

# Pacific Journal of Mathematics

**NONNORMAL BLASCHKE QUOTIENTS**

SHINJI YAMASHITA

## NON-NORMAL BLASCHKE QUOTIENTS

SHINJI YAMASHITA

**A quotient  $B_1/B_2$  of two infinite Blaschke products  $B_1$  and  $B_2$  with no common zero is called a Blaschke quotient. The existence of a Blaschke quotient which is not normal in the open unit disk  $D$ , is well known. We shall show among other things, that, for each  $p, 0 < p < \infty$ , there exists a nonnormal Blaschke quotient  $f$  such that**

$$\iint_D (1 - |z|^p) |f'(z)|^2 / (1 + |f(z)|^2)^2 dx dy < \infty.$$

**This might be of interest because, if  $g$  is meromorphic in  $D$  and if  $\iint_D |g'(z)|^2 / (1 + |g(z)|^2)^2 dx dy < \infty$ , then  $g$  is normal in  $D$ .**

**1. Introduction.** By a Blaschke product we mean a holomorphic function in  $D = \{|z| < 1\}$ , denoted by

$$B(z; \{c_n\}) = \prod_{n=1}^{\infty} \frac{|c_n|}{c_n} \frac{c_n - z}{1 - \bar{c}_n z},$$

where  $\{c_n\}$  is an infinite complex sequence satisfying  $0 < |c_n| < 1$ ,  $n = 1, 2, \dots$ , and  $\sum (1 - |c_n|) < \infty$ . By a Blaschke quotient we mean a meromorphic function in  $D$ , defined by

$$Q(z; \{c_n\}, \{c'_n\}) = B(z; \{c_n\}) / B(z; \{c'_n\}),$$

where the Blaschke products in the right-hand side have no zero in common.

A meromorphic function  $f$  in  $D$  is called normal in  $D$  if  $\sup_{z \in D} (1 - |z|) f^*(z) < \infty$ , where  $f^* = |f'| / (1 + |f|^2)$ ; see [5]. We shall construct nonnormal Blaschke quotients with some additional properties. It is easy to merely construct a nonnormal Blaschke quotient. For example, set  $c_n = 1 - (2n)^{-\lambda}$  and  $c'_n = 1 - (2n + 1)^{-\lambda}$ ,  $n = 1, 2, \dots$ , where  $\lambda > 1$  is a constant. Then  $Q(z) = Q(z; \{c_n\}, \{c'_n\})$  is not normal. Actually, let

$$\sigma(z_1, z_2) = \frac{1}{2} \log \frac{1 + \rho(z_1, z_2)}{1 - \rho(z_1, z_2)}$$

be the non-Euclidean distance between  $z_1$  and  $z_2$  in  $D$ , where

$$\rho(z_1, z_2) = |z_1 - z_2| / |1 - \bar{z}_1 z_2|.$$

Then,  $Q(c_n) = 0$ ,  $Q(c'_n) = \infty$ ,  $n \geq 1$ , and  $\lim_{n \rightarrow \infty} \sigma(c_n, c'_n) = 0$ . Therefore,  $Q$  is not uniformly continuous from  $D$ , endowed with  $\sigma(\cdot, \cdot)$ ,

into the Riemann sphere, endowed with the spherical chordal distance. Consequently,  $Q$  is not normal in  $D$ . Accordingly, J. A. Cima [3, Theorem 4] proved the existence of a nonnormal Blaschke quotient  $Q(z; \{c_n\}, \{c'_n\})$  with  $\inf_{j,k \geq 1} \sigma(c_j, c'_k) > 0$ .

There is another way of finding nonnormal Blaschke quotients. Namely, if a Blaschke quotient  $Q$  has two asymptotic values at one boundary point of  $D$ , then  $Q$  is not normal in  $D$  by [5, Theorem 2]. Therefore, one can easily conclude that the Blaschke quotients found by D. A. Storvick [6, p. 37] and C. Tanaka [7, Theorem 2] both are not normal in  $D$ . A meromorphic function  $f$  in  $D$  is said to have the left angular limit  $w$  (possibly,  $\infty$ ) at 1 if  $f(z) \rightarrow w$  as  $z \rightarrow 1$  within each triangular domain whose vertices are 1 and two points in  $D^+ = \{z \in D \mid \text{Im } z > 0\}$ . Also,  $f$  is said to have the right angular limit  $w$  at 1 if  $\overline{f(\bar{z})}$  has the left angular limit  $\bar{w}$  at 1 (convention:  $\overline{\infty} = \infty$ ). A Blaschke quotient  $Q(z) = Q(z; \{c_n\}, \{c'_n\})$  is called symmetric if  $\bar{c}_n = c'_n$  for each  $n$ . If  $Q$  is symmetric, then  $Q(z)\overline{Q(\bar{z})} \equiv 1$  in  $D$ , so that  $Q$  has the left angular limit  $w$  at 1 if and only if  $Q$  has the right angular limit  $1/\bar{w}$  (convention:  $1/0 = \infty, 1/\infty = 0$ ) at 1. Therefore, if  $Q$  is symmetric and if  $Q$  has the left angular limit 0 at 1, then  $Q$  is never normal in  $D$  because  $Q$  has 0 and  $\infty$  as asymptotic values at 1.

