CERTAIN TRANSFORMATIONS OF BASIC HYPERGEOMETRIC SERIES AND THEIR APPLICATIONS

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We obtain identities of Rogers-Ramanujan type related to the modulus 13. We also obtain the $q$-analogues of the nearly-poised summation theorems and use them for obtaining $q$-analogues of general transformations of nearly-poised hypergeometric series. We also discuss some important applications of the transformations obtained in this note.

Recently, Askey and Wilson [4] derived the transformation

\begin{equation}
\phi_b \left[ a^2, b^2, c, d; q \frac{a b \sqrt{q}}{q}, -a b \sqrt{q}, -c d \right] = \phi_b \left[ a^2 b^2 q, -c d, -c d q \right],
\end{equation}

(provided $a, b, c,$ or $d$ is of the form $q^{-N}, N$ a nonnegative integer). In an earlier paper [11] we have an alternative proof of (1.1). We begin this note by showing in §3 that all the transformations proved by Singh [13], for obtaining the $q$-analogues of identities of the Cayley-Orr type, can be deduced from (1.1). We also show that (1.1) may be used effectively to prove the following transformation:

\begin{equation}
\phi_b \left[ a, q \sqrt{a}, -q \sqrt{a}, i q^a - q^a, -i q^a, -q^a, q^a, o; -aq^{1+n} \right]
\end{equation}

\begin{equation}
\left[ \sqrt{a}, -\sqrt{a}, -iaq^{1+n}, iaq^{1+n}, -aq^{1+n}, aq^{1+n} \right] = \frac{[aq; q]_n^{1+n}}{[aq^{1+n}/a, q^{1+n}/a]},
\end{equation}

due to Andrews [2] which is his key result for obtaining the identities of the Rogers-Ramanujan type of modulus 11. In fact, we shall prove the transformation:

\begin{equation}
\left[ a, q \sqrt{a}, -q \sqrt{a}, c, e, -e, -q^a, q^a, a^3 q^{2+n} \frac{a^2 q^{2+2n}}{c e^2} \right]
\end{equation}

\begin{equation}
\left[ \sqrt{a}, -\sqrt{a}, a q/c, a q/e, -a q/e, -a q^{1+n}, a q^{1+n} \right]
\end{equation}

\begin{equation}
= \frac{[a^2 q^2; q]_n^{1+n} \frac{-a q/e^2; q]_n}{[a^2 q^2/e^2; q]_n [a q; q]_n}}{[aq^{1+n}/a, q^{1+n}/a]},
\end{equation}

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which is a generalization of (1.2) and to which it reduces for \( e = iq^{-n}, c \to 0 \). (1.3) can be used with advantage for obtaining the identities of Rogers-Ramanujan type related to the modulus 13, not given, thus for. In the sequel, we also present a generalization of (1.1) along the lines of a similar result of Burchnall and Chaundy [9].

In § 4, we prove the \( q \)-analogue of the summation theorem for the nearly-poised \(_2\!F_1(1):

\[
(_2\!F_1)_{\nu}(2a, 1 + a, c, -N) = \frac{(2a - 2c)_{\nu}(-c)_{\nu}}{(1 + 2a - c)_{\nu}(-2c)_{\nu}},
\]

in the form

\[
(_2\!F_1)_{\nu}(a, aq/c, c^2q^{1-N}) = \frac{[a^2/q; q]_{\nu}[e^{-1}; q]_{\nu}[-aq/c; q]_{\nu}}{[a^2q/c; q]_{\nu}[e^{-2}; q]_{\nu}[-a/c; q]_{\nu}}.
\]

This result also gives the \( q \)-analogue of the summation theorem for nearly-poised \(_2\!F_1(1), \) viz.

\[
(_2\!F_1)_{\nu}(2a, c, -N; 1 + 2a - c, 1 + 2c - N) = \frac{(2a - 2c)_{\nu}(1+a-c)_{\nu}(-c)_{\nu}}{(1+2a-c)_{\nu}(a-c)_{\nu}(-2c)_{\nu}},
\]
on replacing \('a' \) by \(' -a' \) and then proceeding to the limits in the usual way.

In this connection it may be of interest to note that Andrews had obtained a \( q \)-analogue of (1.6) in the form

\[
(_2\!F_1)_{\nu}(a^2, c, q^{-N}, -a^2q^c; q; q) = \frac{[e^{-1}; q]_{\nu}[a^2e^{-2}; q]_{\nu}}{[a^2q/c; q]_{\nu}[e^{-2}; q]_{\nu}}
\times \frac{(1 + a^2e^{-1})(1 - a^2e^{-2}q^{1-N}) + a^2q^{-1}(1 - q^{-N})(1 + e^{-1})}{(1 - a^2e^{-2})(1 + a^2e^{-1})q}.
\]

However, in view of the identity

\[
(_2\!F_1)_{\nu}(a^2, c, q^{-N}, -a^2q^c; q; q) = \frac{(1+a)}{(1+a^2e^{-1})}(_2\!F_1)_{\nu}(a, -aq, c, q^{-N}; q; q)
\]

\[
- \frac{a(1 - aq^{-1})}{(1 + a^2q^{-1})}(_2\!F_1)_{\nu}(a^2q/c, c^2q^{1-N})
\]

\[
= \frac{(1+a)}{(1+a^2q^{-1})}(_2\!F_1)_{\nu}(a^2, c, q^{-N}; q; q)
\]

\[
- \frac{a(1 - aq^{-1})}{(1 + a^2q^{-1})}(_2\!F_1)_{\nu}(a^2q/c, c^2q^{1-N})
\]

\[
= \frac{(1+a)}{(1+a^2q^{-1})}(_2\!F_1)_{\nu}(a^2, c, q^{-N}; q; q)
\]

