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**EMBEDDING HOMOLOGY 3-SPHERES IN  $S^5$**

STEVEN PAUL PLOTNICK

## EMBEDDING HOMOLOGY 3-SPHERES IN $S^5$

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**The purpose of this note is to give a proof independent of high-dimensional surgery theory of the following embedding result:**

**THEOREM.** Let  $\Sigma^3$  be the homology 3-sphere resulting from a Dehn surgery of type  $1/2a$  on a knot in  $S^3$ . Then  $\Sigma^3$  smoothly embeds in  $S^5$  with complement a homotopy circle.

This theorem illustrates the connection between two major areas of ignorance in low-dimensional topology. For instance, if the homology sphere  $\Sigma^3$  bounds a contractible 4-manifold  $V^4$ , then, using the 5-dimensional Poincaré conjecture, we see that  $\Sigma^3 \times 0 \hookrightarrow \Sigma^3 \times D^2 \cup V^4 \times S^1$  is a smooth embedding of  $\Sigma$  into  $S^5$  with complement homotopy equivalent to a circle. Conversely, if  $\Sigma$  smoothly embeds in  $S^5$  with  $S^5 - \Sigma \simeq S^1$ , and if the Browder-Levine fibering theorem [1] holds in dimension 5, then  $S^5 - \Sigma^3 \times \dot{D}^2$  fibers over  $S^1$ , and the fiber is necessarily contractible.

High dimensional surgery theory can be used to completely solve this problem. Given  $\Sigma^3$ , convert  $\Sigma^3 \times T^2$  to  $K \simeq S^3 \times T^2$  via surgery, with  $\Sigma^3 \subset K$  (see [6]). By work of Kirby-Siebenmann,  $K$  is homeomorphic to  $S^3 \times T^2$ . Lifting to the universal cover, we get  $\mathcal{Y} \subset S^3 \times \mathbb{R}^2 \subset S^5$ , and we see that every homology 3-sphere topologically embeds in  $S^5$  with complement a homotopy circle. However, if  $\Sigma$  has nontrivial Rochlin invariant, a standard argument shows that the embedding cannot be smooth or PL. (If it were smooth (PL), make the homotopy equivalence  $f: S^5 - \Sigma^3 \times \dot{D}^2 \rightarrow S^1$  transverse to a point  $p \in S^1$ . Then  $f^{-1}(p)$  would be a smooth (PL) spin manifold  $V^4$  with zero signature and  $\partial V = \Sigma$ , contradicting the fact that  $\Sigma$  has nontrivial Rochlin invariant.) If  $\Sigma$  has trivial Rochlin invariant, the argument in [8] shows that the embedding can be taken to be smooth or PL. (See [7] for a much deeper analysis of knotting of homology 3-spheres in  $S^5$ .) Nevertheless, it seems desirable to give a more elementary construction for these embeddings when possible. It would be nice if these methods, together with the Kirby-Rolfsen calculus for links in  $S^3$ , could provide the desired embeddings for all  $\Sigma^3$  with zero Rochlin invariant.

This proof grew out of studying Fintushel and Pao's attempt [3] to show that Scharlemann's possibly exotic  $S^3 \times S^1 \# S^2 \times S^2$  is standard [6]. The basic construction is from [3] and will be described below.

I would like to thank the referee for very useful and constructive comments.

*Proof of the theorem.* Let  $K \subset S^3$  be a smooth knot, and let  $\Sigma^3$  be the homology 3-sphere resulting from a Dehn surgery on  $K$  of type  $1/2a$ . Let  $m$  and  $\ell$  be a meridian and preferred longitude of  $K$ . It is not hard to see that surgery on the curve  $\ell \times \{*\}$  in the 4-manifold  $\Sigma^3 \times S^1$  produces a manifold homotopy equivalent to  $S^3 \times S^1 \# S^2 \times S^2$  or  $S^3 \times S^1 \# S^2 \tilde{\times} S^2$ , depending on the framing used, where  $S^2 \tilde{\times} S^2$  is the nontrivial  $S^2$  bundle over  $S^2$ . We will sketch the proof ([3]) that the manifold is in fact diffeomorphic to  $S^3 \times S^1 \# S^2 \times S^2$ , assuming we use the framing which produces an even 4-manifold, and we will also keep track of homology generators for future use.

