SCHAUDER BASES AND FIXED POINTS OF NONEXPANSIVE MAPPINGS

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A fixed point theorem is proved for nonexpansive mappings in Banach spaces which are isomorphic to spaces with certain boundedly complete bases.

1. Introduction. Suppose $X$ and $Y$ are isomorphic Banach spaces with $h \| \cdot \|_Y \leq \| \cdot \|_X \leq k \| \cdot \|_Y$, where $\| \cdot \|_Y$ and $\| \cdot \|_X$ denote the norms in $Y$ and $X$ respectively. Let $t = kh^{-1}$ (this notation will be kept fixed throughout the paper). Suppose also that every convex weakly compact (weak* compact, when $X$ is a dual Banach space) subset $K$ of $X$ has the fixed point property with respect to nonexpansive mappings (i.e., mappings $T: K \rightarrow K$ such that $\|Tx - Ty\|_X \leq \|x - y\|_X$, for all $x, y \in K$). It is not known in general whether, assuming $t$ sufficiently close to 1, convex weakly compact (weak* compact) subsets of $Y$ have the same property (but see Bynum [1]).

In this paper we answer in the affirmative this question when $X$ has a Schauder basis $(b_n)$ which satisfies a condition introduced by Gossez and Lami Dozo [2]. For every positive integer $k$ and $x \in X$ set $U_k(x) = \sum_{n=1}^{k} f_n(x)b_n$, where $(f_n)$ denotes the associated system of linear functionals. We shall always assume that there exists a strictly increasing sequence $(k_n)$ with the following property:

for every $c > 0$ there is $\rho > 0$ such that whenever $x \in X$ and $n$ satisfy

$$\|U_{k_n}(x)\|_X = 1$$
$$\|x - U_{k_n}(x)\|_X \geq c$$

then $\|x\|_X \geq 1 + \rho$.

It is easy to see (Lemma 1 below) that the above condition implies that the basis $(b_n)$ is boundedly complete, so that $X$ is a dual Banach space.

In the next sections it is proved that there exists $t_0 > 1$ such that for $t < t_0$ every weak* compact convex subset of $Y$ has the fixed point property with respect to nonexpansive mappings. For $t = 1$ this follows easily from the results of Karlovitz [3], while for $t > 1$ it can not be deduced from [3]. As a remarkable consequence we obtain that, in every Banach space $Y$ isomorphic to $l^1$ with $t < 2$, weak* compact convex subsets have the fixed point property with respect to nonexpansive mappings.
2. Properties of the space \( X \).

**Lemma 1.** Suppose \( X \) is a Banach space with a Schauder basis \((b_n)\) satisfying the assumptions of the above section. Then the basis \((b_n)\) is boundedly complete and \( X \) is isomorphic to the dual of the Banach space generated by the system of the linear functionals \((f_n)\).

**Proof.** Suppose that \((a_n)\) is a sequence of scalars such that 
\[
\sup_n \|\sum_{j=1}^n a_n b_n\|_X < \infty.
\]
Then, the same argument as in [6, p. 290-291] implies that, for some subsequence \(k_{n_j}\), 
\[
\sum_{n=1}^{k_{n_j}} a_n b_n
\]
converges to a point \(x \in X\). Then, of course, \(f_n(x) = a_n\) for every \(n\), so that 
\[
\sum_{n=1}^{\infty} a_n b_n = x.
\]
The second assertion is proved in [6, Th. II 6.2, 3].

For every positive integer \(n\) and every real \(c > 0\) we set \(r_n(c) = \inf \|x\|_X - 1\), where the infimum is taken over all \(x \in X\) such that 
\[
\|U_{k_n}(x)\|_X = 1, \|x - U_{k_n}(x)\|_X \geq c.
\]
We set also \(r(c) = \inf_n r_n(c)\). Clearly \(r(c) > 0\) for all positive \(c\). We complete the definition of \(r(c)\) by letting \(r(0) = 0\). In the following we set \(V_{k_n}(x) = x - U_{k_n}(x)\).