\[
- \frac{a(1 - aq^{-1})}{(1 + a^2q^{-1})}(_2\!F_1)_{\nu}(a^2q/c, c^2q^{1-N})
\]

\[
= \frac{(1+a)}{(1+a^2q^{-1})}(_2\!F_1)_{\nu}(a^2, c, q^{-N}; q; q)
\]

\[
- \frac{a(1 - aq^{-1})}{(1 + a^2q^{-1})}(_2\!F_1)_{\nu}(a^2q/c, c^2q^{1-N})
\]

\[
= \frac{(1+a)}{(1+a^2q^{-1})}(_2\!F_1)_{\nu}(a^2, c, q^{-N}; q; q)
\]

\[
- \frac{a(1 - aq^{-1})}{(1 + a^2q^{-1})}(_2\!F_1)_{\nu}(a^2q/c, c^2q^{1-N})
\]
The summation (1.5) has been further employed for obtaining two transformations connecting a terminating nearly-poised Saalschützian $\phi_5$ into a terminating well-poised $\phi_n$. It may be remarked that Bailey in his book [7] has mentioned four known transformations of nearly-poised hypergeometric series [7; 4.5 (3-6)]. The $q$-analogues of two of these [7; 4.5(3) and 4.5(6)] only were obtained by Bailey [8]. The above two transformations deduced by us are $q$-analogues of the remaining two transformations 4.5(4) and 4.5(5) given in Bailey's Tract [7]. We conclude the paper by obtaining the summation formula

\[
\alpha, q^\sqrt{\alpha}, -q^\sqrt{\alpha}, \frac{a q^3}{q d}, \frac{a q^3}{q d}, a q^3, -a q^3, \frac{q}{q d}, -a \sqrt{q d}, q^{N}; q; q
\]

which is a $q$-analogue of a summation theorem for well-poised $F_6(1)$ (different from the Dougall's theorem) due to Bailey [7; Ex. 8, p. 98] (see also [6]).

2. Definitions and notations. If we let,

\[ |q| < 1, \, [a; q]_n = (1 - a)(1 - a q) \cdots (1 - a q^{n-1}), \, [a; q]_0 = 1 \]

then we may define the basic hypergeometric series as

\[
P_{p+1q_{p+r}}[a_1, a_2, \cdots, a_{p+1}; q; x]
= \sum_{n=0}^{\infty} [a_1; q]_n \cdots [a_{p+1}; q]_n (-)^n x^n q^{r(2n)(n-1)} [q; q_n[b_1, q]_n \cdots [b_{p+r}; q]_n,
\]
where the series \( p+1 \phi_{p+r}(x) \) converges for all positive integral values of \( r \) and for all \( x \), except when \( r = 0 \), it converges only for \( |x| < 1 \).

Further, we shall denote by

\[
\prod_{s=0}^{\infty} \left( \prod_{j=0}^{\infty} \frac{1 - a_s q^j}{1 - b_s q^j} \right) = \prod_{s=1}^{\infty} \frac{1 - a_s q^j}{1 - b_s q^j} .
\]

3. For obtaining the transformations used by Singh [13] to obtain the \( q \)-analogues of identities of Cayley-Orr, we begin by setting \( b = ak \), \( c = -a^2 b^2 k^2 q^N \) and \( d = q^{-N} \) in (1.1) to obtain

\[
\phi_3 \left[ \frac{a^2, a^2 k^2, a^2 b^2 k^2 q^N, q^{-N}; q^2}{a^2 k^2 \sqrt{q}, -a^2 k^2 \sqrt{q}, a^2 b^2 k^2 q} \right] = \phi_3 \left[ \frac{a^2, a^2 b^2 k^2, a^2 b^4 k^4 q^N, q^{-2N}; q^2}{a^2 b^4 k^4 q, (abk)^2 q, a^4 k^2 q} \right] .
\]

Using the transformation \([12; 8.3]\)

\[
\phi_3 \left[ \frac{a, b, c, q^{-N}; q; q}{e, g, h} \right] = \left[ \frac{g; q}{c} \right]_{N} \left[ \frac{e g^q}{a b} ; q \right]_{N} \frac{e}{a}, \frac{e}{b}, c, q^{-N}; q; q \right] \left[ \frac{g; q}{c} \right]_{N} \left[ \frac{e g^q}{a b} ; q \right]_{N} \frac{e}{a}, \frac{e}{b}, c, q^{-N}; q; q \right] ,
\]

(where \( abc = e g h q^{N+1} \)) on both the sides of (3.2) (in the left hand side with \( a \rightarrow a^2 k^2 \), \( b \rightarrow -a^2 b^2 k^2 q^N \), \( c \rightarrow a^2 \), \( e \rightarrow (abk)^2 \), \( g \rightarrow a^2 k^2 \sqrt{q} \), \( h \rightarrow -a^2 k^2 \sqrt{q} \) and on the right hand side with \( q \rightarrow q^2 \), \( a \rightarrow a^2 \), \( b \rightarrow (abk)^2 q^N \), \( c \rightarrow a^2 k^2 \), \( e \rightarrow (abk)^2 q \), \( g \rightarrow (abk)^2 \), \( h \rightarrow a^4 k^2 q \) ), we get

\[
\phi_3 \left[ \frac{a^2, b^2, -q^{-N}, q^{-N}; q^2}{a^2 k^2 \sqrt{q}, b^2 k^2 q, (abk)^{-2} q^{-2N}, q^{-2N}; q^2} \right] = \phi_3 \left[ \frac{a^2, b^2, -q^{-N}, q^{-N}; q^2}{a^2 b^2 k^2 q, (abk)^{-2} q^{-2N}, q^{-2N}; q^2} \right] \phi_3 \left[ \frac{a^2, b^2, -q^{-N}, q^{-N}; q^2}{a^2 b^2 k^2 q, (abk)^{-2} q^{-2N}, q^{-2N}; q^2} \right] .
\]

