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FIXED POINT SETS OF PRODUCTS AND CONES

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A space X is said to have the complete invariance property (CIP) if every nonempty closed subset of X is the fixed point set of some self-map of X. Examples are given to show that for the class of locally connected continua, the operations of taking products, cones, and strong deformation retractions need not preserve CIP. In fact, it is shown that the operations of taking products and cones do not preserve CIP for LC^∞ continua.

1. Introduction. A subset K of a space X is called a *fixed point set* of X if there is a continuous function $f: X \to X$ such that $f(x) = x$ iff $x \in K$. In [8, p. 553] L. E. Ward, Jr. defines a space X to have the *complete invariance property* (CIP) if every nonempty closed subset of X is a fixed point set of X. Examples 4.3, 5.1 and 6.1 in [4] are examples of non-locally connected continua which show that CIP is not preserved by the operations of taking strong deformation retractions, products and cones. The purpose of this paper is to construct locally connected examples showing that CIP is not preserved by the above operations and, in fact, provide examples which answer Questions 5.2, 6.3 and 6.4 in [4]. In particular, we show that if X is a locally connected continuum possessing CIP, then $X \times I$ and Cone (X) need not have CIP. Moreover, it is shown that it is possible for X to be either a 1-dimensional continuum or an LC^∞ continuum.

2. Notation and terminology. The terminology used in this paper may be found in [1]. In particular, Hilbert space E^ω with metric ρ and Euclidean n-dimensional space E^n are as defined in [1, pp. 10–11]. For $k = 1, 2, 3, \cdots$, let a_k denote the point in E^ω given by the formula $a_k = (1/k, 0, 0, 0, \cdots)$, and let a_0 denote the origin of E^ω. Let S_k^n denote the n-dimensional sphere in E^ω consisting of all the points $x = (x_1, x_2, x_3, \cdots)$ such that $\rho(x, a_k) = 1/k$ and such that $x_i = 0$ for $i > n + 1$. Then we define

$$A^n = \bigcup_{k=1}^{\infty} S_k^n,\ A_m^\infty = \bigcup_{k=m}^{\infty} S_k^k .$$

A point p in a space X is said to be *homotopically stable* if for every deformation $H: X \times I \to X$, $H(p, t) = p$ for all $t \in I$. For instance, a_0 is a homotopically stable point in each of the spaces

A^n and A_m^∞.

3. **Products.** We first consider some variations of a property considered by L. E. Ward, Jr. in [8, Theorem 1].

DEFINITIONS 3.1. (1) A space X has *property* Q if for every nonempty closed subset K of X there is a point $p \in K$, a retract R of X containing K, and a deformation $H: R \times I \to R$ such that $H(x, t) \neq x$ if $x \neq p$ and $t > 0$.

(2) If in (1) we omit p and stipulate that $H(x, t) \neq x$ if $x \notin K$ and $t > 0$, then we say that X has *property* Q (*weak*).

(3) A space X has *property* W if for every point $p \in X$ there is a deformation $H: X \times I \to X$ such that $H(x, t) \neq x$ if $x \neq p$ and $t > 0$.

(4) If in (3) $H(x, t) \neq x$ whenever $t > 0$, we say that X has *property* W (*strong*).

We note that if X is a space having property W (strong) and Y is an arbitrary space, then the product space $X \times Y$ has property W (strong). Since W (strong) $\Rightarrow W \Rightarrow Q \Rightarrow Q$ (weak), the following proposition shows that any metric space satisfying one of these four properties has CIP.

PROPOSITION 3.2. *Any metric space* (X, δ) *having property* Q (*weak*) *has* CIP.

Proof. Let K be a nonempty closed subset of X, and let R and H be as in Definition (2). We may assume that $\delta \leqq 1$, and let $r: X \to R$ be a retraction of X onto R. Define a map $f: X \to X$ by

$$f(x) = H(r(x), \delta(r(x), K)), \ (x \in X) .$$

Then it is easy to check that f is a self-map of X whose fixed point set is K.

PROPOSITION 3.3. *Let* X, Y *be spaces such that one has property* Q *and the other has property* W. *Then* $X \times Y$ *has property* Q.

