POINT-COUNTABLE $k$-SYSTEMS AND PRODUCTS OF $k$-SPACES

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In terms of products of $k$-spaces, we consider spaces having the weak topology with respect to a point-countable covering consisting of compact subsets.

Let $\mathcal{A}$ be a covering (not necessarily closed or open) of a space $X$. Then $X$ is said to have the weak topology with respect to $\mathcal{A}$, if $F \subset X$ is closed in $X$ whenever $F \cap A$ is closed in $A$ for each $A \in \mathcal{A}$. Of course we can replace "closed" by "open". If each element of $\mathcal{A}$ is compact, then such a covering is called a $k$-system according to A. V. Arhangel'skii [1]. Recall that a space $X$ is a $k$-space (resp. sequential space), if it has the weak topology with respect to the cover consisting of all compact (resp. all compact metric) subsets of $X$. Then a space with a $k$-system (resp. $k$-system consisting of metric subspaces) is precisely a $k$-space (resp. sequential space).

As a case where cartesian products are $k$-spaces, E. Michael [8] considered the concept of $k_\omega$-spaces. He pointed out that every product of two $k_\omega$-spaces is a $k_\omega$-space and this is implicit in a result of J. Milnor [10; Lemma 2.1]. A space $X$ is a $k_\omega$-space (K. Morita [11] calls it a space of class $\mathcal{S}$'), if $X$ has a countable $\lambda$-system.

In this paper, as a generalization of $k_\omega$-spaces, we shall investigate spaces with a point-countable $k$-system in terms of products of $k$-spaces. We assume that all spaces are regular $T_1$.

Let us begin with spaces with a star-countable $k$-system. The following gives a characterization of paracompact, locally $k_\omega$-spaces. These spaces will play an important role in connection with the study of products of $k$-spaces.

**Theorem 1.** The following are equivalent.
(a) $X$ has a star-countable $k$-system.
(b) $X$ has a $\sigma$-locally finite $k$-system.
(c) $X$ is a paracompact, locally $k_\omega$-space.
(d) $X^2$ is a paracompact space with a star-countable $k$-system.

**Proof.** (b) $\rightarrow$ (a) and (d) $\rightarrow$ (a) are obvious.
(c) $\rightarrow$ (b). Since $X$ is paracompact, locally $k_\omega$, it has a locally finite closed covering $\mathcal{F} = \{F_\gamma; \gamma \in \Gamma\}$ consisting of $k_\omega$-subspaces. For $\gamma \in \Gamma$, let $\{C_{i\gamma}; i \in N\}$ be a countable $k$-system of $F_\gamma$. Let $\mathcal{G}_1 =$
\( \{C_i ; \gamma \in \Gamma \} \) and \( \mathcal{E} = \bigcup_{i=1}^{n} C_i \). Then it is easy to show that \( \mathcal{E} \) is a \( \sigma \)-locally finite \( k \)-system of \( X \).

(a) \( \rightarrow \) (c) and (d). Let \( \mathcal{E} = \{C_i ; \gamma \in \Gamma \} \) be a star-countable \( k \)-system of \( X \). For \( \gamma, \gamma' \in \Gamma \), define \( \gamma \sim \gamma' \) by \( St^n(C_\gamma, \mathcal{E}) \) contains \( C_\gamma \) for some \( n \in N \). Then, by this equivalence relation \( \sim \), the set \( \Gamma \) can be decomposed as \( \bigcup_{a \in A} \Gamma_a \) for example. For \( \alpha \in A \), let \( X_\alpha = \bigcup \{C_i ; \gamma \in \Gamma_a \} \). Then for each \( C \in \mathcal{E} \), \( X_\alpha \cap C \) is empty or \( C \). Thus, since \( \mathcal{E} \) is a \( k \)-system of \( X \), each \( X_\alpha \) is clopen. Also each \( X_\alpha \) has a countable \( k \)-system \( \{C_i ; \gamma \in \Gamma_a \} \), hence \( X_\alpha \) is \( k_\omega \). Thus \( X \) is the topological sum of \( k_\omega \)-spaces \( X_\alpha \). Hence (c) and (d) follow from [8; (7.5)].

From the previous theorem, we have a generalization of [8; (7.5)].