Now, for  $f$  meromorphic in  $D$ , we set

$$S_p(f) = \iint_D (1 - |z|)^p f^*(z)^2 dx dy, \quad z = x + iy, \quad 0 \leq p < \infty.$$

It is familiar that if  $S_0(f) < \infty$ , then  $f$  is normal in  $D$ . It is not difficult to observe that  $S_1(Q) < \infty$  for each Blaschke quotient  $Q$ . In effect, since  $Q$  is of bounded characteristic in the sense of R. Nevanlinna, it follows from

$$\int_0^1 \left[ \iint_{|z| < r} Q^*(z)^2 dx dy \right] dr < \infty,$$

that  $S_1(Q) < \infty$ ; see (2.10) in §2.

Our first result is

**THEOREM 1.** *Let  $0 < p < 1$ , and let  $0 < q < \infty$ . Then there exists a symmetric Blaschke quotient  $Q(z) = Q(z; \{a_n\}, \{\bar{a}_n\})$  satisfying the following three conditions.*

- (I)  $\inf_{j,k \geq 1} \sigma(a_j, \bar{a}_k) \geq q$ .
- (II)  $Q$  has 0 as the left angular limit at 1.
- (III)  $S_p(Q) < \infty$ .

If we restrict  $p$  in (III) of Theorem 1 as  $1/2 < p < 1$ , then we can construct  $Q$  with an additional property.

By a left horocyclic angle at 1 we mean a domain

$$\{z \in D^+ \mid 1 - x_1 < |z - x_1| \text{ and } 1 - x_2 > |z - x_2|\},$$

where  $0 < x_2 < x_1 < 1$ . A meromorphic function  $f$  in  $D$  is said to have the left horocyclic angular limit  $w$  at 1 if  $f(z) \rightarrow w$  as  $z \rightarrow 1$  within each left horocyclic angle at 1; the notion is essentially due to F. Bagemihl [1]. Also,  $f$  is said to have the right horocyclic angular limit  $w$  at 1 if  $\overline{f(\bar{z})}$  has  $\bar{w}$  as the left horocyclic angular limit at 1. Again, a symmetric Blaschke quotient  $Q$  has the left horocyclic angular limit  $w$  at 1 if and only if  $Q$  has the right horocyclic angular limit  $1/\bar{w}$  at 1. Therefore, if a symmetric  $Q$  has the left horocyclic angular limit 0 at 1, then  $Q$  is never normal in  $D$ .

**THEOREM 2.** *Let  $1/2 < p < 1$ , and let  $0 < q < \infty$ . Then there exists a symmetric Blaschke quotient  $Q(z) = Q(z; \{a_n\}, \{\bar{a}_n\})$  satisfying the same conditions as (I), (II), and (III) in Theorem 1, together with*

(III)  $Q$  has 0 as the left horocyclic angular limit at 1.

Lastly in the present section, we remark that Cima and P. Colwell [4, Theorem 2] proposed a necessary and sufficient condition for a Blaschke quotient to be normal in  $D$  in terms of interpolating sequences.

**2. Proof of Theorem 1.** By the linear transformation  $w = \varphi(z) \equiv (1+z)/(1-z)$ , the disk  $D$  is mapped onto the right half-plane  $R$ , so that,  $R^+ = \varphi(D^+)$  is the first quadrant in the  $w$ -plane. Furthermore, by  $\varphi$ , the chord  $L(\theta) = \{z \in D \mid \arg(1-z) = \theta\}$ ,  $|\theta| < \pi/2$ , is mapped onto the half-line:

$$A(\theta) = \{w = x + iy \in R \mid y = (-\tan \theta)(x + 1)\}.$$

By a simple calculation one obtains

$$(2.1) \quad 1 - |z|^2 = 4 \operatorname{Re} w / |w + 1|^2, \quad w = \varphi(z), \quad z \in D,$$

and

$$(2.2) \quad \rho(z_1, z_2) = |w_1 - w_2| / |\bar{w}_1 + w_2|$$

for  $w_j = \varphi(z_j)$ ,  $z_j \in D$ ,  $j = 1, 2$ .

