

# Pacific Journal of Mathematics

**SCHAUDER BASES AND FIXED POINTS OF NONEXPANSIVE  
MAPPINGS**

PAOLO M. SOARDI

## SCHAUDER BASES AND FIXED POINTS OF NONEXPANSIVE MAPPINGS

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**A fixed point theorem is proved for nonexpansive mappings in Banach spaces which are isomorphic to spaces with certain boundedly complete bases.**

**1. Introduction.** Suppose  $X$  and  $Y$  are isomorphic Banach spaces with  $h\|\cdot\|_Y \leq \|\cdot\|_X \leq k\|\cdot\|_Y$ , where  $\|\cdot\|_Y$  and  $\|\cdot\|_X$  denote the norms in  $Y$  and  $X$  respectively. Let  $t = kh^{-1}$  (this notation will be kept fixed throughout the paper). Suppose also that every convex weakly compact (weak\* compact, when  $X$  is a dual Banach space) subset  $K$  of  $X$  has the fixed point property with respect to nonexpansive mappings (i.e., mappings  $T: K \rightarrow K$  such that  $\|Tx - Ty\|_X \leq \|x - y\|_X$ , for all  $x, y \in K$ ). It is not known in general whether, assuming  $t$  sufficiently close to 1, convex weakly compact (weak\* compact) subsets of  $Y$  have the same property (but see Bynum [1]).

In this paper we answer in the affirmative this question when  $X$  has a Schauder basis  $(b_n)$  which satisfies a condition introduced by Gossez and Lami Dozo [2]. For every positive integer  $k$  and  $x \in X$  set  $U_k(x) = \sum_{n=1}^k f_n(x)b_n$ , where  $(f_n)$  denotes the associated system of linear functionals. We shall always assume that there exists a strictly increasing sequence  $(k_n)$  with the following property:

for every  $c > 0$  there is  $\rho > 0$  such that whenever  $x \in X$  and  $n$  satisfy

$$\begin{aligned} \|U_{k_n}(x)\|_X &= 1 \\ \|x - U_{k_n}(x)\|_X &\geq c \end{aligned}$$

then  $\|x\|_X \geq 1 + \rho$ .

It is easy to see (Lemma 1 below) that the above condition implies that the basis  $(b_n)$  is boundedly complete, so that  $X$  is a dual Banach space.

In the next sections it is proved that there exists  $t_0 > 1$  such that for  $t < t_0$  every weak\* compact convex subset of  $Y$  has the fixed point property with respect to nonexpansive mappings. For  $t = 1$  this follows easily from the results of Karlovitz [3], while for  $t > 1$  it can not be deduced from [3]. As a remarkable consequence we obtain that, in every Banach space  $Y$  isomorphic to  $l^1$  with  $t < 2$ , weak\* compact convex subsets have the fixed point property with respect to nonexpansive mappings.

## 2. Properties of the space $X$ .

LEMMA 1. *Suppose  $X$  is a Banach space with a Schauder basis  $(b_n)$  satisfying the assumptions of the above section. Then the basis  $(b_n)$  is boundedly complete and  $X$  is isomorphic to the dual of the Banach space generated by the system of the linear functionals  $(f_n)$ .*

*Proof.* Suppose that  $(a_n)$  is a sequence of scalars such that  $\sup_N \|\sum_{n=1}^N a_n b_n\|_X < \infty$ . Then, the same argument as in [6, p. 290-291] implies that, for some subsequence  $k_{n_j}$ ,  $\sum_{n=1}^{k_{n_j}} a_n b_n$  converges to a point  $x \in X$ . Then, of course,  $f_n(x) = a_n$  for every  $n$ , so that  $\sum_{n=1}^{\infty} a_n b_n = x$ . The second assertion is proved in [6, Th. II 6.2, 3].

For every positive integer  $n$  and every real  $c > 0$  we set  $r_n(c) = \inf \|x\|_X - 1$ , where the infimum is taken over all  $x \in X$  such that  $\|U_{k_n}(x)\|_X = 1$ ,  $\|x - U_{k_n}(x)\|_X \geq c$ . We set also  $r(c) = \inf_n r_n(c)$ . Clearly  $r(c) > 0$  for all positive  $c$ . We complete the definition of  $r(c)$  by letting  $r(0) = 0$ . In the following we set  $V_{k_n}(x) = x - U_{k_n}(x)$ .

LEMMA 2.  *$r(c)$  is a nondecreasing continuous function of  $c$ .*

*Proof.* Let  $\varepsilon > 0$  be arbitrarily small and  $c_2 > c_1 \geq 0$ . There exist  $n > 0$  and  $x \in X$  such that  $\|U_{k_n}(x)\|_X = 1$ ,  $\|V_{k_n}(x)\|_X \geq c_2$  and  $1 + r(c_2) + \varepsilon > \|x\|_X \geq 1 + r(c_1)$ . Hence  $r(c_2) \geq r(c_1) \geq 0 = r(0)$  and  $r(c)$  is nondecreasing.