**Proposition 2.** If \( X \) has a star-countable \( k \)-system, then \( X^2 \) has a \( k \)-system, hence \( X \) is a \( k \)-space.

In view of the previous proposition, it is desirable to consider a more general case of point-countable \( k \)-systems. However, by the following example, we can not replace "star-countable" by "point-countable" or "point-finite".

**Example 3.** A paracompact space \( X \) with a point-finite \( k \)-system consisting of metric subspaces, but \( X^2 \) does not have any \( k \)-system.

**Proof.** Let \( I \) be the closed unit interval, and \( X \) be \( I^2 \), and define basic neighborhoods \( V_\varepsilon(p), \varepsilon > 0 \), in \( X \) as follows:

For \( p = (x, y), x > 0 \), \( V_\varepsilon(p) = (x - \varepsilon, x + \varepsilon) \times y \), and for \( p = (0, y) \), \( V_\varepsilon(p) = \{0 \times (y - \varepsilon, y + \varepsilon) \} \cup \{0, \beta_\alpha \times \alpha; |\alpha - y| < \varepsilon\} \).

Then \( \{0 \times I, I \times \alpha; \alpha \in I\} \) is a point-finite \( k \)-system consisting of metric subspaces. Let \( Y \) be the quotient space obtained by identifying all points of \( 0 \times I \), and let \( f : X \rightarrow Y \) be the obvious map. Then \( Y \) contains closed copies of spaces \( \mathcal{S}^\gamma, \gamma \leq 2^\gamma \), obtained from the topological sum of \( \gamma \) convergent sequences by identifying all the limit points. Since \( f \) is perfect, \( Y^2 \) is the perfect image of \( X^2 \). Every perfect image of a space with a \( k \)-system has a \( k \)-system, so it is sufficient to show that \( Y^2 \) does not have any \( k \)-system. But \( Y^2 \) contains a closed copy of \( S^* = S_\omega \times S_\omega \). In view of [15; Corollary 2.4], \( S^* \) does not have any \( k \)-system, so that neither does \( Y^2 \).

From the proof of the example, we also have the following.

**Remark.** (i) Not every product of a space having a countable \( k \)-system and a space having a point-finite \( k \)-system has a \( k \)-system.

(ii) Not every perfect image of a space having a point-finite
$k$-system has a point-countable $k$-system (remark that $S_{2\omega}$ does not have a point-countable $k$-system by the later Proposition 8).

Now, Example 3 raises the following question (*): Under what conditions, does $X^2$ have a $k$-system if, or only if $X$ has a point-countable $k$-system?

To consider this question, let us begin with some preliminaries. For $x \in X$, let $(A_n) \downarrow x$ mean a decreasing sequence $\{A_n; n \in N\}$ such that $A_n \supseteq \{x\}$ for $n \in N$. A $k$-sequence due to E. Michael [9] is a decreasing sequence $\{A_n; n \in N\}$ such that $C = \bigcap_{n=1}^{\infty} A_n$ is compact and each neighborhood of $C$ contains some $A_n$.

The following lemma is due to [14; Theorem 4.2]. Recall that a space $X$ has countable tightness, $t(X) \leq \omega$, if $x \in A$ in $X$, then $x \in C$ for some countable $C \subseteq A$. It is well known that every sequential space and every hereditarily separable space has countable tightness.

**Lemma 4.** Suppose that $X \times Y$ has a $k$-system with $t(X) \leq \omega$. Then the following condition (C1) or (C2) holds.

(C1). If $(A_n) \downarrow x$ in $X$, then there exists a nonclosed subset $\{a_n; n \in N\}$ of $X$ with $a_n \in A_n$.

(C2). If $(A_n)$ is a $k$-sequence in $Y$, then some $A_n$ is countably compact.

According to E. Michael [9], a space $X$ is bi-$k$ (resp. countably bi-$k$), if for each filter base $\mathcal{F}$ accumulating at $x$ (resp. each $(F_n) \downarrow x$), there is a $k$-sequence $(A_n)$ in $X$ such that $x \in F \cap A_n$ for $n \in N$ and $F \in \mathcal{F}$ (resp. $x \in F_n \cap A_n$ for $n \in N$), and every bi-$k$-space (resp. countably bi-$k$-space) is characterized as being precisely the bi-quotient image (resp. countably bi-quotient image) of a paracompact $M$-space. Every locally compact space and every first countable space is bi-$k$, and every bi-$k$-space is countably bi-$k$.

