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In terms of products of k-spaces, we consider spaces having the weak topology with respect to a point-countable covering consisting of compact subsets.

Let $\mathscr{A}$ be a covering (not necessarily closed or open) of a space X. Then X is said to have the *weak topology* with respect to $\mathscr{A}$, if $F \subset X$ is closed in X whenever $F \cap A$ is closed in A for each $A \in \mathscr{A}$. Of course we can replace "closed" by "open". If each element of $\mathscr{A}$ is compact, then such a covering is called a *k-system* according to A. V. Arhangel'skii [1]. Recall that a space X is a *k-space* (resp. *sequential space*), if it has the weak topology with respect to the cover consisting of all compact (resp. all compact metric) subsets of X. Then a space with a k-system (resp. k-system consisting of metric subspaces) is precisely a k-space (resp. sequential space).

As a case where cartesian products are k-spaces, E. Michael [8] considered the concept of k_ω-spaces. He pointed out that every product of two k_ω-spaces is a k_ω-space and this is implicit in a result of J. Milnor [10; Lemma 2.1]. A space X is a *k_ω-space* (K. Morita [11] calls it a space of class $\mathfrak{S}'$), if X has a countable k-system.

In this paper, as a generalization of k_ω-spaces, we shall investigate spaces with a point-countable k-system in terms of products of k-spaces. We assume that *all spaces are regular* T_1.

Let us begin with spaces with a star-countable k-system. The following gives a characterization of paracompact, locally k_ω-spaces. These spaces will play an important role in connection with the study of products of k-spaces.

THEOREM 1. *The following are equivalent.*
(a) *X has a star-countable k-system.*
(b) *X has a σ-locally finite k-system.*
(c) *X is a paracompact, locally k_ω-space.*
(d) *X^2 is a paracompact space with a star-countable k-system.*

Proof. (b) $\rightarrow$ (a) and (d) $\rightarrow$ (a) are obvious.

(c) $\rightarrow$ (b). Since X is paracompact, locally k_ω, it has a locally finite closed covering $\mathscr{F} = \{F_\gamma; \gamma \in \Gamma\}$ consisting of k_ω-subspaces. For $\gamma \in \Gamma$, let $\{C_{\gamma i}; i \in N\}$ be a countable k-system of F_γ. Let $\mathscr{C}_i =$

$\{C_{\gamma i};\ \gamma \in \Gamma\}$ and $\mathscr{C} = \bigcup_{i=1}^{\infty} \mathscr{C}_i$. Then it is easy to show that $\mathscr{C}$ is a σ-locally finite k-system of X.

(a) $\rightarrow$ (c) and (d). Let $\mathscr{C} = \{C_\gamma;\ \gamma \in \Gamma\}$ be a star-countable k-system of X. For $\gamma, \gamma' \in \Gamma$, difine $\gamma \sim \gamma'$ by $St^n(C_{\gamma'}, \mathscr{C})$ contains C_γ for some $n \in N$. Then, by this equivalence relation $\sim$, the set Γ can be decomposed as $\bigcup_{\alpha \in A} \Gamma_\alpha$ for example. For $\alpha \in A$, let $X_\alpha = \cup \{C_\gamma;\ \gamma \in \Gamma_\alpha\}$. Then for each $C \in \mathscr{C}$, $X_\alpha \cap C$ is empty or C. Thus, since $\mathscr{C}$ is a k-system of X, each X_α is clopen. Also each X_α has a countable k-system $\{C_\gamma;\ \gamma \in \Gamma_\alpha\}$, hence X_α is k_ω. Thus X is the topological sum of k_ω-spaces X_α. Hence (c) and (d) follow from [8; (7.5)].

From the previous theorem, we have a generalization of [8; (7.5)].

PROPOSITION 2. *If X has a star-countable k-system, then X^2 has a k-system, hence X is a k-space.*

In view of the previous proposition, it is desirable to consider a more general case of point-countable k-systems. However, by the following example, we can not replace "star-countable" by "point-countable" or "point-finite".

EXAMPLE 3. A paracompact space X with a point-finite k-system consisting of metric subspaces, but X^2 does not have any k-system.