To construct  $Q$  we choose  $A$ ,  $0 < A < 1$ , such that

$$(2.3) \quad \frac{1}{2} \log \frac{1+t}{1-t} = q \quad \text{and} \quad t = A/(1+A^2)^{1/2}.$$

Choose  $\theta_0$ ,  $-\pi/2 < \theta_0 < 0$ , so that  $A = -\tan \theta_0$ , and then choose  $s > 1/p > 1$ . Consider the sequence of points  $b_n \in \Lambda(\theta_0)$  such that  $b_n = x_n + iy_n = n^s + iA(n^s + 1)$ ,  $n = 1, 2, \dots$ . Let  $a_n = \varphi^{-1}(b_n)$ ,  $n \geq 1$ . Then  $\{a_n\} \subset L(\theta_0)$ . We then set  $Q(z) = Q(z; \{a_n\}, \{\bar{a}_n\})$ . First of all,  $Q$  is well defined because, by (2.1),

$$(2.4) \quad \begin{aligned} \sum (1 - |a_n|) &= \sum (1 - |\bar{a}_n|) \leq \sum (1 - |a_n|)^p \\ &\leq \sum (1 - |a_n|^{2p}) \leq 4^p \sum n^{-2sp} < \infty . \end{aligned}$$

Further, one observes that

$$(2.5) \quad |Q(z)| = g(w) \equiv \prod_{n=1}^{\infty} g_n(w), \quad w = \varphi(z),$$

where  $g_n(w) = |w^2 - b_n^2|/|w^2 - \bar{b}_n^2|$ ,  $n \geq 1$ .

*Proof of (I).* Let  $w = x + iy \in R$ ,  $\zeta = \xi + i\eta \in R$ , with  $y \geq A(x + 1)$ ,  $\eta \geq A(\xi + 1)$ . Since

$$X \equiv (x + \xi)/(y + \eta) \leq A^{-1},$$

it follows that

$$|w - \bar{\zeta}|/|w + \zeta| \geq (X^2 + 1)^{-1/2} \geq (1 + A^{-2})^{-1/2} = t.$$

In view of (2.2) one can now easily conclude that  $\rho(a_j, \bar{a}_k) \geq t$ , so that  $\sigma(a_j, \bar{a}_k) \geq q$  for all  $j, k \geq 1$ .

*Proof of (II).* To prove that

$$(2.6) \quad \lim_{\substack{z \rightarrow 1 \\ z \in L(\theta_0)}} Q(z) = 0,$$

it suffices by (2.5) to show that

$$(2.7) \quad \lim_{\substack{w \rightarrow \infty \\ w \in \Lambda(\theta_0)}} g(w) = 0.$$

Since  $g_n(w) \leq 1$  for all  $w \in R^+$  and for all  $n \geq 1$ , it follows that

$$(2.8) \quad g(w) \leq g_n(w) \leq 1 \quad \text{for all } w \in R^+ \text{ and all } n \geq 1.$$

Given  $\varepsilon > 0$ , one can find a natural number  $N$  such that  $x_{n+1}/x_n - 1 < \varepsilon$  for all  $n \geq N$ . Then, for each  $w = x + iy \in \Lambda(\theta_0)$  with  $x \geq x_N$ ,

$$(2.9) \quad g(w) \leq A_1 \varepsilon, \quad A_1 = \frac{1}{2}(A + A^{-1}),$$

which proves (2.7). To make sure of (2.9), we first find  $n \geq N$  such that  $x_n \leq x \leq x_{n+1}$ . Then,

$$|w - \bar{b}_n| = (1 + A^2)^{1/2}(x - x_n) \leq (1 + A^2)^{1/2}(x_{n+1} - x_n),$$

$$|w + \bar{b}_n| \geq x + x_n \geq 2x_n,$$

whence

$$|w - \bar{b}_n|/|w + \bar{b}_n| \leq \frac{1}{2}(1 + A^2)^{1/2}\varepsilon.$$

On the other hand,

$$|w + \bar{b}_n|/|w - \bar{b}_n| \leq [(x + x_n)^2 + A^2(x + x_n + 2)^2]^{1/2}/[A(x + x_n + 2)] \leq (1 + A^{-2})^{1/2},$$

so that  $g_n(w) \leq A_1\varepsilon$ . Therefore, in view of (2.8), one can assert (2.9).