The lemma will be used later, but we also have the following application.

**Proposition 5.** Suppose that $t(X) \leq \omega$ and $X$ has a point-countable $k$-system, and that $Y$ is a paracompact bi-$k$-space. Then $X \times Y$ has a $k$-system if and only if $X$ or $Y$ is locally compact.

**Proof.** The "if" part follows from the following well known result due to D. E. Cohen: Every product of a locally compact space and a $k$-space is a $k$-space ([3; Theorem 4.4, p. 263]).
"Only if". Suppose that $Y$ is not locally compact, hence not locally countably compact. Thus there exists $y \in Y$ such that no neighborhood of $y$ has a compact closure. Let $\mathcal{F} = \{X - K; K$ is closed, countably compact in $X\}$. Then $\mathcal{F}$ is a filter base accumulating at $y$. Since $Y$ is bi-k, there is a $k$-sequence $(A_n)$ with each $A_n$ closed and $y \in A_n \cap F$, hence $A_n \cap F \neq \emptyset$ for $n \in N$ and $F \in \mathcal{F}$. This shows that no $A_n$ is countably compact. Thus, by Lemma 4, $X$ satisfies (C).

Now, let $\mathcal{C}$ be a point-countable k-system of $X$. Let $X_0$ be the topological sum of $\mathcal{C}$, and let $f: X_0 \to X$ be the obvious map. Then $f$ is a quotient map such that $f^{-1}(E)$ is Lindelöf for every countable subset $E$ of $X$, for every $f^{-1}(x)$ is countable and $X_0$ is paracompact. Moreover $t(X) \leq \omega$ and $X$ satisfies (C). Thus by [9; Theorem 9.5] for $x \in X$ and an open covering \{ $C \in \mathcal{C}; x \in C$ \} of $f^{-1}(x)$, finitely many $f(C_{\alpha})$ cover a neighborhood of $x$. This implies $X$ is locally compact.

The following lemma will be useful.

**Lemma 6.** Let $X$ be a space with a point-countable k-system $\mathcal{C}$. Then for each k-sequence $(A_n)$ in $X$, some $A_n$ is contained in a finite union of elements of $\mathcal{C}$.

**Proof.** Suppose that no $A_n$ is contained in any finite union of elements of $\mathcal{C}$. For $x \in X$, let $\{C \in \mathcal{C}; x \in C\} = \{C_n(x); n \in N\}$. Beginning with any point $x \in X$, there exists $x_i \in A_i - C_i(x)$. By induction there exists an infinite subset $D = \{x_n; n \in N\}$ of $X$ with $x_n \in A_n - \bigcup_{i \leq n} C_i(x_j)$. Then for each $C \in \mathcal{C}$, $C \cap D$ is at most finite. Thus $D$ is a discrete closed subset of $X$. However, since $x_n \in A_n$, $D$ has an accumulation point in $X$. This is a contradiction. Thus some $A_n$ is contained in a finite union of elements of $\mathcal{C}$.

The previous lemma will be used later, but let us now apply the lemma to two propositions below. Recall that a space $X$ is a $k'$-space (resp. Fréchet space) if, whenever $x \in A$, then there exists a compact subset $C$ of $X$ (resp. a sequence $\{a_n; n \in N\}$ in $A$) with $x \in \overline{A \cap C}$ (resp. $a_n \to x$).

**Proposition 7.** Let $X$ have a point-countable k-system $\mathcal{C}$.

(i) If $X$ is countably compact, then it is compact.
(ii) If $X$ is countably bi-k, then it is locally compact.
(iii) If $X$ is a $k'$-space (resp. separable $k'$-space), then it is locally Lindelöf (resp. Lindelöf).
Proof. (i) follows from the proof of Lemma 6.