Again, using the transformation (3.2) on the right hand side of (3.3) (with \( q \rightarrow q^2 \), \( a \rightarrow a^2 k^2 \), \( b \rightarrow 1/(abk)^2 q^{-2N} \), \( c \rightarrow b^2 k^2 q \), \( e \rightarrow b^{-2} q^{-2N} \), \( g \rightarrow a^{-2} q^{-2N} \), \( h \rightarrow a^2 b^2 k^2 q \) ), we have

\[
\phi_3 \left[ \frac{a^2, b^2, -q^{-N}, q^{-N}; q^2}{a^2 b^2 k^2, k^{-1} q^{-N+1}, -k^{-1} q^{-N+1}} \right] = \phi_3 \left[ \frac{a^2, b^2, -q^{-N}, q^{-N}; q^2}{a^2 b^2 k^2, k^{-1} q^{-N+1}, -k^{-1} q^{-N+1}} \right] \phi_3 \left[ \frac{a^2, b^2, -q^{-N}, q^{-N}; q^2}{a^2 b^2 k^2, k^{-1} q^{-N+1}, -k^{-1} q^{-N+1}} \right] .
\]
Once again using the transformation (3.2) on the right hand side of (3.4) (with $q \rightarrow q^2$, $a \rightarrow b^{\frac{1}{2}}$, $b \rightarrow a^{\frac{1}{2}}$, $c \rightarrow (abk)^2$, $e \rightarrow (abk)^2$, $g \rightarrow b^{-2}q^{2-n}$, $h \rightarrow a^{-2}q^{2-n}$), we get

\[
\phi_n \left[ a^2, b^2, -q^{-N}, q^{-N}; q; q \right]
\]

\[
= \left[ (abk)^2; q \right]_n [b^2; q^2]_n \left( (abk)^2; q^2 \right)_n
\]

\[
\phi_2 \left[ a^2q, b^aq, (abk)^{-2}q^{2-n}, q^{-2n}; q^2; q^2 \right]
\]

\[
\phi_3 \left[ (abk)^2, b^{-2}q^{-2}, a^{-2}q^{-2}, q^{-2n} \right].
\]

Equation (3.5) is one of the results proved by Singh [13]. All the other results due to Singh [13] may be deduced by applying the transformation (3.2) to (3.1) and (3.3) (see [1] for details).

Next, for proving (1.3), we start with the Watson’s transformation [14; 3.4.1.5]:

\[
\phi_n \left[ \frac{a}{c}, \frac{q\sqrt{a}}{e}, -q\sqrt{a}, c, d, e, f, q^{-n}; q; \frac{a^2q^{2+n}}{cdef} \right]
\]

\[
\phi_2 \left[ \frac{a}{e}, q\sqrt{a}, -q\sqrt{a}, c, e, -e, -q^{-n}, q^{-n}; q; \frac{a^2q^{2+n}}{cdef} \right]
\]

\[
\phi_3 \left[ \frac{a}{e}, q\sqrt{a}, -q\sqrt{a}, c, e, -e, -q^{-n}, q^{-n}; q; \frac{a^2q^{2+n}}{cdef} \right]
\]

(3.6)

Reversing the order of the series on the right hand side of (3.6), we obtain (on setting $f = -e$, $d = -q^{-n}$):

\[
\phi_n \left[ \frac{a}{c}, q\sqrt{a}, -q\sqrt{a}, c, e, -e, -q^{-n}, q^{-n}; q; \frac{a^2q^{2+n}}{cdef} \right]
\]

\[
\phi_2 \left[ \frac{a}{e}, q\sqrt{a}, -q\sqrt{a}, c, e, -e, -q^{-n}, q^{-n}; q; \frac{a^2q^{2+n}}{cdef} \right]
\]

\[
\phi_3 \left[ \frac{a}{e}, q\sqrt{a}, -q\sqrt{a}, c, e, -e, -q^{-n}, q^{-n}; q; \frac{a^2q^{2+n}}{cdef} \right]
\]

\[
(-aq)^n [aq; q]_n [e^z; q^z]_n \left[ \frac{a}{c} q^{1+n}; q \right]
\]

\[
e^{-n} [aq^{1+n}; q]_n \left[ \frac{a}{c} q^{1+n}; q \right]_n \left[ \frac{a^2q^2}{e^2}; q^2 \right]_n
\]

\[
\times \phi_2 \left[ \frac{aq}{e^2}, -q^{-2n}, \frac{cq^{-n}}{a}, q^{-n}; q; \frac{aq^{1-n}}{e}, -q^{1-n}, \frac{cq^{-2n}}{a} \right]
\]
\[
\sum_{n=0}^{\infty} \frac{[aq; q]_n [a^2q^2; q^2]_n [\alpha^2q^{2n}; q^n] (-aq^n)}{\left[-aq; q\right]_n \left[\frac{a^2q^2}{e^2}; q^2\right] \left[\frac{a^2q^{2n}}{e^{2n}}; q^n\right] e^{2n}} \times \phi_3 \frac{\left[\frac{aq}{e^2}, -q^{2n}a, -\frac{c^2}{a^2}q^{-2n}, q^{-2n}; q^2\right]}{\left[e^2, -c^2q^{-2n}, -c^2q^{1-2n}\right]},
\]