**Lemma 2.** \(r(c)\) is a nondecreasing continuous function of \(c\).

**Proof.** Let \(\varepsilon > 0\) be arbitrarily small and \(c_2 > c_1 \geq 0\). There exist \(n > 0\) and \(x \in X\) such that 
\[
\|U_{k_n}(x)\|_X = 1, \|x - U_{k_n}(x)\|_X \geq c_2
\]
and 
\[
1 + r(c_1) + \varepsilon > \|x\|_X \geq 1 + r(c_1).\]
Hence \(r(c_2) \geq r(c_1) \geq 0 = r(0)\) and \(r(c)\) is nondecreasing.

Observe now that there exist a sequence of points \(x_j \in X\) and a sequence of positive integers \(n_j\) such that
\[
\|U_{k_{n_j}}(x_j)\|_X = 1, \quad \|V_{k_{n_j}}(x_j)\|_X \geq c_1 \quad \text{and} \quad 1 + r(c_1) + j^{-1} > \|x_j\|_X.
\]
We set \(v_j = \|V_{k_{n_j}}(x_j)\|_X\). After extracting a subsequence if necessary, we may suppose that \(v = \lim_j v_j\) exists. If \(v > c_2\), then, for large values of \(j\), 
\[
1 + r(c_1) + j^{-1} > \|x_j\|_X \geq 1 + r(c_2),
\]
so that, by what has been already proved, \(r(c_1) = r(c_2)\), and we are done. Thus we may assume \(c_1 \leq v \leq c_2\). Let \(y_j = x_j + s_j V_{k_{n_j}}(x_j)\), where \(s_j\) is a scalar such that 
\[
(1 + s_j)v_j = c_2.
\]
Clearly we must have \(\|y_j\|_X \geq 1 + r(c_2)\) and 
\[
\|x_j - y_j\|_X = |s_j| v_j.
\]
Hence 
\[
1 + r(c_1) + j^{-1} > \|x_j\|_X \geq \|y_j\|_X - |s_j|v_j
\]
\[
\geq 1 + r(c_2) - |s_j|v_j.
\]
that is, 
\[
\frac{c_2 - c_1}{v_j + j^{-1}}.
\]
Now, if \(v < c_2\), then \(|s_j| = s_j = (c_2 - v_j)v_j^{-1} \leq (c_2 - c_1)v_j^{-1}\) for \(j\) large enough. If \(v = c_2\) then \(s_j\) tends to 0, so that, if \(j\) is large, \(|s_j| <
(e_i - e_j)v_j^{-1}. In any case, for large values of \( j \), we obtain \( r(e_i) - r(e_j) \leq (e_i - e_j) + j^{-1} \), and the proof is ended.

**Lemma 3.** Suppose that \( (x_n) \subseteq X \) is a sequence of points converging in the weak* topology to a point \( z \in X \). Let \( \gamma = \lim sup \| x_n - z \|_X \). Then, for every \( y \in X \), \( y \neq z \)

\[
\lim sup \| x_n - y \|_X \geq \{ 1 + r(\gamma \| y - z \|^{-1}) \} \| y - z \|_X .
\]

**Proof.** Let \( \varepsilon > 0 \) be arbitrarily small. There exists \( j = j(\varepsilon) \) such that \( \| V_{k_j}(y - z) \|_X < \varepsilon \). Since \( x_n - z \) converges weak* to 0 and the associated functionals \( f_n \) are weak* continuous (Lemma 1), for every fixed \( j \) we can find \( n_0 \) such that \( \| U_{k_j}(x_n - z) \|_X < \varepsilon \) for \( n \) greater than \( n_0 \). Therefore, for \( n > n_0 \), we have by Lemma 2

\[
\begin{align*}
\| y - x_n \|_X & \geq -2\varepsilon + \| U_{k_j}(y - z) \|_X \\
& \geq -2\varepsilon + \| U_{k_j}(y - z) \|_X \{ 1 + r(\| V_{k_j}(z - x_n) \|_X \| U_{k_j}(y - z) \|^{-1}) \} \\
& \geq -2\varepsilon + (\| y - z \|_X - \varepsilon)(1 + r(\| z - x_n \|_X - \varepsilon)(\| y - z \|_X + \varepsilon)^{-1}) .
\end{align*}
\]

By Lemma 2 again

\[
\lim sup \| y - x_n \|_X \\
\geq (\| y - z \|_X - \varepsilon)(1 + r(\gamma - \varepsilon)(\| y - z \|_X + \varepsilon)^{-1})) - 2\varepsilon .
\]

Since \( \varepsilon \) is arbitrary and \( r \) is continuous, the lemma follows.