(ii) Suppose that for some \( x \in X, x \in \text{int} \cup C' \) for any finite subcollection \( C' \) of \( C \). Let \( \{C \in C; x \in C\} = \{C_i; i \in N\} \), and \( F_n = X - \bigcup_{i=1}^{n} C_i \). Then \((F_n) \downarrow x \). Thus there is a k-sequence \((A_n)\) with \( x \in A_n \cap F_n \). By Lemma 6 some \( A_{n_0} \) is contained in a union of finitely many elements \( C^* \) of \( C \). Let \( G = X - \{C^*; x \in C^*\} \). Then \( G \) is a neighborhood of \( x \) which is disjoint from some \( A_{n_1} \cap F_{n_1} \) with \( n_1 \geq n_0 \). But \( x \notin G \cap A_{n_1} \cap F_{n_1} \), a contradiction. Thus each point of \( X \) has a neighborhood which is contained in a finite union of elements of \( C \). Hence \( X \) is locally compact.

(iii) Since the \( k' \) case is proved similarly, we prove the separable \( k' \) case. Let \( X = D \) with \( D \) countable, and \( x \in X \). Then there is a compact subset \( K \) of \( X \) with \( x \in K \cap D \). By Lemma 6, \( K \) is contained in a union of finitely many elements of \( C \). Thus \( x \in K \cap D \) implies \( x \in C \cap D \) for some \( C \in C \). This shows that \( X = \bigcup \{C \cap D; C \in C\} \). Thus \( X \) is \( \sigma \)-compact, hence Lindelöf.

We remark that, in [6], we have a separable space with a point-finite \( k \)-system consisting of metric subspaces, but it is not meta-Lindelöf, hence not Lindelöf. Thus the \( k' \)-ness of the parenthetic part of (iii) is essential. However, I do not know whether or not every separable \( k' \)-space with a point-countable \( k \)-system \( C \) has a countable \( k \)-system. If each element of \( C \) is metric, then such a space has a countable \( k \)-system by the later Corollary 11.

**Proposition 8.** Let \( f : X \to Y \) be a closed map with \( X \) paracompact, countably bi-\( k \). If \( Y \) has a point-countable \( k \)-system, then every \( \partial f^{-1}(y) \) has a countable \( k \)-system.

**Proof.** Let \( C \) be a point-countable \( k \)-system of a space \( Y \). For \( y \in Y \), let \( \{C \in C; y \in C\} = \{C_i; i \in N\} \) and \( F_n = \bigcup_{i=1}^{n} C_i \) for \( n \in N \). For some \( x \in f^{-1}(y) \), assume that \( (X - f^{-1}(F_n)) \downarrow x \). Since \( X \) is countably bi-\( k \), there is a \( k \)-sequence \((A_n)\) in \( X \) such that \( A_n \cap (X - f^{-1}(F_n)) \ni x \) for \( n \in N \). Since \((f(A_n))\) is a \( k \)-sequence in \( Y \), using Lemma 6, as in the proof of Proposition 7(ii), we have a contradiction to the assumption. Thus each point \( x \) of \( f^{-1}(y) \) has a neighborhood \( V_x \) contained in some \( f^{-1}(F_{n_x}) \). Let \( f_i = f | f^{-1}(C_i) \) for \( i \in N \). Then \( \partial f^{-1}(y) \cap V_x \subset \bigcup \{\partial f_i^{-1}(y); i = 1, 2, \ldots, n_x\} \). Let \( \mathcal{V} = \{\partial f_i^{-1}(y) \cap V_x; x \in f^{-1}(y)\} \). Then \( \mathcal{V} \) is an open covering of \( \partial f^{-1}(y) \), hence \( \partial f^{-1}(y) \) has the weak topology with respect to \( \mathcal{V} \). On the other hand, each element of \( \mathcal{V} \) is contained in a finite union of elements of a closed covering \( \mathcal{F} = \{\partial f_i^{-1}(y); i \in N\} \) of \( \partial f^{-1}(y) \). Hence it is easy to show that \( \partial f^{-1}(y) \) has the weak topology with respect to \( \mathcal{F} \). But each \( f_i \) is a closed map of a paracompact space onto a compact space \( C_i \), so
that each \( \partial f^{-1}(y) \) is compact by [7; Theorem 1.1]. Thus each element of \( \mathcal{F} \) is compact. Therefore, each \( \partial f^{-1}(y) \) has a countable k-system.

Now, using Lemmas 4 and 6, we shall prove the following theorem related to the question (*) arised after Example 3.