Proof. Suppose that X has property Q and Y has property W. Let K be a nonempty closed subset of $X \times Y$. Let π denote the natural projection of $X \times Y$ onto X. Then there is a point $p \in \pi(K)$, a retract R of X containing $\pi(K)$, and a homotopy $F: R \times I \to R$ such that

$$F(x, 0) = x ,$$
$$F(x, t) \neq x \text{ if } x \neq p \text{ and } t > 0 .$$

Let q be a point in Y such that $(p, q) \in K$. Then there is a homotopy $G: Y \times I \to Y$ such that

$$G(y, 0) = y ,$$
$$G(y, t) \neq y \text{ if } y \neq q \text{ and } t > 0 .$$

Now $(p, q) \in K \subset R \times Y$ and $R \times Y$ is a retract of $X \times Y$. Define a homotopy $H: (R \times Y) \times I \to R \times Y$ by

$$H((x, y), t) = (F(x, t), G(y, t)), \quad (x \in R, y \in Y, t \in I) .$$

Then $H((x, y), 0) = (x, y)$, and $H((x, y), t) \neq (x, y)$ if $(x, y) \neq (p, q)$ and $t > 0$. Therefore $X \times Y$ has property Q.

As a corollary to Propositions 3.2 and 3.3, we obtain the following theorem.

THEOREM 3.4. *Let X, Y be metric spaces such that one has property Q and the other has property W. Then $X \times Y$ has* CIP.

We remark that the class of spaces satisfying property W includes any space which admits a strongly convex metrization (see [1, p. 219] and [8, p. 554]), compact manifolds without boundary [6], and all compact triangulable manifolds with or without boundary [7]. Furthermore, property W (strong) holds for the case where one of the above manifolds has Euler characteristic equal to zero, or for the case where the space is a metric group which contains an arc [4, p. 1028].

The following example answers (5.2) in [4] and shows that the hypothesis of property Q in Theorem 3.4 cannot be replaced by property Q (weak).

EXAMPLE 3.5. For $k = 1, 2, 3, \cdots$, let p_k denote the unique point on the upper semicircle of $S_1^1 \subset E^2$ whose first coordinate is $1/k$. Let $B_1, B_2, B_3, \cdots$ be a null sequence of disjoint copies of A^1 lying in the disk bounded by S_1^1 such that

(1) For each $k = 1, 2, 3, \cdots, B_k \cap S_1^1 = \{p_k\}$ where p_k is the point in B_k which corresponds to the point a_0 in A^1. Then,

(2) $\lim_{k\to\infty} B_k = \{a_0\} \subset S_1^1$.

Define $Y = S_1^1 \cup \bigcup_{k=1}^{\infty} B_k$.

Then Y is a 1-dimensional planar Peano continuum. It follows from the proof of Theorem 1 in [5] that Y has property Q (weak) and thus has CIP. However, $Y \times I$ does not have CIP. To see this, let

$$C = \{(a_0, 0)\} \cup \bigcup_{k=1}^{\infty} B_{2k-1} \times \{0\} ,$$

$$D = \{(a_0, 1)\} \cup \bigcup_{k=1}^{\infty} B_{2k} \times \{1\} .$$

Suppose $f\colon Y \times I \to Y \times I$ is a mapping whose fixed point set is $C \cup D$. Since the points $a_0, p_1, p_2, p_3, \cdots$ are homotopically stable points of Y, it follows that $f(\{q\} \times I) \subset \{q\} \times I$ for any point $q \in \{a_0, p_1, p_2, p_3, \cdots\}$. Consequently, if $0 < t < 1$ and $f(p_k, t) = (p_k, s)$, it follows that $s < t$ if k is odd, and $s > t$ if k is even. Therefore $f(a_0, t) = (a_0, t)$ for all $t \in I$ which is a contradiction.

The following examples are higher dimensional analogues of Example 3.5.

EXAMPLE 3.6. If we use the same notation as was used in Example 3.5 and let B_k be homeomorphic to $A^n(A_k^\infty)$ for $k = 1, 2, 3, \cdots$, then we obtain an $LC^n(LC^\infty)$ continuum $Y^n(Y^\infty)$ such that $Y^n \times I(Y^\infty \times I)$ does not have CIP. Moreover, it is easy to show that $Y^n(Y^\infty)$ satisfies property Q (weak) and hence has CIP.

