A HOMOGENEOUS EBERLEIN COMPACT SPACE WHICH IS NOT METRIZABLE

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We give an example of a first countable, hereditarily normal, homogeneous Eberlein compact space which is not metrizable. This answers a question of A. V. Arhangel’skii.

1. Introduction. A compact Hausdorff space is called Eberlein compact, if it is homeomorphic to a weakly compact subset of a Banach space. For information concerning Eberlein compact spaces, see [1], [3], [5] and [7].

If \( X \) is Eberlein compact, then \( X \) is metrizable if \( X \) satisfies the countable chain condition, [2], [5], or if \( X \) is linearly orderable, [4]. In view of these facts, the following question due to Arhangel’skii [3, p. 91 problem 5], is quite natural: is there a non-metrizable homogeneous Eberlein compact space? The aim of this paper is to construct such an example which in addition is zero-dimensional, first countable and hereditarily normal. The symbol “\( X \approx Y \)” means that \( X \) and \( Y \) are homeomorphic spaces. I am indebted to Mary Ellen Rudin for spotting some inaccuracies in an earlier version of this paper.

2. Preliminaries. A family \( \mathcal{F} \) of subsets of a topological space \( X \) is called separating provided that for any distinct \( x, y \in X \) there is an \( F \in \mathcal{F} \) such that either \( x \in F \) and \( y \notin F \) or \( y \in F \) and \( x \notin F \). The family \( \mathcal{F} \) is called point-finite if each \( x \in X \) belongs to at most finitely many elements of \( \mathcal{F} \). It is called \( \sigma \)-point-finite if \( \mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n \), where each \( \mathcal{F}_n \) is point-finite.

The following purely topological characterization of Eberlein compacta, due to Rosenthal [8], is convenient for topologists.

**Theorem 2.1.** A compact Hausdorff space is Eberlein compact iff it has a \( \sigma \)-point-finite separating family of open \( F \_n \)-subsets.

Let \( C \) denote the usual Cantor set in \([0, 1]\) (notice that \( 0 \in C \)) and let \( X \) be any space. Topologize \( X \times C \) in the following way:

(a) a basic neighborhood of a point \( \langle x, 0 \rangle \) has the form

\[
(U \times C) - ([x] \times D),
\]

where \( U \subseteq X \) is open, contains \( x \) and \( D \subseteq C - \{0\} \) is compact;  

\(^1\) A space \( X \) is called homogeneous provided that for any two points \( x, y \in X \) there is an autohomeomorphism \( h: X \to X \) with \( h(x) = y \).
(b) a basic neighborhood of a point \( \langle x, c \rangle \) where \( c > 0 \) has the form

\[
\{x\} \times U
\]

where \( U \subset C - \{0\} \) is an open neighborhood of \( c \).

The topological space we obtain in this way will be denoted by \( X(C) \). Observe that the projection \( \pi: X(C) \to X \) onto the first coordinate is continuous. In addition, the function \( f: X \to X(C) \) defined by \( f(x) = \langle x, 0 \rangle \) is an embedding.

**Lemma 2.2.** (1) \( X(C) \) is compact Hausdorff iff \( X \) is compact Hausdorff,

(2) \( X(C) \) is first countable iff \( X \) is first countable,

(3) \( X(C) \) is Eberlein compact iff \( X \) is Eberlein compact.

**Proof.** (1) We only need to show that \( X(C) \) is compact if \( X \) is. Let \( \mathcal{U} \) be an open cover of \( X(C) \) by basis elements. Finitely many elements of \( \mathcal{U} \) cover \( X \times \{0\} \) and the remaining part of \( X(C) \) consists of finitely many compact sets. We conclude that \( \mathcal{U} \) has a finite subcover.

Observe that (2) is trivial and that for (3) we only need to show that \( X(C) \) is Eberlein compact if \( X \) is Eberlein compact (closed subsets of Eberlein compacta are Eberlein compact). To this end, let \( \mathcal{F} = \bigcup_{n=1}^{\infty} \mathcal{F}_n \) be a separating family of open \( F_\sigma \)-subsets of \( X \) such that for all \( n \) the family \( \mathcal{F}_n \) is point-finite. In addition, let \( \{C_m: m = 1, 2, \cdots\} \) be a countable basis for \( C - \{0\} \) consisting of compact open sets. For all \( n, m \in \mathbb{N} \) define

\[
\mathcal{F}_n = \{F \times C: F \in \mathcal{F}_n\},
\]

and

\[
\mathcal{C}_m = \{\{x\} \times C_m: x \in X\}
\]

respectively. Observe that both \( \mathcal{F}_n \) and \( \mathcal{C}_m \) are point-finite, that \( \mathcal{F}_n \) consists of open \( F_\sigma \)-subsets of \( X(C) \) and that \( \mathcal{C}_m \) consists of clopen ( = closed and open) subsets of \( X(C) \). Since trivially,

\[
\mathcal{G} = \bigcup_{n=1}^{\infty} \mathcal{F}_n \cup \bigcup_{m=1}^{\infty} \mathcal{C}_m
\]

is separating, Theorem 2.1 implies that \( X(C) \) is Eberlein compact. \( \square \)