(using (1.1) with \(a^2 \rightarrow -aq/e^2, b^2 \rightarrow -q^{-2n}/a, c \rightarrow (c/a)q^{-n}, d \rightarrow q^{-n}\))

\[
\sum_{n=0}^{\infty} \frac{[aq; q]_n [a^2q^2; q^2]_n [\alpha^2q^{2n}; q^n] (-aq^n)}{\left[-aq; q\right]_n \left[\frac{a^2q^2}{e^2}; q^2\right] \left[\frac{a^2q^{2n}}{e^{2n}}; q^n\right] e^{2n}} \times \phi_3 \frac{\left[\frac{aq}{e^2}, -q^{2n}a, -\frac{c^2}{a^2}q^{-2n}, q^{-2n}; q^2\right]}{\left[e^2, -c^2q^{-2n}, -c^2q^{1-2n}\right]},
\]

(using (3.2) with \(q \rightarrow q^2, a \rightarrow -a^{-1}q^{-2n}, b \rightarrow c^2a^{-2}q^{-2n}, c \rightarrow -aqe^{-2}, e \rightarrow e^{-2}q^{2-2n}, g \rightarrow -ca^{-1}q^{-2n}, h \rightarrow -ca^{-1}q^{1-2n}\)).

Reversing the order of the series \(\phi_3\) in the right hand side of the above expression, we get (1.3).

Furthermore, using (1.3) we prove the following three transformations. These transformations on specialization yield identities of Rogers-Ramanujan type related to the moduli 11 and 13:

\[
[a^\prime q^\prime; q^\prime] \sum_{n=0}^{\infty} \frac{[aq; q]_n [a^2q^2; q^2]_n [\alpha^2q^{2n}; q^n] (-a^{-n}a^{4n}q^{2n(2n-2p)})}{[q; q^n][a^\prime q^\prime; q^n][\alpha^\prime q^{2n}; q^n]} \times \phi_3 \frac{[\alpha^2a^{-2}q^{4n-1}, -\alpha^2a^{-2}q^{4n}, -q^{-2n}, a^{-1}q^{-2n}; q^2]}{[q^{-2n}a^{-2}q^{4n-1}, a^{-1}q^{-4n}, -a^{-1}q^{-4n}]} \times \phi_3 \frac{[\alpha^2a^{-2}q^{4n-1}, a^{-1}q^{-4n}, -a^{-1}q^{-4n}]}{[q^{-2n}a^{-2}q^{4n-1}, a^{-1}q^{-4n}, -a^{-1}q^{-4n}]} \times \phi_3 \frac{[\alpha^2a^{-2}q^{4n-1}, -\alpha^2a^{-2}q^{4n}, -q^{-2n}, a^{-1}q^{-2n}; q^2]}{[a; q^n][1 - a^{-1}q^{-2n}][\alpha^{-1}q^{2n}; q^n]}, \]

(3.7)

and

\[
[a^\prime q^\prime; q^\prime] \sum_{n=0}^{\infty} \frac{[aq; q]_n [a^2q^2; q^2]_n [\alpha^2q^{2n}; q^n] (-a^{-n}a^{4n}q^{2n(2n-2p+4s)})}{[q; q^n][a^\prime q^\prime; q^n][\alpha^\prime q^{2n}; q^n]} \times \phi_3 \frac{[\alpha^2a^{-2}q^{4n-1}, -\alpha^2a^{-2}q^{4n}, -q^{-2n}, a^{-1}q^{-2n}; q^2]}{[q^{-2n}a^{-2}q^{4n-1}, a^{-1}q^{-4n}, -a^{-1}q^{-4n}]} \times \phi_3 \frac{[\alpha^2a^{-2}q^{4n-1}, a^{-1}q^{-4n}, -a^{-1}q^{-4n}]}{[q^{-2n}a^{-2}q^{4n-1}, a^{-1}q^{-4n}, -a^{-1}q^{-4n}]} \times \phi_3 \frac{[\alpha^2a^{-2}q^{4n-1}, -\alpha^2a^{-2}q^{4n}, -q^{-2n}, a^{-1}q^{-2n}; q^2]}{[a; q^n][1 - a^{-1}q^{-2n}][\alpha^{-1}q^{2n}; q^n] e^n}, \]

(3.8)
\[ [a'q^4; q^n] \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[aq; q]_{a+n+2r}(\ldots)q^{4n+6r}}{[q^4; q^n][q^n]} \cdot \]

(3.9)

\[ = \sum_{n=0}^{\infty} \frac{[aq; q]_n}{[q; q]_n} (1-a) [a'q^4; q^n] \cdot \]

Proof of (3.7). Setting \( e = iq^{-n} \) in (1.3), we get

\[ \sum_{r=0}^{n} [aq; q]_n (1-a) \frac{[aq; c]}{[q; q]_n} [a'q^4; q^n + r][q]_n \cdot \]

(3.10)

\[ \times \phi_n \left[ -ca^{-2}q^{4n-1}, -ca^{-2}q^{4n}, -q^{2n}, q^{2}, q^{2} \right]. \]

Now, in Bailey's transformation [14] choosing

\[ u_\alpha = \frac{1}{[q^n; q^n]}, \quad v_\alpha = \frac{1}{[a'q^4; q^n]} \cdot \]

and

\[ \delta_\alpha = \frac{[x; q^n][y; q^n]a^{4q^{4n-1}}}{a^{4q^{4n-1}}} \cdot \]

and evaluating \( \langle \beta_n \rangle, \langle \gamma_n \rangle \) by using (3.10) and following formula [15]

(3.11)

\[ \phi_n \left[ a, b; \frac{ec}{ab} \right] = \prod_{e} \left[ \frac{e}{a}, \frac{e}{b}; \frac{q}{a} \right] \left[ \frac{e}{a}, \frac{e}{ab} \right] \phi_n \left[ \frac{abq}{e}, 0 \right] \cdot \]

(where, either \( a, b, \) or \( c \) is of the form \( q^{-p} \), \( p \) a nonnegative integer. In case only \( c \) is of the form \( q^{-p} \) then (3.8) is valid only if \( |ec/ab| < 1 \), we get (3.7) on letting \( x, y \to \infty \).