To see this for Y^∞, suppose C is a nonempty closed subset of $A_1^\infty = \bigcup_{k=1}^\infty S_k^k$. Let R denote the retract of A_1^∞ defined by

$$R = S_1^1 \cup \bigcup \{S_k^k \mid C \cap (S_k^k - \{a_0\}) \neq \emptyset\} .$$

We first construct a deformation of R which shows that A_1^∞ has property Q (weak) by constructing deformations on each of the S_k^k lying in R. For each $k > 1$ such that $S_k^k \subset R$, choose one point $c_k \in C \cap (S_k^k - \{a_0\})$. If $a_0 \in C$, consider a deformation of S_k^k which fixes c_k and a_0 for all values of t, and for $t > 0$ moves points along radial rays from the point c_k to the point a_0 at infinity. To obtain the required deformation of R we use the above deformations together with a deformation of S_1^1 whose terminal map is a translation of S_1^1 which fixes the point a_0 at infinity. If $a_0 \notin C$, then there are only finitely many values for k such that $S_k^k \subset R$. In this case, we consider a deformation of S_1^1 whose terminal map is a rotation of S_1^1, and we adjust the previous arguments so as to obtain deformations of the S_k^k into R which agree with the new deformation of S_1^1. This shows that A_1^∞ has property Q (weak) and these arguments can be modified to show that Y^∞ has property Q (weak).

4. Cones. The following example answers Questions 6.3 and 6.4 in [4] by showing that the cone over a 1-dimensional Peano continuum having CIP need not have CIP.

EXAMPLE 4.1. First we construct a 1-dimensional planar Peano continuum X such that every point in X is homotopically stable. We start the construction by considering the Hawaiian earring $A^1 =$

$\bigcup_{k=1}^{\infty} S_k^1$. Let $B_1^n, B_2^n, B_3^n, \cdots$ be a null sequence of disjoint copies of A^1 lying in the disk bounded by S_n^1 and lying in the exterior of the disk bounded by S_{n+1}^1 such that

(1) For each $k = 1, 2, 3, \cdots, B_k^n \cap S_n^1 = \{b_k^n\}$ where b_k^n denotes the unique homotopically stable point of B_k^n.

(2) The set $\{b_k^n | k = 1, 2, 3, \cdots\}$ is a dense subset of S_n^1.

The first stage of our construction yields the space $A^1 \cup \bigcup_{k,n=1}^{\infty} B_k^n$. In the second stage, the above process is repeated for each of the Hawaiian earrings B_k^n. This process is continued and we obtain a continuum X with the required properties. It follows from the proof of Theorem 1 in [5] that X has property Q (weak) and hence has CIP. We now show that the cone over X, denoted by Cone (X), does not have CIP.

By Cone (X) we mean the identification space obtained by taking the disjoint union of $X \times I$ and a set consisting of a single point v, and then identifying each point of the form $(x, 1)$ with the point v. The point v is called the vertex of Cone (X) and, if $0 \leqq t < 1$, we shall regard $X \times [0, t]$ as a subspace of Cone (X). Moreover, since X is a compact metric space, we may assume that Cone (X) is embedded in E^{ω}.

Suppose Cone (X) has CIP and f: Cone $(X) \to$ Cone (X) is a mapping whose fixed point set is $X \times \{0\}$. Since the fixed point set of f is $X \times \{0\}$, it follows that there is a number q such that $0 < q < 1$ and $f^{-1}(v) \subset Y$ where

$$Y = \{(x, t) | (x, t) \in \text{Cone}\,(X) \text{ and } t \geqq q\} .$$

We note that for a point of the form (x, q), $f(x, q) = (x, s)$ for some $s > q$. Let α: Cone $(X) \to Y$ be a retraction of Cone (X) onto Y defined by

$$\alpha(x, t) = \begin{cases} (x, t) \text{ if } q \leqq t \leqq 1 , \\ (x, q) \text{i f } 0 \leqq t \leqq q . \end{cases}$$

Let g: $Y \to Y$ be the mapping defined by $g(x, t) = \alpha f(x, t)$ for all $(x, t) \in Y$. It follows that g is a fixed point free map and, consequently, there is a number $\varepsilon > 0$ such that $\rho(g(x, t), (x, t)) \geqq \varepsilon$ for every point $(x, t) \in Y$. From the construction of X it follows that there is a polyhedron $P \subset X$ and a retraction r_1: $X \to P$ from X onto P such that $\rho(r_1(x), x) < \varepsilon/2$ for all $x \in X$. Define a mapping r_2: $Y \to Y$ by $r_2(x, t) = (r_1(x), t)$ for all $(x, t) \in Y$. Then $K = r_2(Y)$ is an AR-space which is homeomorphic to Cone (P). Let h: $K \to K$ be the map defined by $h(x, t) = r_2 g(x, t)$ for all $(x, t) \in K$. Then $\rho(h(x, t), (x, t)) > \varepsilon/2$ for all $(x, t) \in K$. But this contradicts the fact that K has the fixed point property and, therefore, Cone (X) does not have

CIP.