Observe now that there exist a sequence of points  $x_j \in X$  and a sequence of positive integers  $n_j$  such that

$$\|U_{k_{n_j}}(x_j)\|_X = 1, \quad \|V_{k_{n_j}}(x_j)\|_X \geq c_1 \quad \text{and} \quad 1 + r(c_1) + j^{-1} > \|x_j\|_X.$$

We set  $v_j = \|V_{k_{n_j}}(x_j)\|_X$ . After extracting a subsequence if necessary, we may suppose that  $v = \lim_j v_j$  exists. If  $v > c_2$ , then, for large values of  $j$ ,  $1 + r(c_1) + j^{-1} > \|x_j\|_X \geq 1 + r(c_2)$ , so that, by what has been already proved,  $r(c_1) = r(c_2)$ , and we are done. Thus we may assume  $c_1 \leq v \leq c_2$ . Let  $y_j = x_j + s_j V_{k_{n_j}}(x_j)$ , where  $s_j$  is a scalar such that  $(1 + s_j)v_j = c_2$ . Clearly we must have  $\|y_j\|_X \geq 1 + r(c_2)$  and  $\|x_j - y_j\|_X = |s_j|v_j$ . Hence

$$\begin{aligned} 1 + r(c_1) + j^{-1} > \|x_j\|_X &\geq \|y_j\|_X - |s_j|v_j \\ &\geq 1 + r(c_2) - |s_j|v_j \end{aligned}$$

that is,

$$r(c_2) - r(c_1) \leq |s_j|v_j + j^{-1}.$$

Now, if  $v < c_2$ , then  $|s_j| = s_j = (c_2 - v_j)v_j^{-1} \leq (c_2 - c_1)v_j^{-1}$  for  $j$  large enough. If  $v = c_2$  then  $s_j$  tends to 0, so that, if  $j$  is large,  $|s_j| <$

$(c_2 - c_1)v_j^{-1}$ . In any case, for large values of  $j$ , we obtain  $r(c_2) - r(c_1) \leq (c_2 - c_1) + j^{-1}$ , and the proof is ended.

**LEMMA 3.** *Suppose that  $(x_n) \subseteq X$  is a sequence of points converging in the weak\* topology to a point  $z \in X$ . Let  $\gamma = \limsup_n \|x_n - z\|_X$ . Then, for every  $y \in X, y \neq z$*

$$\limsup_n \|x_n - y\|_X \geq \{1 + r(\gamma\|y - z\|_X^{-1})\}\|y - z\|_X.$$

*Proof.* Let  $\varepsilon > 0$  be arbitrarily small. There exists  $j = j(\varepsilon)$  such that  $\|V_{k_j}(y - z)\|_X < \varepsilon$ . Since  $x_n - z$  converges weak\* to 0 and the associated functionals  $f_n$  are weak\* continuous (Lemma 1), for every fixed  $j$  we can find  $n_0$  such that  $\|U_{k_j}(x_n - z)\|_X < \varepsilon$  for  $n$  greater than  $n_0$ . Therefore, for  $n > n_0$ , we have by Lemma 2

$$\begin{aligned} & \|y - x_n\|_X \\ & \geq -2\varepsilon + \|U_{k_j}(y - z) + V_{k_j}(z - x_n)\|_X \\ & \geq -2\varepsilon + \|U_{k_j}(y - z)\|_X\{1 + r(\|V_{k_j}(z - x_n)\|_X \cdot \|U_{k_j}(y - z)\|_X^{-1})\} \\ & \geq -2\varepsilon + (\|y - z\|_X - \varepsilon)\{1 + r((\|z - x_n\|_X - \varepsilon)(\|y - z\|_X + \varepsilon)^{-1})\}. \end{aligned}$$

By Lemma 2 again

$$\begin{aligned} & \limsup_n \|y - x_n\|_X \\ & \geq (\|y - z\|_X - \varepsilon)\{1 + r((\gamma - \varepsilon)(\|y - z\|_X + \varepsilon)^{-1})\} - 2\varepsilon. \end{aligned}$$

Since  $\varepsilon$  is arbitrary and  $r$  is continuous, the lemma follows.