Proof of (3.8). In (3.10), letting \( c \to \infty \), we have

\[ \frac{[aq; q]_n(-)^{-n}q^{-2n^2}}{[a'q^4; q^n][q^n]} \sum_{r=0}^{\infty} \frac{[aq; q]_n}{[q; q][a'q^4; q^n][q^n]} \cdot \]

(3.12)

\[ \sum_{r=0}^{n} \frac{[aq; q]_n(1-aq^r)(-)^{a}q^{(1/2)r(5r-1)}}{[q; q][a'q^4; q^n][q^n]} \cdot \]

(3.13)

\[ + \sum_{r=1}^{n} \frac{[aq; q]_n(1-q^r)(-)^{a}q^{(1/2)r(5r-1)}}{[q; q][a'q^4; q^n][q^n]} \cdot \]
Identities of Rogers-Ramanujan type related to the modulus 13.

(3.7) for \( c \to \infty \), \( a = 1 \) and \( p = 0 \) yields
\[ \frac{[q^a; q^b]}{[q; q^c]} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q; q^{1+2r}](-)^n q^{2n+2r^2+4nr+4r}}{[q^a; q^b][q^a]_n [q^a; q^a]_n} = \prod_{n=0,6,7 \text{ (mod 13)}} (1-q^n)^{-1}. \]

But, (3.7) for \( c \to \infty \), \( a = 1 \) and \( p = 1 \), gives
\[ \frac{[q^a; q^b]}{[q; q^c]} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q; q^{1+2r}](-)^n q^{2n+2r^2+4nr-4n+3r}}{[q^a; q^b][q^a]_n [q^a; q^a]_n} = \prod_{n=0,6,11 \text{ (mod 13)}} (1-q^n)^{-1}. \]

On the other hand, (3.7) for \( c \to \infty \), \( a = q \) and \( p = 0 \), reduces to
\[ \frac{[q^a; q^b]}{[q; q^c]} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q; q^{1+2r+1}](-)^n q^{2n+2r^2+4nr+4n+3r}}{[q^a; q^b][q^a]_n [q^a; q^a]_n} = \prod_{n=0,1,11 \text{ (mod 13)}} (1-q^n)^{-1}. \]

Next, on setting \( a = 1 \), (3.8) yields
\[ \frac{[q^a; q^b]}{[q; q^c]} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q; q^{1+2r+1}](-)^n q^{2n+2r^2+4nr+4n+3r}}{[q^a; q^b][q^a]_n [q^a; q^a]_n} = \prod_{n=0,5,11 \text{ (mod 13)}} (1-q^n)^{-1}. \]

Whereas, in (3.8) setting \( a = q^{-1} \) and using (3.15), we get
\[ \frac{[q^a; q^b]}{[q; q^c]} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q; q^{1+2r+1}](-)^n q^{2n+2r^2+4nr+4n+3r}}{[q^a; q^b][q^a]_n [q^a; q^a]_n} = \prod_{n=0,5,9 \text{ (mod 13)}} (1-q^n)^{-1}. \]

Lastly, in (3.8) setting \( a = q \) and using (3.19), we have
\[ \frac{[q^a; q^b]}{[q; q^c]} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \frac{[q; q^{1+2r+1}](-)^n q^{2n+2r^2+4nr+4n+3r}}{[q^a; q^b][q^a]_n [q^a; q^a]_n} = \prod_{n=0,4,9 \text{ (mod 13)}} (1-q^n)^{-1}. \]

Similarly the five identities of Rogers-Ramanujan type related to the modulus 11 due to Andrews [2] may be obtain from (3.7) and (3.9).

In view of the above applications of (1.1), it may be of interest to record a generalization of (1.1). In fact we prove that if \( a, b, e, f \) is of the form \( q^{-N} \), then
\[ \left[ a^z b^y f^z q^z \right] = \sum_{n=0}^{\infty} \frac{[a^z b^y q^z; q^z]_n}{[q^a; q^b][c^z; q^z]_n} \times \frac{[e; q]_n[f; q]_n e^n q^{n(n-1)/2}}{[-e; q]_n(ab)^n} \left[ a^z b^y q^z; q^z \right]_n \times \left[ a^z b^y q^z; q^z \right]_n \times \left[ a^z b^y q^z; q^z \right]_n. \]
(3.21) reduces to (1.1) for \( c = ab\sqrt{q} \).

We complete the proof of (3.21) by evaluating

\[
S = \sum_{r \geq 0} \frac{[a^2; q], [b^2; q], [c; q], [f; q], c^{2r}q^{-r(1/2)r_{r-1}}}{[q; q], [c^2; q], [-ef; q], (ab)^{2r}}
\]

(3.22)

\[
\times s_{\phi_2} \left[ q^{-r}, q^{1-r}, q^{-2r}; c^2; q^2 \right],
\]

in two different ways. Firstly, if we substitute the series definition of \( s_{\phi_2} \), change the order of summations and then diagonalize the two series, we get

\[
S = \sum_{r \geq 0} \frac{[a^2; q], [b^2; q], [c; q], [f; q], c^{2r}q^{-r(1/2)r_{r-1}}}{[q; q], [c^2; q], [-ef; q], (ab)^{2r}}
\]

\[
\times s_{\phi_2} \left[ q^{-r}, eq^r, fq^r; q, q \right].
\]

Summing the inner \( s_{\phi_2} \) by the \( q \)-analogue of Saalschütz summation theorem [14; 3.3.2.2], we get the left hand side of (3.21).