Proof. Let I be the closed unit interval, and X be I^2, and define basic neighborhoods $V_\varepsilon(p)$, $\varepsilon > 0$, in X as follows:

For $p = (x, y)$, $x > 0$, $V_\varepsilon(p) = (x - \varepsilon, x + \varepsilon) \times y$, and for $p = (0, y)$, $V_\varepsilon(p) = \{0 \times (y - \varepsilon, y + \varepsilon)\} \cup \{[0, \beta_\alpha) \times \alpha;\ |\alpha - y| < \varepsilon\}$.

Then $\{0 \times I, I \times \alpha;\ \alpha \in I\}$ is a point-finite k-system consisting of metric subspaces. Let Y be the quotient space obtained by identifying all points of $0 \times I$, and let $f: X \rightarrow Y$ be the obvious map. Then Y contains closed copies of spaces S_γ, $\gamma \leqq 2^\omega$, obtained from the topological sum of γ convergent sequences by identifying all the limit points. Since f is perfect, Y^2 is the perfect image of X^2. Every perfect image of a space with a k-system has a k-system, so it is sufficient to show that Y^2 does not have any k-system. But Y^2 contains a closed copy of $S^* = S_\omega \times S_{2^\omega}$. In view of [15; Corollary 2.4], S^* does not have any k-system, so that neither does Y^2.

From the proof of the example, we also have the following.

REMARK. (i) Not every product of a space having a countable k-system and a space having a point-finite k-system has a k-system.

(ii) Not every perfect image of a space having a point-finite

k-system has a point-countable k-system (remark that S_{2^ω} does not have a point-countable k-system by the later Proposition 8).

Now, Example 3 raises the following question (*): Under what conditions, does X^2 have a k-system if, or only if X has a point-countable k-system?

To consider this question, let us begin with some preliminalies. For $x \in X$, let $(A_n) \downarrow x$ mean a decreasing sequence $\{A_n; n \in N\}$ such that $\overline{A_n - \{x\}} \ni x$ for $n \in N$. A *k-sequence* due to E. Michael [9] is a decreasing sequence $\{A_n; n \in N\}$ such that $C = \bigcap_{n=1}^{\infty} A_n$ is compact and each neighborhood of C contains some A_n.

The follwing lemma is due to [14; Theorem 4.2]. Recall that a space X has *countable tightness*, $t(X) \leqq \omega$, if $x \in \bar{A}$ in X, then $x \in \bar{C}$ for some countable $C \subset A$. It is well known that every sequential space and every hereditarily separable space has countable tightness.

LEMMA 4. *Suppose that $X \times Y$ has a k-system with $t(X) \leqq \omega$. Then the following condition* (C_1) *or* (C_2) *holds.*

(C_1). *If $(A_n) \downarrow x$ in X, then there exists a nonclosed subset $\{a_n; n \in N\}$ of X with $a_n \in A_n$.*

(C_2). *If (A_n) is a k-sequence in Y, then some $\bar{A}_n$ is countably compact.*

According to E. Michael [9], a space X is *bi-k* (resp. *countably bi-k*), if for each filter base $\mathscr{F}$ accumulating at x (resp. each $(F_n) \downarrow x$), there is a k-sequence (A_n) in X such that $x \in \overline{F \cap A_n}$ for $n \in N$ and $F \in \mathscr{F}$ (resp. $x \in \overline{F_n \cap A_n}$ for $n \in N$), and every *bi-k*-space (resp. countably *bi-k*-space) is characterized as being precisely the *bi*-quotient image (resp. countably *bi*-quotient image) of a paracompact M-space. Every locally compact space and every first countable space is *bi-k*, and every *bi-k*-space is countably *bi-k*.

The lemma will be used later, but we also have the following application.

PROPOSITION 5. *Suppose that $t(X) \leqq \omega$ and X has a point-countable k-system, and that Y is a paracompact bi-k-space. Then $X \times Y$ has a k-system if and only if X or Y is locally compact.*

Proof. The "if" part follows from the following well known result due to D. E. Cohen: Every product of a locally compact space and a k-space is a k-space ([3; Theorem 4.4, p. 263]).

"Only if". Suppose that Y is not locally compact, hence not locally countably compact. Thus there exists $y \in Y$ such that no neighborhood of y has a compact closure. Let $\mathscr{F} = \{X - K; K$ is closed, countably compact in $X\}$. Then $\mathscr{F}$ is a filter base accumulating at y. Since Y is *bi-k*, there is a k-sequence (A_n) with each A_n closed and $y \in \overline{A_n \cap F}$, hence $A_n \cap F \neq \emptyset$ for $n \in N$ and $F \in \mathscr{F}$. This shows that no A_n is countably compact. Thus, by Lemma 4, X satisfies (C_1).

