappings ............................................................................................................... 193
Yoshio Tanaka, Point-countable k-systems and products of k-spaces .......... 199
Fausto A. Toranzos, The points of local nonconvexity of starshaped sets ........ 209
Lorenzo Traldi, The determinantal ideals of link modules. I ......................... 215
P. C. Trombi, Invariant harmonic analysis on split rank one groups with
applications ....................................................................................................... 223
Shinji Yamashita, Nonnormal Blaschke quotients .......................................... 247