Secondly, we may rewrite (3.22) as

\[
S = \sum_{r \geq 0} \frac{[a^2; q], [b^2; q], [c; q], [f; q], c^{2r}q^{-r(1/2)r_{r-1}}}{[q; q], [c^2; q], [-ef; q], (ab)^{2r}}
\]

\[
\times s_{\phi_2} \left[ q^{-r}, q^{2r-1}, q^{2r}; q^2 \right],
\]

(3.23)

\[
\times s_{\phi_2} \left[ q^{-r}, q^{2r-1}, q^{2r}; q^2 \right] + \sum_{r \geq 0} \frac{[a^2; q], [b^2; q], [c; q], [f; q], c^{2r}q^{-r(1/2)r_{r-1}}}{[q; q], [c^2; q], [-ef; q], (ab)^{2r+2}}
\]

\[
\times s_{\phi_2} \left[ q^{-r}, q^{2r-1}, q^{2r}; q^2 \right].
\]

In the transformation

(3.24)

\[
\left[ b, c, q^{-N}; q; \frac{eqq^N}{bc} \right] = \frac{[g; q]_N}{[g; q]N} e_{\phi_2} \left[ \frac{e}{b}, c, q^{-N}; q \right],
\]

(3.25)

\[
\left[ b, c, q^{-N}; q; \frac{eqq^{-N}}{bc} \right] = \frac{[g; q]_N}{[g; q]N} e_{\phi_2} \left[ \frac{g}{b}, c, q^{-N}; q \right].
\]

(which is obtained from (3.2) by substituting for \( h \) and then letting \( a \rightarrow \infty \)), transforming the \( s_{\phi_2} \) on the left hand side by the same formula (3.24) (with \( e \) replaced by \( g \)), we get

(3.25)

\[
\left[ b, c, q^{-N}; q; \frac{eqq^{-N}}{bc} \right] = \frac{[g; q]_N}{[g; q]N} e_{\phi_2} \left[ \frac{g}{b}, c, q^{-N}; q \right].
\]
Now, using (3.25) for transforming the two \( \phi_2 \) series in (3.23) [to transform the first of the two \( \phi_2 \) in (3.23), we use (3.25) with \( q \rightarrow q^2 \), \( N = r, e \rightarrow q^{-ir}/a^2, e/b \rightarrow q^{-2r}/c^2, c \rightarrow q^{-2r}/b^2 \) and for transforming the second \( \phi_2 \) in (3.23), we use (3.25) with \( q \rightarrow q^7, N = r, e \rightarrow q^{-7r}/a^2, e/b \rightarrow q^{-7r}/c^2, c \rightarrow q^{-7r}/b^2 \), we get

\[
S = \sum_{r \geq 0} \frac{[a_r^2; q_{2r}][b_r^2; q_{2r}][e; q_{2r}][f; q_{2r}][q; q_{2r}]_{c^2}q^{2r}}{[q; q^2_{2r}][c^2; q_{2r}][e; q_{2r}][f; q_{2r}][q; q_{2r}]_{c^2}q^{2r}} \cdot \frac{\frac{a^2b^2q}{c^2}, q^{1-2r}, q^{-2r}; q^2; q^2}{\frac{b^2q, a^2q}{b^2q, a^2q}}
\]

Writing the series definition for inner \( \phi_2 \) and then interchanging the order of summations of the two series, we get the right hand side of (3.21).

If \( a \) or \( b \) is of the form \( q^{-N}, e = x, f = o \), then (3.21) yields

\[
\phi_2 \left[ a^2, b^2, x^2; q^2; \frac{c^2q}{a^2b^2} \right] = \sum_{n \geq 0} \frac{[a^2; q^2][b^2; q^2][c^2; q^2][e; q^2][f; q^2][q; q^2]_{c^2}q^{2n}}{[q; q^2][c^2; q^2][e^2; q^2][f^2; q^2][q; q^2]_{c^2}q^{2n}} \cdot \frac{\frac{a^2b^2q}{c^2}, q^{1-r}, q^{-r}; q^2; q^2}{\frac{b^2q, a^2q}{b^2q, a^2q}}
\]

In which replacing \( a, b, c \), by \( q^{-N}, q^h, q^r \) respectively and letting \( q \rightarrow 1 \), we only get a terminating version of the following formula of Burchnell and Chaundy [9; 5.7] (with \( x \) replaced by \( 1 - 2x \)):

\[
\sum_{n=0}^\infty (a)_n(b)_n \left( a + b - c + \frac{1}{2} \right)_n 4^x x^{2n}
\]

(3.26)

On the other hand to obtain the non-terminating version of (3.26), we start (3.21) by replacing \( e \) by \( -e, f = q^{-N} \) and then
replace $a, b, c, e$ by $q^a, q^b, q^c, q^e$ respectively and let $q \to 1$ to obtain

\[
\frac{a, b, e, -N;}{c, \frac{1}{2}(e-N), \frac{1}{2}(e-N+1)}
\]

(3.27)

\[
\sum_{n=0}^{\infty} \binom{n}{(a)_n(b)_n c, a + b - c + \frac{1}{2}} (-N)_n q^n
\]

\[
\times \sum_{n=0}^{\infty} [2a + 2n, 2b + 2n, -N + 2n; c + 2n, e - N + 2n]
\]

In (3.27) on replacing $e$ by $N(1 - 1/x)$ and letting $N \to \infty$, we get the non-terminating version of (3.26).

4. We begin this section by proving a $q$-analogue of the transformation due to Bailey [5, 2.5] in the form:

\[
\phi_4
\]

\[
\frac{a^2}{q}, a\sqrt{q}, -a\sqrt{q}, \frac{a}{b} \sqrt{q}; q; b^2 z
\]

(4.1)

\[
\frac{a}{\sqrt{q}}, \frac{a}{\sqrt{q}}, \frac{a}{\sqrt{q}}, \frac{a}{\sqrt{q}};
\]

\[
\frac{b}{\sqrt{q}}, -b\sqrt{q}, -b\sqrt{q}, -b\sqrt{q}; q, a^2 z
\]

provided $|a^2 z| < 1, |b^2 z| < 1$.