Now, let $\mathscr{C}$ be a point-countable k-system of X. Let X_0 be the topological sum of $\mathscr{C}$, and let $f: X_0 \to X$ be the obvious map. Then f is a quotient map such that $f^{-1}(E)$ is Lindelöf for every countable subset E of X, for every $\overline{f^{-1}(x)}$ is countable and X_0 is paracompact. Moreover $t(X) \leqq \omega$ and X satisfies (C_1). Thus by [9; Theorem 9.5] for $x \in X$ and an open covering $\{C_\alpha \in \mathscr{C}; x \in C_\alpha\}$ of $f^{-1}(x)$, finitely many $f(C_\alpha)$ cover a neighborhood of x. This implies X is locally compact.

The following lemma will be useful.

LEMMA 6. *Let X be a space with a point-countable k-system $\mathscr{C}$. Then for each k-sequence (A_n) in X, some A_n is contained in a finite union of element of $\mathscr{C}$.*

Proof. Suppose that no A_n is contained in any finite union of elements of $\mathscr{C}$. For $x \in X$, let $\{C \in \mathscr{C}; x \in C\} = \{C_n(x); n \in N\}$. Beginning with any point $x \in X$, there exists $x_1 \in A_1 - C_1(x)$. By induction there exists an infinite subset $D = \{x_n; n \in N\}$ of X with $x_n \in A_n - \bigcup_{i,j \leqq n} C_i(x_j)$. Then for each $C \in \mathscr{C}$, $C \cap D$ is at most finite. Thus D is a discrete closed subset of X. However, since $x_n \in A_n$, D has an accumulation point in X. This is a contradiction. Thus some A_n is contained in a finite union of elements of $\mathscr{C}$.

The previous lemma will be used later, but let us now apply the lemma to two propositions below. Recall that a space X is a *k′-space* (resp. *Fréchet space*) if, whenever $x \in \bar{A}$, then there exists a compact subset C of X (resp. a sequence $\{a_n; n \in N\}$ in A) with $x \in \overline{A \cap C}$ (resp. $a_n \to x$).

PROPOSITION 7. *Let X have a point-countable k-system $\mathscr{C}$.*

(i) *If X is countably compact, then it is compact.*

(ii) *If X is countably bi-k, then it is locally compact.*

(iii) *If X is a k′-space (resp. separable k′-space), then it is locally Lindelöf (resp. Lindelöf).*

Proof. (i) follows from the proof of Lemma 6.

(ii) Suppose that for some $x \in X$, $x \notin \operatorname{int} \cup \mathscr{C}'$ for any finite subcollection $\mathscr{C}'$ of $\mathscr{C}$. Let $\{C \in \mathscr{C}; x \in C\} = \{C_i; i \in N\}$, and $F_n = X - \bigcup_{i=1}^{n} C_i$. Then $(F_n) \downarrow x$. Thus there is a k-sequence (A_n) with $x \in \overline{A_n \cap F_n}$. By Lemma 6 some A_{n_0} is contained in a union of finitely many elements C^* of $\mathscr{C}$. Let $G = X - \{C^*; x \notin C^*\}$. Then G is a neighborhood of x which is disjoint from some $A_{n_1} \cap F_{n_1}$ with $n_1 \geqq n_0$. But $x \in \overline{G \cap A_{n_1} \cap F_{n_1}}$, a contradiction. Thus each point of X has a neighborhood which is contained in a finite union of elements of $\mathscr{C}$. Hence X is locally compact.

(iii) Since the k' case is proved similarly, we prove the separable k' case. Let $X = \bar{D}$ with D countable, and $x \in X$. Then there is a compact subset K of X with $x \in \overline{K \cap D}$. By Lemma 6, K is contained in a union of finitely many elements of $\mathscr{C}$. Thus $x \in \overline{K \cap D}$ implies $x \in \overline{C \cap D}$ for some $C \in \mathscr{C}$. This shows that $X = \cup \{\overline{C \cap D}; C \in \mathscr{C}\}$. Thus X is σ-compact, hence Lindelöf.