**3. The example.** Let \( X \) be any space. Define \( X_1 = X \) and \( X_{n+1} = X_n(C) \). The projection from \( X_{n+1} \) onto its first coordinate is
denoted by $f_{n,x}$. Put

$$\tilde{X} = \lim_{n \to \infty} (X_n, f_{n,x})$$

(i.e., $\tilde{X} = \{x \in \prod_{n=1}^{\infty} X_n : f_{n,x}(x_{n+1}) = x_n \text{ for all } n \in \mathbb{N}\}$). Let $\pi_{n,x} : \tilde{X} \to X_n$ be the projection. Observe that

$$\pi_{n,x} = f_{n,x} \circ \pi_{n+1,x}.$$  

**Lemma 3.1.** (1) If $A \subset X$, then $\tilde{A} \approx (\pi_{i+1}^{-1})^\sim(A) \subset \tilde{X}$.

(2) $\tilde{X}_n \approx \tilde{X}$ for all $n \in \mathbb{N}$.

**Proof.** Obvious. 

We claim that $\tilde{C}$ is homogeneous, nonmetrizable and Eberlein compact. By a repeated application of Lemma 2.2(3) it follows that each $C_n$ is Eberlein compact. Consequently, by Theorem 2.1,

$$\prod_{n=1}^{\infty} C_n$$

is Eberlein compact which implies that $\tilde{C}$ is Eberlein compact, being a closed subspace of $\prod_{n=1}^{\infty} C_n$. Similarly, each $C_n$ is first countable and consequently, $\tilde{C}$ is first countable. It is clear that $\tilde{C}$ is not metrizable, since it maps onto the nonmetrizable space $C_2$ ($C_2$ is not metrizable since it contains an uncountable family of pairwise disjoint nonempty open subsets). Obviously, $\tilde{C}$ is zerodimensional.

**Theorem 3.2.** $\tilde{C}$ has the property that all of its nonempty clopen subspaces are homeomorphic (hence $\tilde{C}$ is strongly homogeneous in the sense of [8]).

**Proof.** By induction on $n$ we will show that $\pi_{n,c}(U) \approx \tilde{C}$ for all nonempty clopen $U \subset C_n$. This is clearly true for $n = 1$ since all nonempty clopen subsets of $C$ are homeomorphic to $C$ which implies that

$$\pi_{1,c}^{-1}(U) \approx \tilde{U} \approx \tilde{C}$$

for all clopen $U \subset \tilde{C}_1$ (Lemma 3.1(1)). Now suppose the statement to be true for $n$ and take a nonempty clopen $U \subset C_{n+1}$ arbitrarily. If $U \cap (C_n \times \{0\}) = \emptyset$ then $U$ is homeomorphic to $C$ by definition of the topology of $C_{n+1}$. Consequently, by Lemma 3.1(1), (2) it then follows that

$$\pi_{n+1,c}(U) \approx \tilde{U} \approx \tilde{C}.$$ 

Therefore assume that $U \cap (C_n \times \{0\}) \neq \emptyset$. By definition of the
topology of $C_{n+1}$ there is a finite $F \subset C_n$ and for each $x \in F$ a clopen $S_x \subset C$ not containing 0 such that $F \times \{0\} \subset V = U \cap (C_n \times \{0\})$ while moreover
\[
E = (V \times C) - (\bigcup_{x \in F} \{x\} \times S_x) \subset U.
\]
For each $x \in F$ let $h_x : C - S_x \to C$ be a homeomorphism such that in a fixed neighborhood of 0 each $h_x$ is the identity. Define $h : E \to V \times C$ by
\[
\begin{align*}
  h(\langle a, b \rangle) &= \langle a, b \rangle \quad \text{if } a \notin F, \\
  h(\langle a, b \rangle) &= \langle a, h_x(b) \rangle \quad \text{if } a \in F.
\end{align*}
\]
Clearly, $h$ is a homeomorphism. Therefore
\[
\pi_{n+1,c}(E) \approx \pi_{n,c}(V) \approx \tilde{C},
\]
by induction hypothesis. Put $G = U - E$. Then $G$ is a clopen subset of $C_{n+1}$ which misses $C_n \times \{0\}$. If $G = \emptyset$ then we are done, and if $G \neq \emptyset$ then observe that
\[
\pi_{n+1,c}(G) \approx \tilde{C}
\]
since $G \approx C$ (cf. the above remarks). Consequently, $\pi_{n+1,c}(U)$ is the disjoint union of two clopen copies of $\tilde{C}$, hence is itself homeomorphic to $\tilde{C}$ since $C$ is the disjoint union of two clopen copies of itself. This completes the induction.