Proof of (4.1). Using the $q$-analogue of a nearly-poised summation theorem due to Bailey [8, (3)] in the form

\[
\phi_4
\]

\[
\frac{b}{q}, b\sqrt{q}, -b\sqrt{q}, \frac{b}{a\sqrt{q}}; q, q^{-n}; q, q
\]

\[
\frac{b}{q}, -b\sqrt{q}, -b\sqrt{q}, -b\sqrt{q}; q, q^{-n}; q, q
\]

\[
\frac{a^{-2n}; q, a^2 q; q, a^2 q^{-1-n}}{a^{-2n}; q, a^2 q; q, a^2 q^{-1-n}}
\]

the left hand side of (4.1) may be rewritten as:

\[
\sum_{n=0}^{\infty} \binom{a^2 b^{-2}; q, a^2 b^2 q^n}{q, q, q, q}
\]

\[
\frac{b}{q}, b\sqrt{q}, -b\sqrt{q}, \frac{b}{a\sqrt{q}}; q, q^{-n}; q, q
\]

\[
\frac{b}{q}, b\sqrt{q}, -b\sqrt{q}, \frac{b}{a\sqrt{q}}; q, q^{-n}; q, q
\]
\[
= \sum_{r=0}^{\infty} \frac{[b^r q^{-1}; q]_{r+1} [b^r q; q]_r}{[q; q]_r [b^r q^{-1}; q^n] [ab^r q; q]_r} \phi_{b} \left[ \frac{a^2}{b^2}; q; b^2 \right].
\]

Summing the \( \phi_q \), we get the right hand side of (4.1).

Augmenting parameters on both sides of \( \phi \)-series (4.1) by using \( q \)-beta transform \([10]\), we get

\[
Z, aV q, -aV q, -\beta, c, d; q; b^2 z
\]

(4.2)

\[
\phi_q \left[ \frac{a^2}{q}, aV q, -aV q, \frac{a}{bV q}, c, d; q; b^2 z \right] = \sum_{n=0}^{\infty} \frac{[b^n q^{-1}; q]_n [b^n q; q^n]_n}{[q; q]_n [b^n q^{-1}; q^n]_n} \times \frac{[b a^{n+1}; q]_{n+d} [c; q]_{n+d} [a^2 z^{n+1}; q]_{n+1}}{[abV q; q]_n [e; q]_{n+d} [f; q]_n} \phi_q \left[ \frac{a^2}{b^2}, cq^n, dq^n; q; b^2 z \right].
\]

In (4.2) setting \( d = q^{-N}, z = qb^{-2}, f = a^2 c b^{-2} e^{-1} q^{-1-N} \) and summing the inner \( \phi_q \) on the right hand side by the \( q \)-analogue of Saalschütz summation theorem, we get

\[
\phi_q \left[ \frac{a^2}{q}, aV q, -aV q, \frac{a}{bV q}, c, q^{-N}; q; q \right] = \left[ ec^{-1}; q \right]_{e} \left[ eb^2 a^{-2}; q \right]_{e} \left[ e; q \right]_{e} \left[ eb^2 c^{-1} a^{-2}; q \right]_{e} \left( 4.3 \right)
\]

\[
= \phi_q \left[ \frac{a^2}{q}, aV q, -aV q, \frac{a}{bV q}, c, q^{-N}; q; q \right] \times \phi_q \left[ \frac{b^2}{q}, bV q, -bV q, \frac{b}{aV q}, c, q^{-N}; q; q \right]
\]

(4.3) for \( N \to \infty \) yields the \( q \)-analogue of a non-terminating version of a transformation due to Bailey \([5; 2.51]\) in the form (with \( e \) replaced by \( a^2 e \)):

\[
\frac{[a^2 e; q]_{\infty}}{[a^2 e c^{-1}; q]_{\infty}} \phi_q \left[ \frac{a^2}{q}, aV q, -aV q, \frac{a}{bV q}, c; q; \frac{b^2 e}{c} \right] = \phi_q \left[ \frac{b^2}{q}, bV q, -bV q, \frac{b}{aV q}, c; q; \frac{a^2 e}{c} \right]
\]

(4.4)
On the other hand (4.3), for \( b = -1 \), reduces to the summation theorem:

\[
\phi_0\left[ \frac{a^2}{q}, a\sqrt{q}, c, q^{-N}; q; q \right] = \frac{e; q}{[e; q][ec^{-1}a^{-2}; q]_N}\left\{ 1 + \frac{(1 - c)(1 - q^{-N})a\sqrt{q}}{(a^2 - e)(1 - c/q^{-N})} \right\}.
\]

(4.5)

It may be worthwhile to remark that (4.5) could have been obtained directly by transforming the Saalschützian \( _4\phi_3 \) in (4.5) by using (3.2) with \( a \rightarrow a^2q^{-1}, b \rightarrow a\sqrt{q}, e \rightarrow a\sqrt{q}, g \rightarrow e \) and \( h \rightarrow (c/e)a^2q^{-1-N} \).

Now, if we specialize \( e = a^2/c \) in (4.5), we get (1.5) (on replacing \( a \) by \( a\sqrt{q} \)).