**Theorem 9.** Let \( f: X \to Y \) be a quotient s-map (i.e., every \( f^{-1}(y) \) is separable), and let \( X \) have a point-countable base. If \( Y \) is a k'-space, then the following are equivalent.

(a) \( Y^e \) has a k-system.

(b) \( Y \) has a point-countable base or a point-countable k-system.

**Proof.** (a) \( \to \) (b). \( Y \) is sequential so that it has countable tightness. Thus by Lemma 4, \( Y \) satisfies \( (C_1) \) or \( (C_2) \). If \( Y \) satisfies \( (C_1) \), by [9; Theorem 9.8] \( Y \) has a point-countable base. So, suppose that \( Y \) satisfies \( (C_2) \). Here we remark that every closed countably compact subset of \( Y \) is compact metric. Indeed, since \( Y \) is assumed to be countably compact, \( Y \) satisfies \( (C_1) \) so that \( Y \) has a point-countable base. Thus by [12; Corollary 1.6], \( Y \) is compact metric. Now, let \( \mathcal{B} \) be a point-countable base of \( X \) and assume that \( \mathcal{B} \) is closed under finite intersections. For \( x \in X \), suppose that \( \{V(x, n) \in \mathcal{B}; n \in \mathbb{N}\} \) is a decreasing local base at \( x \), hence is a k-sequence. Then by [9; Proposition 1.4], \( (f(V(x, n))) \) is a k-sequence in \( Y \), so is \( (f(V(x, n))) \). Thus by \( (C_2) \) some \( f(V(x, n)) \) is countably compact, hence separable metric. But, \( X \) has the weak topology with respect to a point-countable open covering \( \mathcal{B}' = \{V(x, n); x \in X\} \). Since \( f \) is a quotient and s-map, \( Y \) has the weak topology with respect to a point-countable covering \( f(\mathcal{B}') \) consisting of separable metric subspaces. On the other hand, \( Y \) is a Fréchet space, because every compact subset of a k'-space \( Y \) is metric. Thus, by [6], \( Y \) is the topological sum of spaces with a countable k-network. Hence, to complete the proof, it suffices to show that every k-space \( Z \) having a countable k-network and satisfying \( (C_2) \) has a countable k-system. Here we shall recall that a covering \( \mathcal{F} \) of \( Z \) is a countable k-network, if \( C \subseteq U \) with \( C \) compact and \( U \) open in \( Z \), then there exists a finite subcovering \( \mathcal{F}' \) of \( \mathcal{F} \) such that \( C \subseteq \cup \mathcal{F}' \subseteq U \). We can assume that each element of \( \mathcal{F} \) is closed and \( \mathcal{F} \) is closed under finite unions and intersections. Let \( K \) be any compact subset of \( Z \), and \( \{F \in \mathcal{F}; F \ni K\} = \{F; i \in \mathbb{N}\} \). Let \( K_n = \bigcap_{i \geq n} F_i \) for \( n \in \mathbb{N} \). Then each \( K_n \in \mathcal{F} \), and \( (K_n) \) is a k-sequence with \( K = \bigcap_{n=1}^{\infty} K_n \). Thus by \( (C_2) \) some \( K_n \) is countably compact, hence compact. This shows that \( \mathcal{F} = \{F \in \mathcal{F}; F \text{ is compact in } Z\} \) is still a countable k-network. Since \( Z \) is a k-space, \( \mathcal{F} \) is obviously a countable k-system of \( Z \). That completes the proof.
(b) \implies (a). If \( Y \) has a point-countable base, then \( Y^2 \) is first countable. Thus \( Y^2 \) has a \( k \)-system. Suppose that \( Y \) has a point-countable \( k \)-system \( \mathcal{E} \). Since every compact subset of \( Y \) is metric, a Fréchet space \( Y \) has the weak topology with respect to the point-countable covering \( \mathcal{E} \) consisting of separable metric subspaces. On the other hand, \( Y \) satisfies \( (C_2) \) by Lemma 6. Hence, by the proof of (a) \implies (b), \( Y \) is the topological sum of \( k_\omega \)-subspaces. Hence, \( Y^2 \) has a \( k \)-system by Proposition 2.