3. Main results. The following lemma is a variant of a result of [5].

**Lemma 4.** Suppose \( Y \) is a dual Banach space, \( K \subseteq Y \) is a convex weak* compact subset, \( T:K \to K \) is a nonexpansive mapping. Then, for every \( x \in K \) there is a closed convex subset \( H(x) \subseteq K \) which is invariant under \( T \) and satisfies

(a) \( \text{diam } H(x) \leq \sup_n \| x - T^n x \|_Y \)

(b) \( \sup_{y \in H(x)} \| x - y \|_Y \leq 2 \sup_n \| x - T^n x \|_Y \).

**Proof.** For \( x \in K \), set \( d(x) = \sup_n \| x - T^n x \|_Y \) and denote by \( O(x) \) the orbit of \( x \) (i.e., \( O(x) = \{ x, Tx, T^2x, \ldots, T^n x, \ldots \} \)). Set also

\[
A_0 = \text{cl* co } O(x) \quad A_{n+1} = \text{cl* co } T(A_n) , \quad n = 0, 1, 2, \ldots
\]

where \( \text{cl* co} \) denotes the weak* closure of the convex hull. Clearly \( A_n \subseteq K \), \( O(T^{n+1} x) \subseteq T(A_n) \subseteq A_{n+1} \), \( \text{diam } A_n \leq d(x) \). Since \( K \) is weak* compact, \( B_k = \bigcap_{n \geq k} A_n \) is nonvoid for every \( k = 0, 1, 2, \ldots \). Moreover \( B_k \) is closed and convex, \( \text{diam } B_k \leq d(x) \), \( B_k \subseteq B_{k+1} \), \( T(B_k) \subseteq B_{k+1} \).
It follows that $H(x) = \bigcup_{k=0}^{\infty} B_k$ satisfies (a). Property (b) follows from the fact that $H(x)$ contains the set $\bigcap_{n=0}^{\infty} \text{cl}^* O(T^n x)$. It is also clear that $H(x)$ is invariant.

The following theorem is our main result announced in § 1.

**THEOREM.** Suppose $X$ is a Banach space with a Schauder basis satisfying the assumptions of § 1. Let $Y$ denote an isomorphic Banach space with $t < 1 + r(1)$. Then, every convex weak* compact subset $K$ of $Y$ has the fixed point property with respect to nonexpansive mappings.

**Proof.** Suppose $T: K \rightarrow K$ is a nonexpansive mapping. There is a sequence $(x_n^0) \subseteq K$ such that $\lim_n \|x_n^0 - Tx_n^0\|_Y = 0$. After passing to a subsequence if necessary, we may assume that $x_n^0$ is weak* convergent to a point $z^0 \in K$, and that $\alpha_0 = \lim_n \|x_n^0 - z^0\|_Y$ exists. By nonexpansiveness, for every positive integer $k$ we have $\|z^0 - T^k z^0\|_Y \leq \limsup_n \|x_n^0 - T^k z^0\|_Y \leq \alpha_0$. Thus $d(z^0) \leq \alpha_0$. By Lemma 4 there is a closed convex invariant subset $H(z^0) \subseteq K$ such that $\text{diam } H(z^0) \leq \alpha_0$. Then there exists a sequence $(x_n^1)$ contained in $H(z^0)$ such that $\|x_n^1 - z^0\|_Y \leq 2\alpha_0$ and also $\gamma_1 = \lim_n \|x_n^1 - z^1\|_X$ exists. We then have (recall the notation introduced in § 1) for every $m$

$$\alpha_0 \geq \limsup_n \|x_n^1 - x_n^0\|_Y \geq k^{-1} \limsup_n \|x_n^1 - x_n^0\|_X$$

$$\geq k^{-1} \|x_m^1 - z^1\|_X (1 + r(\gamma_1 \|x_m^1 - z^1\|^{-1}))$$

$$\geq k^{-1} h \|x_m^0 - z^1\|_Y (1 + r(\gamma_1 \|x_m^0 - z^1\|^{-1}))$$

by Lemma 3. Letting $m$ tend to infinity we get

$$\alpha_0 \geq \limsup_m (\limsup_n \|x_m^1 - x_n^0\|_Y)$$

$$\geq t^{-1} \alpha_0 \alpha_1 (1 + r(1))$$

that is,

$$\alpha_1 \leq t(1 + r(1))^{-1} \alpha_0.$$

Moreover, since $z^1$ belongs to the weak* closure $H(z^0)$, Lemma 4, (b) implies $\|z^0 - z^1\|_Y \leq 2\alpha_0$.