Next, using the summation theorem (1.5), we can prove the \( q \)-analogue of a transformation of Bailey [7; 4.5(4)] in the form if \( k = a^2q/bcd \) then

\[
\phi_b\left[ k, q\sqrt{k}, -q\sqrt{k}, aq, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{k^2}{a}q^{-N}; q; q \right] = \frac{k; q}{[k; q][k^2a^{-2}; q]_N}\left\{ -\frac{kq}{a\sqrt{k}}; q \right\}_N
\]

(4.6)

Further replacing \( "\sqrt{a}" \) by \( "-\sqrt{a}" \) in (4.6), we get the \( q \)-analogue of another result of Bailey [7; 4.5(5)].

**Proof of (4.6).** Using the \( q \)-analogue of Dougall's theorem [14; 3.3.1.1] in the form

\[
\phi_2\left[ k, q\sqrt{k}, -q\sqrt{k}, \frac{kb}{a}, \frac{kc}{a}, \frac{kd}{a}, aq^n, q^{-n}; q; q \right]
\]

\[
\phi_1\left[ -\sqrt{a}, k, q\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{k}{\sqrt{a}}, -\frac{kq}{\sqrt{a}}, k\sqrt{a}; q \right]
\]

\[
\times \phi_1\left[ -k\sqrt{q}, aq, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{k}{\sqrt{q}}, -\frac{kq}{\sqrt{q}}, k\sqrt{q}; q^{-n}, kq^{1+n} \right]
\]
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\[ \frac{[kg; q_n][b; q_n][c; q_n][d; q_n]}{[a; q_n][aq; b; q_n][aq; c; q_n][aq; d; q_n]}, \]

we may rewrite the left hand side of (4.6) (denoted by \( S \)) in the form:

\[
S = \sum_{n=0}^{\infty} \frac{[a; q_n][q^\sqrt{\alpha}; q_n]\left(\frac{\alpha}{k}; q\right)_n}{[q; q_n][q^\sqrt{\alpha}; q_n][a^k q^{-\alpha^2}; q_n][k^q; q_n]}
\]

\[
\times \phi^r_1 \left[ k, q^\sqrt{k}, -q^\sqrt{k}, \frac{kb}{a}, \frac{kc}{a}, \frac{kd}{a}, aq^n, q^{-n}; q \right]
\]

\[
= \sum_{r=0}^{\infty} \frac{[k; q_n][k^q; q_n]^r \left(\frac{\alpha}{k}; q\right)_r}{[q; q_r][k; q^r]^r \left(\frac{\alpha}{k}; q\right)_r \left(\frac{aq}{c}; q\right)_r \left(\frac{aq}{d}; q\right)_r \left(\frac{aq}{\bar{d}}; q\right)_r \left(\frac{\bar{a}q}{\bar{c}}; q\right)_r \left(\frac{\bar{a}q}{\bar{d}}; q\right)_r \left(\frac{\bar{a}q}{\bar{\bar{d}}}; q\right)_r \left(\frac{\bar{a}q}{\bar{\bar{c}}}; q\right)_r \left(\frac{\bar{a}q}{\bar{\bar{d}}}; q\right)_r}
\]

\[
\times \phi^r_1 \left[ [q^{-N_r}a, a^* q^r \frac{\alpha}{k}, q^{-N_r}, a^* q^{N_r+q}; q, q \right]
\]

summing the inner \( \phi^r_1 \) by (1.5), we get the desired result.
Lastly, we prove the formula (1.9).

**Proof of (1.9).** In view of the q-analogue of Dougall's theorem in the form:

\[
\phi^r_1 \left[ a, q^\sqrt{a}, -q^\sqrt{a}, c, dq^n, eq^{-n}, kq^n, q^{-n}; q \right]
\]

\[
\times \left[ aq; q \right] \left[ \frac{aq}{c}; q \right] \left[ \frac{aq}{e}; q \right] \left[ \frac{aq}{d}; q \right] \left[ \frac{\bar{a}q}{\bar{c}}; q \right] \left[ \frac{\bar{a}q}{\bar{e}}; q \right] \left[ \frac{\bar{a}q}{\bar{d}}; q \right] \left[ \frac{\bar{a}q}{\bar{\bar{c}}}; q \right] \left[ \frac{\bar{a}q}{\bar{\bar{d}}}; q \right] \left[ d; q \right] \left[ k; q \right] \left[ q^{-N}; q \right] \left[ q^n \right]
\]

(\( k = a^2 q(cde)^{-1} \)), we have

\[
\sum_{n=0}^{\infty} \frac{[cd; q] \left[ \frac{aq}{d}; q \right] \left[ \frac{aq}{\bar{e}}; q \right] \left[ \frac{aq}{\bar{d}}; q \right] \left[ d; q \right] \left[ k; q \right] \left[ q^{-N}; q \right] \left[ q^n \right]}{\left[ \frac{aq}{c}; q \right] \left[ \frac{aq}{\bar{e}}; q \right] \left[ \frac{aq}{\bar{d}}; q \right] \left[ d; q \right] \left[ k; q \right] \left[ q^{-N}; q \right] \left[ q^n \right]}
\]

\[
= \sum_{n=0}^{\infty} \frac{[cd; q] \left[ \frac{aq}{d}; q \right] \left[ \frac{aq}{\bar{e}}; q \right] \left[ \frac{aq}{\bar{d}}; q \right] \left[ d; q \right] \left[ k; q \right] \left[ q^{-N}; q \right] \left[ q^n \right]}{\left[ \frac{aq}{c}; q \right] \left[ \frac{aq}{\bar{e}}; q \right] \left[ \frac{aq}{\bar{d}}; q \right] \left[ d; q \right] \left[ k; q \right] \left[ q^{-N}; q \right] \left[ q^n \right]}
\]
In (4.7) taking \( c = a/d, \ k = aq/e \) and then summing the inner \( \phi_2 \) on the right hand side by the \( q \)-analogue of Saalschütz summation theorem [14; 3.3.2.2], we get (1.9).

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