Since  $|Q(z)| = g(\varphi(z)) \leq 1$  in  $D^+$  by (2.8), and since (2.6) holds, it follows from E. Lindelöf's theorem [8, Theorem VIII. 10, p. 306], together with an obvious conformal homeomorphism from the upper half-disk onto  $D^+$ , mapping 0 to 1, that  $Q$  has the left angular limit zero at 1.

*Proof of (III).* We remember L. Carleson's family  $T_\alpha$  of meromorphic functions  $h$  in  $D$  such that

$$I_\alpha(h) \equiv \int_0^1 (1 - r)^{-\alpha} \left[ \iint_{|z| < r} h^*(z)^2 dx dy \right] dr < \infty,$$

where  $0 \leq \alpha < 1$ ; see [2, p. 19]. Letting  $X_r(z)$  be the characteristic function of the disk  $\{|z| < r\}$ , one observes that

$$(2.10) \quad \begin{aligned} I_\alpha(h) &= \int_0^1 (1 - r)^{-\alpha} \left[ \iint_D X_r(z) h^*(z)^2 dx dy \right] dr \\ &= \iint_D \left[ \int_0^1 (1 - r)^{-\alpha} X_r(z) dr \right] h^*(z)^2 dx dy = (1 - \alpha)^{-1} S_{1-\alpha}(h). \end{aligned}$$

For a Blaschke quotient  $Q_1(z) = Q(z; \{c_n\}, \{c'_n\})$  we assume that

$$\sum (1 - |c_n|)^{1-\alpha} < \infty \quad \text{and} \quad \sum (1 - |c'_n|)^{1-\alpha} < \infty.$$

Then it follows from [2, Theorem 2.2, p. 24] that  $Q_1 \in T_\alpha$ .

Returning to our  $Q$ , we can easily conclude from (2.4) that  $Q \in T_{1-p}$ , whence  $S_p(Q) < \infty$  by (2.10).

REMARK. The Blaschke quotient  $Q$ , described in the second paragraph in § 1, satisfies  $S_p(Q) < \infty$ , for a  $p$ ,  $0 < p < 1$ , provided that  $\lambda < 1/p$ .

3. **Proof of Theorem 2.** Let  $\lambda > (1/2)(p^{-1} + 1)$  and  $1/(2p) < \mu < 1$ , and  $y_{n,m} = n^\lambda m^\mu$  ( $n, m = 1, 2, \dots$ ). Let  $t$  and  $A$  be as in (2.3).

Then, for each fixed  $n \geq 1$ , we may find a natural number  $M_n$  such that

$$y_{n,m} \geq A(n+1) \geq A(n^{-1}+1) \quad \text{for all } m \geq M_n.$$

Then, for each fixed  $n \geq 1$ , the points  $b_{n,m} = n + iy_{n,m}$ ,  $m \geq M_n$ , lie on the half-line  $\Gamma(n) = \{w \in R^+ \mid \text{Re } w = n\}$ , so that  $a_{n,m} = \varphi^{-1}(b_{n,m})$  ( $m \geq M_n$ ) lie on the half-oricycle  $C(n) = \varphi^{-1}(\Gamma(n))$ . Similarly, for each fixed  $n \geq 2$ , the points  $b_{n,m}^* = n^{-1} + iy_{n,m}$ ,  $m \geq M_n$ , lie on the half-line  $\Gamma^*(n) = \{w \in R^+ \mid \text{Re } w = n^{-1}\}$ , so that  $a_{n,m}^* = \varphi^{-1}(b_{n,m}^*)$  ( $m \geq M_n$ ) lie on the half-oricycle  $C^*(n) = \varphi^{-1}(\Gamma^*(n))$ . Let  $\{a_n\} = \{a_{n,m}\} \cup \{a_{n,m}^*\}$ . Then  $Q(z) = Q(z; \{a_n\}, \{\bar{a}_n\})$  is the requested. We first observe that, for  $n \geq 1$ ,

$$\beta_n \equiv \sum_{m \geq M_n} [\text{Re } b_{n,m} / |b_{n,m} + 1|^2]^p \leq n^{p(1-2\lambda)} \sum_{m=1}^{\infty} m^{-2p\mu},$$

and for  $n \geq 2$ ,

$$\beta_n^* \equiv \sum_{m \geq M_n} [\text{Re } b_{n,m}^* / |b_{n,m}^* + 1|^2]^p \leq n^{-p(1+2\lambda)} \sum_{m=1}^{\infty} m^{-2p\mu}.$$