Think of surgery on  $\ell \times \{*\}$  in  $\Sigma^3 \times S^1$  as follows: First remove a tubular neighborhood  $T \approx S^1 \times D^2 \times S^1$  of  $\ell \times S^1$  in  $\Sigma^3 \times S^1$ , leaving  $(S^3 - K \times \dot{D}^2) \times S^1$ . Let  $X \approx S^1 \times D^3$  be a tubular neighborhood of  $\ell \times \{*\}$ , where  $X$  sits in  $T$  in the obvious fashion, so that  $\overline{T - X} = S^1 \times D^2 \times I$ . To surger  $\ell$ , replace  $X$  by  $D^2 \times S^2$ , identifying  $S^1 \times \{\text{polar caps}\} \subset D^2 \times S^2$  with  $S^1 \times D^2 \times \{\pm 1\} \subset S^1 \times D^2 \times I$ .

The identification  $D^2 \times S^2 \bigcup_{S^1 \times D^2 \times \{\pm 1\}} S^1 \times D^2 \times I$  produces a 4-manifold  $P^4$  which can be identified as the result of plumbing two copies of  $S^2 \times D^2$  at two points. The boundary of  $P^4$  is  $T^3$  with homology generators  $e_1, e_2, e_3$  as follows:  $e_1$  is a meridian of  $S^1 \times D^2 \times I$ ,  $e_2$  is that longitude of  $S^1 \times D^2 \times I$  which, after being isotoped across a plumbing point, becomes a meridian to  $D^2 \times \text{equator} \subset D^2 \times S^2$ , and  $e_3$  generates  $H_1(P^4) \cong \mathbb{Z}$ . Actually,  $e_3$  is defined only modulo multiples of  $e_1$  and  $e_2$ , but  $P^4$  admits self-diffeomorphisms taking any generator of  $H_1(P)$  to any other generator (see [2], Lemma 3.3), so we can ignore this point.

If we let  $N$  denote the result of surgery on  $\ell \times \{*\}$  in  $\Sigma^3 \times S^1$  (using the framing induced from the zero framing of  $\ell$  in  $\Sigma^3$ ), we see that  $N$  is the union of  $P^4$  and  $(S^3 - K \times \dot{D}^2) \times S^1$  defined by the matrix

$$\begin{matrix} & e_1 & e_2 & e_3 \\ \begin{matrix} m \\ \ell \\ h \end{matrix} & \begin{pmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{matrix},$$

where  $m$  is a meridian to  $K$  in  $S^3$ , and  $h$  generates the circle factor in  $(S^3 - K \times D^2) \times S^1$ .

Notice that there are two natural 2-spheres in  $P^4$ , the cores of

the two copies of  $S^2 \times D^2$ . We have  $H_2(P) \cong \mathbf{Z}^2$ , generated by the cores, which we denote  $A$  and  $B$ , where  $A$  corresponds to the  $S^2$  added in the surgery, and  $B$  is

$$D^2 \times \{\text{north and south poles}\} \cup S^1 \times \{0\} \times I$$

in the decomposition  $P = D^2 \times S^2 \cup S^1 \times D^2 \times I$ . Also,  $H_2(T^3 = \partial P^4) \cong \mathbf{Z}^3$ , generated by  $e_1 \wedge e_2$ ,  $e_1 \wedge e_3$ , and  $e_2 \wedge e_3$ , which we write as  $e_{12}$ ,  $e_{13}$ ,  $e_{23}$ . The inclusion  $T^3 \hookrightarrow P^4$  induces  $e_{12} \mapsto 0$ ,  $e_{13} \mapsto A$ ,  $e_{23} \mapsto B$ . Finally,

$$H_2((S^3 - K \times \dot{D}^2) \times S^1) \cong \mathbf{Z},$$

generated by  $m \wedge h$ .