Now let $A \subset \tilde{C}$ be clopen and nonempty. There is clearly an index $n \in N$ and a nonempty clopen $B \subset C_n$ such that
\[
\pi_{n,c}(B) = A.
\]
Therefore $A = \pi_{n,c}(B) \approx \tilde{C}$. □

The above theorem shows that $\tilde{C}$ is homogeneous, for any zero-dimensional strongly homogeneous first countable space $X$ is homogeneous. This is well-known and for completeness sake we will include the trivial proof. Take $x, y \in X$. Since $X$ is first countable, there is a clopen neighborhood basis $\{V_n : n \in N\}$ for $x$ and a clopen neighborhood basis $\{W_n : n \in N\}$ for $y$ such that
\[
\begin{align*}
  (1) \quad V_1 &= W_1 = X, \\
  (2) \quad V_{n+1} \text{ is properly contained in } V_n, \text{ and} \\
  (3) \quad W_{n+1} \text{ is properly contained in } W_n.
\end{align*}
\]
For each $n \in N$ let $h_n : V_n - V_{n+1} \to W_n - W_{n+1}$ be any homeomorphism. The function $h : X \to X$ defined by
\[
\begin{align*}
  h(x) &= y, \\
  h(a) &= h_n(a) \quad \text{if } a \in V_n - V_{n+1}
\end{align*}
\]
is clearly a homeomorphism mapping \( x \) onto \( y \).

**Remark 3.3.** It is not by accident that our example is first countable. By [5, 4.3] every Eberlein compact space is first countable at a dense set of points, consequently, a homogeneous Eberlein compact space must be first countable. Notice however that we used the first countability of \( C \) to show it is homogeneous.

4. \( \tilde{C} \) is hereditarily normal. In this section we will show that \( \tilde{C} \) is a continuous image of a compact linearly orderable topological space. This implies that \( \tilde{C} \) is hereditarily normal (even monotonically normal).

Let \( L_1 = C \) and let \( L_2 = C \times C \) with topology generated by the lexicographical ordering. Let \( g_1: L_2 \to L_1 \) be the projection onto the first coordinate. Observe that \( g_1 \) is order preserving. Let \( \varphi_1: L_1 \to C_1 \) be the identity and let \( h: C \to C \) be an arbitrary onto map such that

\[
h(0) = 0 \quad \text{and} \quad h(1) = 0.
\]

Define \( \varphi_2: L_2 \to C_2 \) by

\[
\varphi_2((a, b)) = (a, h(b)).
\]

Because \( h(0) = 0 = h(1) \), \( \varphi_2 \) is continuous.

It is easily seen that the diagram

\[
\begin{array}{ccc}
L_1 & \xleftarrow{g_1} & L_2 \\
\downarrow{\varphi_1} & & \downarrow{\varphi_2} \\
C_1 & \xleftarrow{\varphi_{1,C}} & C_2
\end{array}
\]

commutes. Suppose that we have defined \( L_n \) and \( \varphi_n \). Let \( L_{n+1} = L_n \times C \) with topology generated by the lexicographical ordering and let \( g_n: L_{n+1} \to L_n \) be the projection. Define \( \varphi_{n+1}: L_{n+1} \to C_{n+1} \) by \( \varphi_{n+1}((a, b)) = (a, h(b)) \), where \( h \) is defined as above. Observe that \( g_n \) is order preserving and that

\[
f_{n,C} \circ \varphi_{n+1} = \varphi_n \circ g_n.
\]

Put \( L = \lim (L_n, g_n) \). Since the maps \( g_n \) are all order preserving, \( L \) can be ordered in a natural way (It is easy to describe the ordering of \( L \). Alternatively, the orderability theorems given in [6] or [9] are also easily applied.). By (1), the space \( L \) maps onto \( \tilde{C} \) so that \( \tilde{C} \) is hereditarily normal.

As was pointed out to me by Dave Lutzer, it is also easily seen
that $\tilde{C}$ is hereditarily paracompact, since $L$ is a first countable compact LOTS and $L$ maps onto $\tilde{C}$.

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Received February 11, 1981 and in revised form July 6, 1981.

Subfaculteit Wiskunde
Vrije Universiteit
De Boelelaan 1081
Amsterdam, The Netherlands
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