Since  $p(1+2\lambda) > p(2\lambda-1) > 1$  and  $2p\mu > 1$ , it follows from (2.1) that

$$\begin{aligned} (3.1) \quad \sum (1 - |a_n|) &\leq \sum (1 - |a_n|^2)^p \\ &\leq 4^p \left( \sum_{n=1}^{\infty} \beta_n + \sum_{n=2}^{\infty} \beta_n^* \right) < \infty, \end{aligned}$$

so that  $Q$  is well defined. Now, one observes that

$$(3.2) \quad |Q(z)| = G(w) \equiv \prod_{n=1}^{\infty} G_n(w) \prod_{n=2}^{\infty} G_n^*(w), \quad w = \varphi(z),$$

where

$$\begin{aligned} G_n &= \prod_{m=M_n}^{\infty} g_{n,m}, & G_n^* &= \prod_{m=M_n}^{\infty} g_{n,m}^*, \\ g_{n,m}(w) &= |w^2 - b_{n,m}^2| / |w^2 - \bar{b}_{n,m}^2|, \\ g_{n,m}^*(w) &= |w^2 - b_{n,m}^{*2}| / |w^2 - \bar{b}_{n,m}^{*2}|. \end{aligned}$$

*Proof of (I).* The same as that of (I) of Theorem 1.

*Proofs of (II) and (III).* We shall first show that

$$(3.3) \quad \lim_{\substack{z \rightarrow 1 \\ z \in C(n)}} Q(z) = 0 \quad \text{for all } n \geq 1,$$

and

$$(3.4) \quad \lim_{\substack{z \rightarrow 1 \\ z \in C^*(n)}} Q(z) = 0 \quad \text{for all } n \geq 2.$$

Since  $g_{n,m}(w) \leq 1$  and  $g_{n,m}^*(w) \leq 1$  for all  $w \in R^+$  and for all possible pairs  $n, m$ , it follows that

$$(3.5) \quad G(w) \leq g_{n,m}(w) \leq 1, \quad w \in R^+, \quad n \geq 1, \quad m \geq M_n,$$

and

$$(3.6) \quad G(w) \leq g_{n,m}^*(w) \leq 1, \quad w \in R^+, \quad n \geq 2, \quad m \geq M_n.$$

For the proof of (3.3), it suffices by (3.2) to show that

$$(3.7) \quad \lim_{\substack{w \rightarrow \infty \\ w \in \Gamma(n)}} G(w) = 0, \quad n \geq 1.$$

Since  $\mu < 1$ , it follows that, for each  $n \geq 1$  and for a given  $\varepsilon > 0$  there exists a natural number  $M'_n \geq M_n$  such that  $y_{n,m+1} - y_{n,m} < \varepsilon$  for all  $m \geq M'_n$ . Then, for each  $w = n + iy \in \Gamma(n)$  with  $y \geq y_{n,M'_n}$ , there exists  $m \geq M'_n$  such that  $y_{n,m} \leq y \leq y_{n,m+1}$ . Consequently,

$$|w - b_{n,m}| / |w + \bar{b}_{n,m}| \leq (y_{n,m+1} - y_{n,m}) / (2n)$$

and

$$\left| \frac{w + b_{n,m}}{w - \bar{b}_{n,m}} \right| \geq \sqrt{1 + \frac{4n^2}{(2y_{n,m})^2}} \leq \sqrt{1 + n^{2-2\lambda}},$$

so that, by (3.5),  $G(w) \leq g_{n,m}(w) \leq k_n \varepsilon$ , where  $k_n$  is a constant depending only on  $n$ . The proof of (3.7) is thus complete. Similarly we can prove, via (3.6), that

$$\lim_{\substack{w \rightarrow \infty \\ w \in \Gamma^*(n)}} G(w) = 0, \quad n \geq 2,$$

which, together with (3.2), shows (3.4). By the Lindelöf theorem [8, Theorem VIII. 10, p. 306] again, (II) is established. For the proof of the horocyclic part, we first note that  $|Q| \leq 1$  in  $D^+$ . Set  $\mathcal{C} = \{C(n) | n \geq 1\} \cup \{C^*(n) | n \geq 2\}$ . Then for each left horocyclic angle  $H$  at 1, we may find members  $C_1$  and  $C_2$  of  $\mathcal{C}$  such that the left horocyclic angle  $H_1$  at 1, bounded by  $C_1$  and  $C_2$  and a line segment on the real axis, contains  $H$ . Since