**3. Main results.** The following lemma is a variant of a result of [5].

**LEMMA 4.** *Suppose  $Y$  is a dual Banach space,  $K \subseteq Y$  is a convex weak\* compact subset,  $T: K \rightarrow K$  is a nonexpansive mapping. Then, for every  $x \in K$  there is a closed convex subset  $H(x) \subseteq K$  which is invariant under  $T$  and satisfies*

- (a)  $\text{diam } H(x) \leq \sup_n \|x - T^n x\|_Y$
- (b)  $\sup_{y \in H(x)} \|x - y\|_Y \leq 2 \sup_n \|x - T^n x\|_Y$ .

*Proof.* For  $x \in K$ , set  $d(x) = \sup_n \|x - T^n x\|_Y$  and denote by  $O(x)$  the orbit of  $x$  (i.e.,  $O(x) = \{x, Tx, T^2x, \dots, T^n x, \dots\}$ ). Set also

$$A_0 = \text{cl}^* \text{co } O(x) \quad A_{n+1} = \text{cl}^* \text{co } T(A_n), \quad n = 0, 1, 2, \dots$$

where  $\text{cl}^* \text{co}$  denotes the weak\* closure of the convex hull. Clearly  $A_n \subseteq K, O(T^{n+1}x) \subseteq T(A_n) \subseteq A_{n+1}, \text{diam } A_n \leq d(x)$ . Since  $K$  is weak\* compact,  $B_k = \bigcap_{n \geq k} A_n$  is nonvoid for every  $k = 0, 1, 2, \dots$ . Moreover  $B_k$  is closed and convex,  $\text{diam } B_k \leq d(x), B_k \subseteq B_{k+1}, T(B_k) \subseteq B_{k+1}$ .

It follows that  $H(x) = \overline{\bigcup_{k=0}^{\infty} B_k}$  satisfies (a). Property (b) follows from the fact that  $H(x)$  contains the set  $\bigcap_{n=0}^{\infty} \text{cl}^* O(T^n x)$ . It is also clear that  $H(x)$  is invariant.

The following theorem is our main result announced in § 1.

**THEOREM.** *Suppose  $X$  is a Banach space with a Schauder basis satisfying the assumptions of § 1. Let  $Y$  denote an isomorphic Banach space with  $t < 1 + r(1)$ . Then, every convex weak\* compact subset  $K$  of  $Y$  has the fixed point property with respect to nonexpansive mappings.*

*Proof.* Suppose  $T: K \rightarrow K$  is a nonexpansive mapping. There is a sequence  $(x_n^0) \subseteq K$  such that  $\lim_n \|x_n^0 - Tx_n^0\|_Y = 0$ . After passing to a subsequence if necessary, we may assume that  $x_n^0$  is weak\* convergent to a point  $z^0 \in K$ , and that  $\alpha_0 = \lim_n \|x_n^0 - z^0\|_Y$  exists. By nonexpansiveness, for every positive integer  $k$  we have  $\|z^0 - T^k z^0\|_Y \leq \limsup_n \|x_n^0 - T^k z^0\|_Y \leq \alpha_0$ . Thus  $d(z^0) \leq \alpha_0$ . By Lemma 4 there is a closed convex invariant subset  $H(z^0) \subseteq K$  such that  $\text{diam } H(z^0) \leq \alpha_0$ . Then there exists a sequence  $(x_n^1)$  contained in  $H(z^0)$  such that  $\|x_n^1 - Tx_n^1\|_Y$  tends to 0,  $x_n^1$  converges weak\* to  $z^1 \in K$ ,  $\alpha_1 = \lim_n \|x_n^1 - z^1\|_Y$  exists and also  $\gamma_1 = \lim_n \|x_n^1 - z^1\|_X$  exists. We then have (recall the notation introduced in § 1) for every  $m$

$$\begin{aligned} \alpha_0 &\geq \limsup_n \|x_m^1 - x_n^1\|_Y \geq k^{-1} \limsup_n \|x_m^1 - x_n^1\|_X \\ &\geq k^{-1} \|x_m^1 - z^1\|_X \{1 + r(\gamma_1 \|x_m^1 - z^1\|_{\bar{X}}^{-1})\} \\ &\geq k^{-1} h \|x_m^1 - z^1\|_Y \{1 + r(\gamma_1 \|x_m^1 - z^1\|_{\bar{X}}^{-1})\} \end{aligned}$$

by Lemma 3. Letting  $m$  tend to infinity we get

$$\begin{aligned} \alpha_0 &\geq \limsup_m (\limsup_n \|x_m^1 - x_n^1\|_Y) \\ &\geq t^{-1} \alpha_1 (1 + r(1)) \end{aligned}$$

that is,

$$\alpha_1 \leq t(1 + r(1))^{-1} \alpha_0.$$

Moreover, since  $z^1$  belongs to the weak\* closure  $H(z^0)$ , Lemma 4, (b) implies  $\|z^0 - z^1\|_Y \leq 2\alpha_0$ .