The following examples are higher dimensional analogues of Example 4.1.

EXAMPLE 4.2. If we follow the construction used in Example 4.1 and replace A^1 by A^n, then we obtain an n-dimensional LC^{n-1} continuum X^n such that Cone (X^n) does not have CIP. In a similar fashion, one can replace A^1 by A_1^∞, B_k^n by a copy of A_{n+k}^∞, and modify the construction of Example 4.1 to obtain an LC^∞ continuum X^∞ such that Cone (X^∞) does not have CIP. A modification of the proof of Theorem 1 in [5] as applied to Example 4.1, together with the ideas introduced in 3.6, can be used to prove that X^n and X^∞ have property Q (weak) and hence have CIP. We omit the details.

5. Deformation retracts. In [4, p. 1024] it is shown that CIP is not preserved by strong deformation retractions of non-locally connected continua. The following example shows that for each positive integer $n = 1, 2, 3, \cdots$, there is a contractible $(n+2)$-dimensional LC^{n-1} continuum Z^n having CIP which contains a strong deformation retract not having CIP.

EXAMPLE 5.1. Let X^n denote the n-dimensional LC^{n-1} continuum defined in 4.2 and let B^n denote the $(n + 1)$-dimensional LC^{n-1} continuum defined by R. J. Knill in [2, p. 37]. Define Z^n to be the wedge obtained by taking the disjoint union of Cone (X^n) and Cone (B^n), and then identifying the vertex of Cone (X^n) with the vertex of Cone (B^n). Clearly, Z^n is a contractible $(n + 2)$-dimensional LC^{n-1} continuum containing Cone (X^n) as a strong deformation retract. Since Cone (X^n) does not have CIP, to complete the proof we must show that Z^n has CIP. But, since Cone (B^n) is an arcwise connected continuum without the fixed point property [2, p. 40], it follows from Proposition 3.6 of [4, p. 1022] that Z^n has CIP if Cone (B^n) has CIP. We now show that Cone (B^n) has CIP.

Let v denote the vertex of Cone (B^n) and let K be a nonempty closed subset of Cone (B^n). If $v \in K$, then K is a fixed point set of Cone (B^n) by Theorem 1 of [8, p. 554]. Thus, without loss of generality, we shall assume that $K \subset B^n \times [0, 1/4]$. It is easy to show, using Lemma 2.1 of [4, p. 1018], that there is a mapping $f_0: B^n \times [0, 1/2] \to \text{Cone}(B^n)$ such that the fixed point set of f_0 is K and $f_0(B^n \times \{1/2\}) = v$. For $0 < t \leqq 1/4$, let

$$C_t^n = \{(x, s) \mid (x, s) \in \text{Cone}(B^n) \text{ and } s \geqq 1 - t\}$$

and let $h_t: \text{Cone}(B^n) \to C_t^n$ be the homeomorphism defined by

$$h_t(x, s) = (x, ts + 1 - t) \text{ for all } (x, s) \in \text{Cone}(B^n) .$$

Let g^n: Cone $(B^n) \to$ Cone (B^n) denote the fixed point free map defined by R. J. Knill in [2, p. 40]. Then $f_t = h_t g^n h_t^{-1}$ is a fixed point free self-map of C_t^n. Let $\sigma(t) = t - 1/2$ if $1/2 < t < 3/4$, and $\sigma(t) = 1/4$ if $3/4 \leqq t \leqq 1$. Define a function f: Cone $(B^n) \to$ Cone (B^n) by

$$f(x, t) = \begin{cases} f_0(x, t) & \text{if } 0 \leqq t \leqq \frac{1}{2} , \\ f_{\sigma(t)}\left(x, \frac{3}{2} - t\right) & \text{if } \frac{1}{2} < t \leqq \frac{3}{4} , \\ f_{\sigma(t)}(x, t) & \text{if } \frac{3}{4} \leqq t \leqq 1 . \end{cases}$$

It follows that f is continuous and that the fixed point set of f is K.

REMARKS 6. We end the paper with the following three remarks.

6.1. By using only spheres of dimension $\geqq n + 1$ in the constructions found in 3.6 and 4.2, it is possible to construct examples which are both C^n and LC^∞.

6.2. The authors do not know of any locally contractible examples.

6.3. The question posed in [3, p. 165] as to whether every AR-space (ANR-space) has CIP remains unsolved.

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