We remark that, in [6], we have a separable space with a point-finite k-system consisting of metric subspaces, but it is not meta-Lindelöf, hence not Lindelöf. Thus the k'-ness of the parenthetic part of (iii) is essential. However, I do not know whether or not every separable k'-space with a point-countable k-system $\mathscr{C}$ has a countable k-system. If each element of $\mathscr{C}$ is metric, then such a space has a countable k-system by the later Corollary 11.

PROPOSITION 8. *Let $f: X \to Y$ be a closed map with X paracompact, countably bi-k. If Y has a point-countable k-system, then every $\partial f^{-1}(y)$ has a countable k-system.*

Proof. Let $\mathscr{C}$ be a point-countable k-system of a space Y. For $y \in Y$, let $\{C \in \mathscr{C}; y \in C\} = \{C_i; i \in N\}$ and $F_n = \bigcup_{i=1}^{n} C_i$ for $n \in N$. For some $x \in f^{-1}(y)$, assume that $(X - f^{-1}(F_n)) \downarrow x$. Since X is countably *bi-k*, there is a k-sequence (A_n) in X such that $\overline{A_n \cap (X - f^{-1}(F_n))} \ni x$ for $n \in N$. Since $(f(A_n))$ is a k-sequence in Y, using Lemma 6, as in the proof of Proposition 7(ii), we have a contradiction to the assumption. Thus each point x of $f^{-1}(y)$ has a neighborhood V_x contained in some $f^{-1}(F_{n_x})$. Let $f_i = f \mid f^{-1}(C_i)$ for $i \in N$. Then $\partial f^{-1}(y) \cap V_x \subset \cup \{\partial f_i^{-1}(y); i = 1, 2, \cdots, n_x\}$. Let $\mathscr{V} = \{\partial f^{-1}(y) \cap V_x; x \in f^{-1}(y)\}$. Then $\mathscr{V}$ is an open covering of $\partial f^{-1}(y)$, hence $\partial f^{-1}(y)$ has the weak topology with respect to $\mathscr{V}$. On the other hand, each element of $\mathscr{V}$ is contained in a finite union of elements of a closed covering $\mathscr{F} = \{\partial f_i^{-1}(y); i \in N\}$ of $\partial f^{-1}(y)$. Hence it is easy to show that $\partial f^{-1}(y)$ has the weak topology with respect to $\mathscr{F}$. But each f_i is a closed map of a paracompact space onto a compact space C_i, so that each $\partial f_i^{-1}(y)$ is compact by [7; Theorem 1.1]. Thus each element of $\mathscr{F}$ is compact. Therefore, each $\partial f^{-1}(y)$ has a countable k-system.

Now, using Lemmas 4 and 6, we shall prove the following theorem related to the question $(*)$ arised after Example 3.

THEOREM 9. *Let* $f: X \to Y$ *be a quotient s-map* (i.e., *every* $f^{-1}(y)$ *is separable*), *and let* X *have a point-countable base. If* Y *is a* k'-*space, then the following are equivalent.*

(a) Y^2 *has a k-system.*

(b) Y *has a point-countable base or a point-countable k-system.*

Proof. (a) $\to$ (b). Y is sequential so that it has countable tightness. Thus by Lemma 4, Y satisfies (C_1) or (C_2). If Y satisfies (C_1), by [9; Theorem 9.8] Y has a point-countable base. So, suppose that Y satisfies (C_2). Here we remark that every closed countably compact subset of Y is compact metric. Indeed, since Y is assumed to be countably compact, Y satisfies (C_1) so that Y has a point-countable base. Thus by [12; Corollary 1.6], Y is compact metric. Now, let $\mathscr{B}$ be a point-countable base of X and assume that $\mathscr{B}$ is closed under finite intersections. For $x \in X$, suppose that $\{V(x, n) \in \mathscr{B}; n \in N\}$ is a decreasing local base at x, hence is a k-sequence. Then by [9; Proposition 1.4], $(f(V(x, n)))$ is a k-sequence in Y, so is $(\overline{f(V(x, n))})$. Thus by (C_2) some $\overline{f(V(x, n_x))}$ is countably compact, hence separable metric. But, X has the weak topology with respect to a point-countable open covering $\mathscr{B}' = \{V(x, n_x); x \in X\}$. Since f is a quotient and s-map, Y has the weak topology with respect to a point-countable covering $f(\mathscr{B}')$ consisting of separable metric subspaces. On the other hand, Y is a Fréchet space, because every compact subset of a k'-space Y is metric. Thus, by [6], Y is the topological sum of spaces with a countable k-network. Hence, to complete the proof, it suffices to show that every k-space Z having a countable k-network and satisfying (C_2) has a countable k-system. Here we shall recall that a covering $\mathscr{F}$ of Z is a countable k-network, if $C \subset U$ with C compact and U open in Z, then there exists a finite subcovering $\mathscr{F}'$ of $\mathscr{F}$ such that $C \subset \cup \mathscr{F}' \subset U$. We can assume that each element of $\mathscr{F}$ is closed and $\mathscr{F}$ is closed under finite unions and intersections. Let K be any compact subset of Z, and $\{F \in \mathscr{F}; F \supset K\} = \{F_i; i \in N\}$. Let $K_n = \bigcap_{i \geq n} F_i$ for $n \in N$. Then each $K_n \in \mathscr{F}$, and (K_n) is a k-sequence with $K = \bigcap_{n=1}^{\infty} K_n$. Thus by (C_2) some K_n is countably compact, hence compact. This shows that $\mathscr{C} = \{F \in \mathscr{F}; F$ is compact in $Z\}$ is still a countable k-network. Since Z is a k-space, $\mathscr{C}$ is obviously a countable k-system of Z. That completes the proof.