Examination of the Mayer-Vietoris sequence for  $N$  yields  $H_2(N) \cong \mathbf{Z}^2$ , with explicit generators. The 2-sphere  $A$  is one generator. Since  $e_2$  bounds a disk in  $P$ , and is glued to  $\mathcal{C}$ , which bounds a Seifert surface in  $S^3 - K \times \dot{D}^2$ , we may glue the disk to the surface to produce the other generator, which we refer to as the generator arising from  $e_2$ . Notice that  $B$  is trivial in  $H_2(N)$ .

Now create  $W^5$  by adding a 2-handle to  $\Sigma^3 \times S^1 \times I$  along  $\mathcal{C} \times \{*\} \times \{1\}$ , producing a cobordism from  $\Sigma^3 \times S^1$  to  $N$ . The class of  $A$  in  $H_2(N)$  dies in  $H_2(W)$ , while the class arising from  $e_2$  lives in  $H_2(W)$ . In fact, it is easy to see that

$$H_i(W) = \begin{cases} \mathbf{Z}, & i = 0, 1, 2, 3, 4 \\ 0, & i = 5 \end{cases}$$

with all of  $H_*(W)$  coming from  $H_*(N)$ .

Now, as first observed by Pao [5],  $P^4$  admits the following self-diffeomorphism: remove one copy of  $S^2 \times D^2$  and replace it by an element in the kernel of  $\pi_1 SO(2) \rightarrow \pi_1 SO(3)$ . This idea can easily be used to produce a self-diffeomorphism  $f$  which fixes  $e_3$  and one of  $e_1, e_2$  (say  $e_2$ ), and takes  $e_1$  to  $e_1$  plus an even multiple of  $e_2$ . (To do this we remove and replace  $B$ .) This gives the following diagram:

$$\begin{array}{ccc} P \longleftarrow \partial P \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 2\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} (S^3 - K \times \dot{D}^2) \times S^1 & & \\ \downarrow f & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 2\alpha & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \parallel \\ P \longleftarrow \partial P \xrightarrow{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}} (S^3 - K \times \dot{D}^2) \times S^1 & & \end{array}$$

The top row gives  $N$ , the bottom  $S^3 \times S^1 \# S^2 \times S^2$ , yielding

$$N \xrightarrow[\cong]{f} S^3 \times S^1 \# S^2 \times S^2 .$$

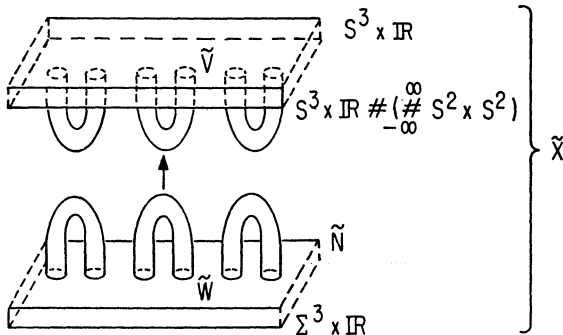
We can also create  $V^5$ , a cobordism from  $S^3 \times S^1$  to  $S^3 \times S^1 \# S^2 \times S^2$ , by attaching a 2-handle along  $\not\subset \times \{*\} \times \{0\}$  in  $S^3 \times S^1 \times I$ . Glue  $W$  to  $V$  using the diffeomorphism  $f$ , creating a cobordism  $X^5$  from  $\Sigma^3 \times S^1$  to  $S^3 \times S^1$ .

The point is this: The class of  $A$  in  $H_2(N)$  is taken to the corresponding class in  $H_2(S^3 \times S^1 \# S^2 \times S^2)$ , which dies in  $H_2(V)$ . This is certainly not true geometrically, since  $f$  takes  $A$  to  $A + 2aB$  (essentially,  $A$  is altered by the ‘‘belt trick’’), but  $B$  is homologically trivial. The class in  $H_2(N)$  arising from  $e_2$  is geometrically taken to the corresponding class in  $H_2(S^3 \times S^1 \# S^2 \times S^2)$ .