$$\lim_{\substack{z \rightarrow 1 \\ z \in C_j}} Q(z) = 0, \quad j = 1, 2,$$

by (3.3) and/or (3.4), it follows from another theorem of Lindelöf [8, Theorem VIII. 7, p. 304], via an obvious conformal homeomorphism, that  $Q(z)$  has the limit 0 as  $z \rightarrow 1$  within  $H_1$  containing  $H$ . We have thus established (IIH).



*Proof of (III).* The same as that of (III) of Theorem 1.

#### REFERENCES

1. Frederick Bagemihl, *Horocyclic boundary properties of meromorphic functions*, Ann. Acad. Sci. Fenn. Ser. AI, Math., **385** (1966), 1-18.
2. Lennart Carleson, *On a class of meromorphic functions and its associated exceptional sets*, Thesis, Uppsala, 1950.
3. Joseph A. Cima, *A nonnormal Blaschke-quotient*, Pacific J. Math., **15** (1965), 767-773.
4. Joseph A. Cima and Peter Colwell, *Blaschke quotients and normality*, Proc. Amer. Math. Soc., **19** (1968), 796-798.
5. Olli Lehto and Kaarlo I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math., **97** (1957), 47-65.
6. David A. Storvick, *On meromorphic functions of bounded characteristic*, Proc. Amer. Math. Soc., **8** (1957), 32-38.
7. Chuji Tanaka, *On the boundary values of Blaschke products and their quotients*, Proc. Amer. Math. Soc., **14** (1963), 472-476.
8. Masatsugu Tsuji, *Potential Theory in Modern Function Theory*, Maruzen Co., Ltd., Tokyo, 1959.

Received April 6, 1979.

TOKYO METROPOLITAN UNIVERSITY  
FUKAZAWA, SETAGAYA-KU,  
TOKYO, 158 JAPAN

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DONALD BABBITT (Managing Editor)

University of California  
Los Angeles, California 90024

HUGO ROSSI

University of Utah  
Salt Lake City, UT 84112

C. C. MOORE and ARTHUR AGUS

University of California  
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, California 90007

R. FINN and J. MILGRAM

Stanford University  
Stanford, California 94305

## ASSOCIATE EDITORS

R. ARNES

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF HAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

Natália Bebiano, On the evaluation of permanents .....	1
David Borwein and Bruce Brigham Watson, Tauberian theorems between the logarithmic and Abel-type summability methods .....	11
Leo George Chouinard, II, Hermite semigroup rings .....	25
Kun-Jen Chung, Remarks on nonlinear contractions .....	41
Lawrence Jay Corwin, Representations of division algebras over local fields. II .....	49
Mahlon M. Day, Left thick to left lumpy—a guided tour .....	71
M. Edelstein and Mo Tak Kiang, On ultimately nonexpansive semigroups .....	93
Mary Rodriguez Embry, Semigroups of quasinormal operators .....	103
William Goldman and Morris William Hirsch, Polynomial forms on affine manifolds .....	115
S. Janakiraman and T. Soundararajan, Totally bounded group topologies and closed subgroups .....	123
John Rowlay Martin, Lex Gerard Oversteegen and Edward D. Tymchatyn, Fixed point set of products and cones .....	133
Jan van Mill, A homogeneous Eberlein compact space which is not metrizable .....	141
Steven Paul Plotnick, Embedding homology 3-spheres in $S^5$ .....	147
Norbert Riedel, Classification of the $C^*$ -algebras associated with minimal rotations .....	153
Benedict Seifert, Combinatorial and geometric properties of weight systems of irreducible finite-dimensional representations of simple split Lie algebras over fields of 0 characteristic .....	163
James E. Simpson, Dilations on locally convex spaces .....	185
Paolo M. Soardi, Schauder bases and fixed points of nonexpansive mappings .....	193
Yoshio Tanaka, Point-countable $k$ -systems and products of $k$ -spaces .....	199
Fausto A. Toranzos, The points of local nonconvexity of starshaped sets .....	209
Lorenzo Traldi, The determinantal ideals of link modules. I .....	215
P. C. Trombi, Invariant harmonic analysis on split rank one groups with applications .....	223
Shinji Yamashita, Nonnormal Blaschke quotients .....	247