Carrying on this process we produce a sequence of nonnegative numbers  $\alpha_n$  such that  $\alpha_{n+1} \leq t(1 + r(1))^{-1} \alpha_n \leq \{t(1 + r(1))^{-1}\}^{n+1} \alpha_0$ , and a sequence of points  $z^n \in K$  such that  $\|z^{n+1} - z^n\|_Y \leq 2\alpha_n$ ,  $\|z^n - Tz^n\|_Y \leq \alpha_n$ . Hence  $z^n$  is strongly convergent to a fixed point of  $T$ .

If  $X = l^p$ , it is easy to see that  $1 + r(1) = 2^{1/p}$ . Therefore we have the following remarkable corollary.

**COROLLARY.** *Suppose  $Y$  is isomorphic to  $l^t$  with  $t < 2$ . Then every weak\* compact convex subset of  $Y$  has the fixed point property with respect to nonexpansive mappings.*

This corollary generalizes a result of Karlovitz ([3, Corollary]). In [4] an example was given of a space isomorphic to  $l^t$  with  $t = 2$ , whose unit ball has not the fixed property with respect to nonexpansive mappings. Hence our corollary is the best result possible.

**4. Concluding remarks and comparison with previous results.** If  $X$  is reflexive, then the above theorem can be proved in a much simpler way. This case however is not new, because it is easily seen that, under our assumption on  $Y$ , every convex weakly compact subset of  $Y$  has normal structure. If  $X$  is not reflexive, we were not able to decide whether every weak\* compact convex subset of  $Y$  has normal structure (of course when  $t < 1 + r(1)$ ). Recall that a weak\* closed convex subset  $C \subseteq Y$  has normal structure if every weak\* compact convex subset  $K \subseteq C$  (containing more than one point) has a nondiametral point (see ([4])). A sufficient condition for  $C$  to admit normal structure was also given in [4]. The condition is as follows.

Suppose there exists a functions  $\delta: (R^+)^2 \rightarrow R^+$  such that

- (i) for each fixed  $s$ ,  $\delta(r, s)$  is continuous and strictly increasing
- (ii)  $\delta(s, s) > s$  for all  $s$
- (iii) if  $x_n$  tends to 0 weak\* and  $\|x_n\|_Y$  tends to  $s$ , then, for all  $y \in K$ ,  $\|y - x_n\|_Y$  tends to  $\delta(\|y\|_Y, s)$ .

It is easy to see that this condition is not satisfied in the space  $Y$  obtained by renorming  $l^t$  with the norm  $\|y\|_Y = \max(\|y\|_{l^\infty}, t^{-1}\|y\|_{l^t})$ , where  $1 < t < 2$ . Indeed, if  $(b_n)$  is the natural basis of  $l^t$ , take  $y = b_1$ . Assume that the condition of [4] is satisfied, say, for the unit ball of  $Y$ . We have  $\|y\|_Y = 1$ . Set  $x_n = (t - 1)b_n$ . Then  $\|x_n\|_Y = t - 1$ ,  $\|y - x_n\|_Y = 1$ , so that, by (iii),  $\delta(1, (t - 1)) = 1$ . On the other hand, if we choose  $z = b_1 + (t - 1)b_2$ , we have  $\|z\|_Y = 1$  and  $\|z - x_n\|_Y = t^{-1}\|z - x_n\|_{l^t} = t^{-1}(2t - 1)$ . Hence, by (iii) we should have  $\delta(1, t - 1) = 2 - t^{-1}$ , a contradiction.

Analogous arguments show also that the relation  $\perp$  is not approximately uniformly symmetric in  $Y$  (in the sense of [3]) and our result cannot be deduced from [3].

For other examples concerning spaces  $X$  satisfying our assumptions, we refer to [2] and [6].

REFERENCES

1. W. L. Bynum, *Normal structure coefficients for Banach spaces*, Pacific J. Math., **86** (1980), 427-436.

2. P. Gossez and E. Lami Dozo, *Structure normale et base de Scauder*, Bull. Cl. Sc. Ac. R. Belgique, **55** (1969), 673-681.
3. L. A. Karlovitz, *On nonexpansive mappings*, Proc. Amer. Math. Soc., **55** (1969), 321-325.
4. T. C. Lim, *Asymptotic centers and nonexpansive mappings in some conjugate spaces*, preprint.
5. D. Roux and Paolo M. Suardi, *Alcune generalizzazioni del teorema di Browder-Göhde-Kirk*, Atti Accad. Naz. Lincei, Rend. Cl. Sc. Mat. Fis. Nat., **52** (1972), 424-430.
6. I. Singer, *Bases in Banach Spaces I*, Springer Verlag, Berlin, Heidelberg, New York, 1970.

Received August 17, 1980.

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