As a generalization of closed maps and open maps, we recall that a map \( f : X \to Y \) is \textit{pseudo-open} if for any neighborhood \( U \) of \( f^{-1}(y) \), \( y \in \text{int} \, f(U) \). Every pseudo-open map is quotient. Every pseudo-open image of a metric space is obviously Fréchet. Thus we have the following corollary from Theorem 9 and the fact that every quotient \( s \)-image of a locally separable metric space is metrizable if it has a point-countable base [4; Corollary 1].

**Corollary 10.** Let \( X \) be the pseudo-open \( s \)-image of a metric space (resp. locally separable, metric space). Then \( X^2 \) has a \( k \)-system if and only if \( X \) has a point-countable base (resp. \( X \) is metric) or \( X \) has a point-countable \( k \)-system.

**Corollary 11.** Suppose that \( X \) has a point-countable \( k \)-system consisting of metric subspaces. If \( X \) is a \( k' \)-space (resp. separable \( k' \)-space), then \( X \) is the topological sum of \( k_\omega \)-subspaces (resp. \( X \) is a \( k_\omega \)-space), hence \( X^2 \) is a \( k \)-space.

**Proof.** Let \( \mathcal{E} \) be a point-countable \( k \)-system consisting of metric subspaces. Let \( X_0 \) be the topological sum of \( \mathcal{E} \) and \( f : X_0 \to X \) be the obvious map. Then \( f \) is a quotient \( s \)-map. Thus, since \( X \) is Fréchet, \( X \) is the topological sum of \( k_\omega \)-subspaces by the proof of (b) \implies (a) of Theorem 9. If \( X \) is moreover separable, by Proposition 7(iii), \( X \) is Lindelöf. Thus \( X \) is a \( k_\omega \)-space.

The \( k' \)-ness of the previous corollary is essential by Example 3. However, we have the following question in connection with whether or not we can omit the metric "pieces".

**Question 12.** Suppose that \( X \) is a \( k' \)-space with a point-countable \( k \)-system. Then does \( X^2 \) have a \( k \)-system?

As is well known, every \( k' \)-space is precisely the pseudo-open image of a locally compact paracompact space ([2; Chapter III, Theorem 3.3]). As for Question 12, if \( X \) is the closed image of a locally
compact paracompact space, then the answer is affirmative. More

generally we have

**Theorem 13.** Let \( f: X \to Y \) be a closed map. If \( X \) is a par-

acompact countably bi-k-space, then (a), (b) and (c) below are equiva-

lent. Moreover (a) implies (d).

(a) \( Y \) has a point-countable \( k \)-system.

(b) \( Y \) is a paracompact, locally \( k_\omega \)-space.

(c) \( Y^2 \) has a point-countable \( k \)-system.

(d) \( Y^2 \) has a paracompact space with a \( k \)-system.

**Proof.** (b) \( \to \) (c) and (b) \( \to \) (d) follow from Theorem 1, and (c) \( \to \)

(a) is clear.

(a) \( \to \) (b). The paracompactness of \( Y \) follows from the well known

results due to E. Michael: Every closed image of a paracompact
spaces is paracompact ([3; Theorem 2.4, p. 165]). We prove \( Y \)

is locally \( k_\omega \). Let \( y \in Y \). Then every \( \partial f^{-1}(y) \) is Lindelöf by Proposition

8, and by the proof there, each point of \( f^{-1}(y) \) has a neighborhood

contained in the inverse image of some compact subset of \( Y \). Now,

since each closed subset of \( X \) is countably bi-k, as in the proof

of [7; Corollary 1.2] we can assume that every \( f^{-1}(y) \) is Lindelöf.

Hence there exists a neighborhood \( W \) of \( y \), open subsets \( V_n \) of \( X \),

and compact subsets \( C_n \) of \( Y \) such that \( f^{-1}(W) \subseteq \bigcup_{n=1}^{\infty} V_n \), \( V_n \subseteq f^{-1}(C_n) \)

for \( n \in \mathbb{N} \). Let \( F = f^{-1}(W) \) and \( \mathcal{V} = \{ F \cap V_n; n \in \mathbb{N} \} \). Then \( \mathcal{V} \)

is an open covering of \( F \) and \( F \cap V_n \subseteq F \cap f^{-1}(C_n) \) for \( n \in \mathbb{N} \). Thus \( F \)

has the weak topology with respect to \( \{ F \cap f^{-1}(C_n); n \in \mathbb{N} \} \). Since \( f|F \)

is closed, hence quotient, so \( f(F) = W \) has the weak topology

with respect to \( \{ W \cap C_n; n \in \mathbb{N} \} \). This shows that \( Y \) is a locally \( k_\omega \)-

space.