Now examine  $H_*(X)$ . Since  $A$  bounds  $D^3$  in  $W$ , and  $A + 2aB$  bounds a 3-chain in  $V$ , we produce a generator in  $H_3(X)$ . This 3-cycle has intersection number  $\pm 1$  with the generator of  $H_2(W)$  arising from  $e_2$ , and the generator of  $H_2(W)$  arising from  $e_2$  is identified with a class in  $H_2(V)$  which we can represent by an embedded 2-sphere (with trivial normal bundle), since  $\not\subset$  bounds a singular disk in  $S^3$ .

Now surger the generator of  $H_2(X)$ . Standard sequences for this surgery show that this simultaneously kills the generator of  $H_2(X)$  and its dual in  $H_3(X)$ . The result is a homology product,  $Y$ , from  $\Sigma^3 \times S^1$  to  $S^3 \times S^1$ , and  $\pi_1 Y \cong \mathbb{Z}$ , coming from the circle factor in either boundary component. If we now glue  $D^4 \times S^1$  to  $Y$  along  $S^3 \times S^1$ , and glue  $\Sigma^3 \times D^2$  along  $\Sigma^3 \times S^1$ , we have a simply-connected homology 5-sphere, hence  $S^5$ . Thus, we have a smooth embedding of  $\Sigma^3$  in  $S^5$  with  $\pi_1(S^5 - \Sigma^3 \times \dot{D}^2) \cong \mathbb{Z}$ .

Actually, it follows from [6] that for every homology 3-sphere  $\Sigma$ ,  $\Sigma \times S^1$  is homology-cobordant to  $S^3 \times S^1$ . The argument is as follows: embed  $\Sigma^3$  in  $S^5$  and remove a tubular neighborhood  $\Sigma^3 \times D^2$



of  $\Sigma$  and a tubular neighborhood  $S^1 \times D^4$  of a meridian to the knotted  $\Sigma^3$ . The result is a homology-cobordism  $Y^5$  from  $\Sigma^3 \times S^1$  to  $S^3 \times S^1$ , and  $\pi_1(\Sigma) \rightarrow \pi_1(Y)$  is trivial. In general,  $\pi_1 Y$  will be mysterious.

Consider the universal cover  $\tilde{X}$ : We have  $H_2(\tilde{X}) \cong \mathbf{Z}(\mathbf{Z})$  and  $H_3(\tilde{X}) \cong \mathbf{Z} \oplus \mathbf{Z}(\mathbf{Z})$ . If we do  $\mathbf{Z}$  surgeries equivariantly, killing the  $\mathbf{Z}(\mathbf{Z})$  factors, the result is  $\tilde{Y}$ . To create  $\widetilde{S^5 - \Sigma \times D^2}$ , attach  $D^4 \times \mathbf{R}$  to  $\tilde{Y}$  along  $S^3 \times \mathbf{R}$ . This kills  $H_3(\tilde{Y})$ , and thus  $\widetilde{S^5 - \Sigma \times D^2}$  is contractible, so that  $S^5 - \Sigma \times D^2$  is a  $K(\mathbf{Z}, 1)$ . This proves the theorem.

REMARKS. (1) Surgery of type  $1/2a$  on a knot in  $S^3$  results in a homology 3-sphere with zero Rochlin invariant, by [4].

(2) The proof is equally valid for (a) knots in homology spheres which bound contractible 4-manifolds, or (b) surgeries of type  $1/2a_i$ ,  $i = 1, \dots, n$ , on a link of  $n$  components, provided the components are algebraically unlinked (by doing  $n$  times as many surgeries). In particular, the theorem is valid for connected sums of  $\Sigma$ 's as above.

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UNIVERSITY OF CHICAGO  
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