(b) $\to$ (a). If Y has a point-countable base, then Y^2 is first countable. Thus Y^2 has a k-system. Suppose that Y has a point-countable k-system $\mathscr{C}$. Since every compact subset of Y is metric, a Fréchet space Y has the weak topology with respect to the point-countable covering $\mathscr{C}$ consisting of separable metric subspaces. On the other hand, Y satisfies (C_2) by Lemma 6. Hence, by the proof of (a) $\to$ (b), Y is the topological sum of k_ω-subspaces. Hence, Y^2 has a k-system by Proposition 2.

As a generalization of closed maps and open maps, we recall that a map $f: X \to Y$ is *pseudo-open* if for any neighborhood U of $f^{-1}(y)$, $y \in \operatorname{int} f(U)$. Every pseudo-open map is quotient. Every pseudo-open image of a metric space is obviously Fréchet. Thus we have the following corollary from Theorem 9 and the fact that every quotient s-image of a locally separable metric space is metrizable if it has a point-countable base [4; Corollary 1].

COROLLARY 10. *Let X be the pseudo-open s-image of a metric space (resp. locally separable, metric space). Then X^2 has a k-system if and only if X has a point-countable base (resp. X is metric) or X has a point-countable k-system.*

COROLLARY 11. *Suppose that X has a point-countable k-system consisting of metric subspaces. If X is a k'-space (resp. separable k'-space), then X is the topological sum of k_ω-subspaces (resp. X is a k_ω-space), hence X^2 is a k-space.*

Proof. Let $\mathscr{C}$ be a point-countable k-system consisting of metric subspaces. Let X_0 be the topological sum of $\mathscr{C}$ and $f: X_0 \to X$ be the obvious map. Then f is a quotient s-map. Thus, since X is Fréchet, X is the topological sum of k_ω-subspaces by the proof of (b) $\to$ (a) of Theorem 9. If X is moreover separable, by Proposition 7(iii), X is Lindelöf. Thus X is a k_ω-space.

The k'-ness of the previous corollary is essential by Example 3. However, we have the following question in connection with whether or not we can omit the metric "pieces".

Question 12. Suppose that X is a k'-space with a point-countable k-system. Then does X^2 have a k-system?

As is well known, every k'-space is precisely the pseudo-open image of a locally compact paracompact space ([2; Chapter III, Theorem 3.3]). As for Question 12, if X is the closed image of a locally compact paracompact space, then the answer is affirmative. More generally we have

THEOREM 13. *Let $f: X \to Y$ be a closed map. If X is a paracompact countably bi-k-space, then* (a), (b) *and* (c) *below are equivalent. Moreover* (a) *implies* (d).

(a) *Y has a point-countable k-system.*

(b) *Y is a paracompact, locally k_ω-space.*

(c) *Y^2 has a point-countable k-system.*

(d) *Y^2 has a paracompact space with a k-system.*

Proof. (b) $\to$ (c) and (b) $\to$ (d) follow from Theorem 1, and (c) $\to$ (a) is clear.