Carrying on this process we produce a sequence of nonnegative numbers $\alpha_n$ such that $\alpha_{n+1} \leq t(1 + r(1))^{-1} \alpha_n \leq t(1 + r(1))^{-1} \alpha_{n+1} \alpha_0$, and a sequence of points $z^n \in K$ such that $\|z^{n+1} - z^n\|_Y \leq 2\alpha_n$, $\|z^n - Tz^n\|_Y \leq \alpha_n$. Hence $z^n$ is strongly convergent to a fixed point of $T$.

If $X = l^p$, it is easy to see that $1 + r(1) = 2^{1/p}$. Therefore we have the following remarkable corollary.
Corollary. Suppose $Y$ is isomorphic to $l^1$ with $t < 2$. Then every weak* compact convex subset of $Y$ has the fixed point property with respect to nonexpansive mappings.

This corollary generalizes a result of Karlovitz ([3, Corollary]). In [4] an example was given of a space isomorphic to $l^1$ with $t = 2$, whose unit ball has not the fixed property with respect to nonexpansive mappings. Hence our corollary is the best result possible.

4. Concluding remarks and comparison with previous results. If $X$ is reflexive, then the above theorem can be proved in a much simpler way. This case however is not new, because it is easily seen that, under our assumption on $Y$, every convex weakly compact subset of $Y$ has normal structure. If $X$ is not reflexive, we were not able to decide whether every weak* compact convex subset of $Y$ has normal structure (of course when $t < 1 + r(1)$). Recall that a weak* closed convex subset $C \subseteq Y$ has normal structure if every weak* compact convex subset $K \subseteq C$ (containing more than one point) has a nondiametral point (see ([4])). A sufficient condition for $C$ to admit normal structure was also given in [4]. The condition is as follows.

Suppose there exists a function $\delta: (R^+)^2 \rightarrow R^+$ such that

(i) for each fixed $s$, $\delta(r, s)$ is continuous and strictly increasing
(ii) $\delta(s, s) > s$ for all $s$
(iii) if $x_n$ tends to 0 weak* and $\|x_n\|_Y$ tends to $s$, then, for all $y \in K$, $\|y - x_n\|_Y$ tends to $\delta(\|y\|_Y, s)$.

It is easy to see that this condition is not satisfied in the space $Y$ obtained by renorming $l^1$ with the norm $\|y\|_Y = \max (\|y\|_1, \ t^{-1}\|y\|_1)$, where $1 < t < 2$. Indeed, if $(b_n)$ is the natural basis of $l^1$, take $y = b_1$. Assume that the condition of [4] is satisfied, say, for the unit ball of $Y$. We have $\|y\|_Y = 1$. Set $x_n = (t - 1)b_n$. Then $\|x_n\|_Y = t - 1$, $\|y - x_n\|_Y = 1$, so that, by (iii), $\delta(1, (t - 1)) = 1$. On the other hand, if we choose $z = b_1 + (t - 1)b_2$, we have $\|z\|_Y = 1$ and $\|z - x_n\|_Y = t^{-1}\|z - x_n\|_1 = t^{-1}(2t - 1)$. Hence, by (iii) we should have $\delta(1, t - 1) = 2 - t^{-1}$, a contradiction.

Analogous arguments show also that the relation $\perp$ is not approximately uniformly symmetric in $Y$ (in the sense of [3]) and our result cannot be deduced from [3].

For other examples concerning spaces $X$ satisfying our assumptions, we refer to [2] and [6].

References


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**Università di Milano**

"Federigo Enriques"

**Via C Saldini, 50**

20133 Milano, Italy
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