Concerning the implication (d) \( \to \) (a) of the previous theorem, we

have

**Theorem 14.** (CH). Let \( f: X \to Y \) be a closed map with \( X \) para-

acompact bi-k (resp. paracompact locally compact). Suppose that \( t(Y) \leq \omega \). Then the following are equivalent. When \( Y \) is sequential,

(CH) can be omitted.

(a) \( Y \) has a point-countable \( k \)-system, or \( Y \) is bi-k (resp. \( Y \) has

a point-countable \( k \)-system).

(b) \( Y \) has a point-countable \( k \)-system, or every \( \partial f^{-1}(y) \) is compact

(resp. every \( \partial f^{-1}(y) \) is Lindelöf).

(c) \( Y^2 \) has a \( k \)-system.

**Proof.** (a) \( \to \) (b). If \( Y \) is bi-k, then every \( \partial f^{-1}(y) \) is compact
by [9; Theorem 9.9]. The parenthetic part follows from Proposition 8.

(b) → (c). If \( Y \) has a point-countable \( k \)-system, then \( Y^2 \) has a \( k \)-system by Theorem 13. If every \( \partial f^{-1}(y) \) is compact, we can assume that every \( f^{-1}(y) \) is compact. Thus \( Y^2 \) is a \( k \)-space by [9; Proposition 3.E.4]. If \( X \) is locally compact and every \( \partial f^{-1}(y) \) is Lindelöf, then \( Y^2 \) is a \( k \)-space by [15; Lemma 2.5].

(c) → (a). Suppose that \( X \) is paracompact \( bi-k \) and \( t(Y) \leq \omega \). Then by [5; Theorem 2.11], \( Y \) is paracompact locally \( k_\omega \), or \( bi-k \) under (CH), and if \( Y \) is sequential, (CH) can be omitted. Thus we have (a) by Theorem 1. If \( X \) is paracompact locally compact and \( Y \) is \( bi-k \), since every \( \partial f^{-1}(y) \) is compact, \( Y \) is paracompact locally compact. Hence \( Y \) has a point-countable \( k \)-system.

From Theorem 14 and Proposition 8, we have

**Corollary 15.** Let \( f: X \to Y \) be a closed map with \( X \) paracompact and first countable. If \( Y^2 \) has a \( k \)-system, then every \( \partial f^{-1}(y) \) has a countable \( k \)-system.

Finally we shall consider the product \( X^\omega \) of countably many copies of \( X \).

**Theorem 16.** (i) \( X^\omega \) has a point-countable \( k \)-system if and only if \( X \) is compact.

(ii) Suppose that \( X \) has a point-countable \( k \)-system and \( t(X) \leq \omega \). Then \( X^\omega \) has a \( k \)-system if and only if \( X \) is locally compact.

**Proof.** (i) The “if” part is clear.

“Only if”. Suppose that \( X \) is not compact. Hence \( X \) is not countably compact by Proposition 7(i). Then \( X \) contains a closed copy of \( N \), hence \( X^\omega \) contains a closed copy of \( N^\omega \). Since a metric space \( N^\omega \) has a point-countable \( k \)-system, \( N^\omega \) must be locally compact by Proposition 7(ii). This is a contradiction. Hence \( X \) is compact.

(ii) “If”. Since \( X \) is locally compact, \( X^\omega \) is a \( k \)-space by [2; Chaper III, Theorems 3.7 and 3.9].

“Only if”. We may assume that \( X \) is not compact, hence not countably compact. Then \( X^\omega \) contains a closed copy of \( X \times N^\omega \). Thus by Proposition 5, \( X \) is locally compact.

The previous theorem suggests the following question.

**Question 17.** Suppose that \( X \) has a point-countable or countable \( k \)-system. Then is \( X \) locally compact if \( X^\omega \) has a \( k \)-system?
References


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