(a) $\to$ (b). The paracompactness of Y follows from the well known results due to E. Michael: Every closed image of a paracompact spaces is paracompact ([3; Theorem 2.4, p. 165]). We prove Y is locally k_ω. Let $y \in Y$. Then every $\partial f^{-1}(y)$ is Lindelöf by Proposition 8, and by the proof there, each point of $f^{-1}(y)$ has a neighborhood contained in the inverse image of some compact subset of Y. Now, since each closed subset of X is countably *bi-k*, as in the proof of [7; Corollary 1.2] we can assume that every $f^{-1}(y)$ is Lindelöf. Hence there exists a neighborhood W of y, open subsets V_n of X, and compact subsets C_n of Y such that $f^{-1}(\bar{W}) \subset \bigcup_{n=1}^{\infty} V_n$, $V_n \subset f^{-1}(C_n)$ for $n \in N$. Let $F = f^{-1}(\bar{W})$ and $\mathscr{V} = \{F \cap V_n; n \in N\}$. Then $\mathscr{V}$ is an open covering of F and $F \cap V_n \subset F \cap f^{-1}(C_n)$ for $n \in N$. Thus F has the weak topology with respect to $\{F \cap f^{-1}(C_n); n \in N\}$. Since $f|F$ is closed, hence quotient, so $f(F) = \bar{W}$ has the weak topology with respect to $\{\bar{W} \cap C_n; n \in N\}$. This shows that Y is a locally k_ω-space.

Concerning the implication (d) $\to$ (a) of the previous theorem, we have

THEOREM 14. (CH). *Let $f: X \to Y$ be a closed map with X paracompact bi-k (resp. paracompact locally compact). Suppose that $t(Y) \leqq \omega$. Then the following are equivalent. When Y is sequential, (CH) can be ommited.*

(a) *Y has a point-countable k-system, or Y is bi-k (resp. Y has a point-countable k-system).*

(b) *Y has a point-countable k-system, or every $\partial f^{-1}(y)$ is compact (resp. every $\partial f^{-1}(y)$ is Lindelöf).*

(c) *Y^2 has a k-system.*

Proof. (a) $\to$ (b). If Y is *bi-k*, then every $\partial f^{-1}(y)$ is compact by [9; Theorem 9.9]. The parenthetic part follows from Proposition 8.

(b) $\rightarrow$ (c). If Y has a point-countable k-system, then Y^2 has a k-system by Theorem 13. If every $\partial f^{-1}(y)$ is compact, we can assume that every $f^{-1}(y)$ is compact. Thus Y^2 is a k-space by [9; Proposition 3.E.4]. If X is locally compact and every $\partial f^{-1}(y)$ is Lindelöf, then Y^2 is a k-space by [15; Lemma 2.5].

(c) $\rightarrow$ (a). Suppose that X is paracompact *bi-k* and $t(Y) \leqq \omega$. Then by [5; Theorem 2.11], Y is paracompact locally k_ω, or *bi-k* under (CH), and if Y is sequential, (CH) can be ommited. Thus we have (a) by Theorem 1. If X is paracompact locally compact and Y is *bi-k*, since every $\partial f^{-1}(y)$ is compact, Y is paracompact locally compact. Hence Y has a point-countable k-system.

From Theorem 14 and Proposition 8, we have

COROLLARY 15. *Let $f: X \rightarrow Y$ be a closed map with X paracompact and first countable. If Y^2 has a k-system, then every $\partial f^{-1}(y)$ has a countable k-system.*

Finally we shall consider the product X^ω of countably many copies of X.

THEOREM 16. (i) *X^ω has a point-countable k-system if and only if X is compact.*

(ii) *Suppose that X has a point-countable k-system and $t(X) \leqq \omega$. Then X^ω has a k-system if and only if X is locally compact.*

Proof. (i) The "if" part is clear.

"Only if". Suppose that X is not compact. Hence X is not countably compact by Proposition 7(i). Then X contains a closed copy of N, hence X^ω contains a closed copy of N^ω. Since a metric space N^ω has a point-countable k-system, N^ω must be locally compact by Proposition 7(ii). This is a contradiction. Hence X is compact.

(ii) "If". Since X is locally compact, X^ω is a k-space by [2; Chaper III, Theorems 3.7 and 3.9].

"Only if". We may assume that X is not compact, hence not countably compact. Then X^ω contains a closed copy of $X \times N^\omega$. Thus by Proposition 5, X is locally compact.

The previous theorem suggests the following question.

Question 17. Suppose that X has a point-countable or countable k-system. Then is X locally compact if X^ω has a k-system?

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