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We obtain identities of Rogers-Ramanujan type related to the modulus 13. We also obtain the q-analogues of the nearly-poised summation theorems and use them for obtaining q-analogues of general transformations of nearly-poised hypergeometric series. We also discuss some important applications of the transformations obtained in this note.

Recently, Askey and Wilson [4] derived the transformation

$$(1.1)\qquad {}_4\phi_3\left[\begin{matrix}a^2, b^2, c, d; q; q\\ ab\sqrt{q}, \ -ab\sqrt{q}, \ -cd\end{matrix}\right] = {}_4\phi_3\left[\begin{matrix}a^2, b^2, c^2, d^2; q^2; q^2\\ a^2b^2q, \ -cd, \ -cdq\end{matrix}\right],$$

(provided a, b, c, or d is of the form q^{-N}, N a nonnegative integer). In an earlier paper [11] we have an alternative proof of (1.1). We begin this note by showing in § 3 that all the transformations proved by Singh [13], for obtaining the q-analogues of identities of the Cayley-Orr type, can be deduced from (1.1). We also show that (1.1) may be used effectively to prove the following transformation:

$$(1.2)\qquad \begin{aligned}&{}_8\phi_8\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, iq^{-n}, -iq^{-n}, -q^{-n}, q^{-n}, o; q; -aq^{1+4n}\\ \sqrt{a}, -\sqrt{a}, -iaq^{1+n}, iaq^{1+n}, -aq^{1+n}, aq^{1+n}\end{matrix}\right]\\ &\qquad = \frac{[aq; q]_{4n}}{[a^4q^{4+4n}; q^4]_n}\, {}_3\phi_2\left[\begin{matrix}-q^{-2n}, q^{-2n}, o; q^2; q^2\\ q^{-4n}/a, q^{1-4n}/a\end{matrix}\right],\end{aligned}$$

due to Andrews [2] which is his key result for obtaining the identities of the Rogers-Ramanujan type of modulus 11. In fact, we shall prove the transformation:

$$(1.3)\qquad \begin{aligned}&{}_8\phi_7\left[\begin{matrix}a, q\sqrt{a}, -q\sqrt{a}, c, e, -e, -q^{-n}, q^{-n}; q; \dfrac{a^2q^{2+2n}}{ce^2}\\ \sqrt{a}, -\sqrt{a}, aq/c, aq/e, -aq/e, -aq^{1+n}, aq^{1+n}\end{matrix}\right]\\ &\qquad = \frac{[a^2q^2; q^2]_n[-aq/e^2 : q]_{2n}}{[a^2q^2/c^2; q^2]_n[-aq; q]_{2n}}\\ &\qquad\quad \times \frac{[a^2q^2/c^2e^2; q^2]_n e^{2n}}{[a^2q^2/e^2; q^2]_n}\, {}_4\phi_3\left[\begin{matrix}\dfrac{ce^2}{a^2}q^{-2n-1}, \dfrac{ce^2}{a^2}q^{-2n}, e^2, q^{-2n}; q^2; q^2\\ \dfrac{e^2}{a^2}c^2q^{-2n}, \ -\dfrac{e^2}{a}q^{-2n}, \ -\dfrac{e^2}{a}q^{1-2n}\end{matrix}\right]\end{aligned}$$

which is a generalization of (1.2) and to which it reduces for $e = iq^{-n}$, $c \to 0$. (1.3) can be used with advantage for obtaining the identities of Rogers-Ramanujan type related to the modulus 13, not given, thus for. In the sequel, we also present a generalization of (1.1) along the lines of a similar result of Burchnall and Chaundy [9].

In § 4, we prove the q-analogue of the summation theorem for the nearly-poised ${}_4F_3(1)$:

$$(1.4)\qquad {}_4F_3\left[\begin{matrix}2a,\ 1+a,\ c,\ -N: \\ a,\ 1+2a-c,\ 1+2c-N\end{matrix}\right] = \frac{(2a-2c)_N(-c)_N}{(1+2a-c)_N(-2c)_N},$$

in the form

$$(1.5)\qquad {}_4\phi_3\left[\begin{matrix}a^2,\ aq,\ c,\ q^{-N};\ q;\ q \\ a,\ a^2q/c,\ c^2q^{1-N}\end{matrix}\right] = \frac{[a^2/c^2;\ q]_N[c^{-1};\ q]_N[-aq/c;\ q]_N}{[a^2q/c;\ q]_N[c^{-2};\ q]_N[-a/c;\ q]_N}.$$

This result also gives the q-analogue of the summation theorem for nearly-posed ${}_3F_2$, viz.

$$(1.6)\qquad {}_3F_2\left[\begin{matrix}2a,\ c,\ -N; \\ 1+2a-c,\ 1+2c-N\end{matrix}\right] = \frac{(2a-2c)_N(1+a-c)_N(-c)_N}{(1+2a-c)_N(a-c)_N(-2c)_N},$$

on replacing 'a' by '$-a$' and then proceeding to the limits in the usual way.

In this connection it may be of interest to note that Andrews had obtained a q-analogue of (1.6) in the form

$$(1.7)\qquad \begin{aligned} {}_4\phi_3&\left[\begin{matrix}a^2,\ c,\ q^{-N},\ -a^2q^2/c;\ q;\ q \\ a^2q/c,\ c^2q^{1-N},\ -a^2q/c\end{matrix}\right] = \left[\frac{[c^{-1};\ q]_N[a^2c^{-2};\ q]_N}{[a^2qc^{-1};q]_N[c^{-2};\ q]_N}\right] \\ &\times\frac{\{(1+a^2c^{-1})(1-a^2c^{-2}q^{1+N}) + a^2qc^{-1}(1-q^{N-1})(1+c^{-1})\}}{(1-a^2c^{-2})(1+a^2c^{-1}q)}. \end{aligned}$$

However, in view of the identity

$$(1.8)\qquad \begin{aligned} {}_4\phi_3&\left[\begin{matrix}a^2,\ c,\ q^{-N},\ -a^2q^2/c;\ q;\ q \\ a^2q/c,\ c^2q^{1-N},\ -a^2q/c\end{matrix}\right] = \frac{(1+a)}{(1+a^2c^{-1}q)}{}_4\phi_3\left[\begin{matrix}a^2,\ -aq,\ c,\ q^{-N};\ q;\ q \\ -a,\ a^2q/c,\ c^2q^{1-N}\end{matrix}\right] \\ &- \frac{a(1-aqc^{-1})}{(1+a^2qc^{-1})}{}_3\phi_2\left[\begin{matrix}a^2,\ c,\ q^{-N};\ q;\ q^2 \\ a^2q/c,\ c^2q^{1-N}\end{matrix}\right] \\ &= \frac{(1+a)}{(1+a^2qc^{-1})}{}_4\phi_3\left[\begin{matrix}a^2,\ -aq,\ c,\ q^{-N};\ q;\ q \\ -a,\ a^2q/c,\ c^2q^{1-N}\end{matrix}\right] \\ &- \frac{a(1-aqc^{-1})}{(1+a^2qc^{-1})}{}_3\phi_2\left[\begin{matrix}a^2,\ cq,\ q^{-N};\ q;\ q \\ a^2q/c,\ c^2q^{1-N}\end{matrix}\right] \end{aligned}$$

$$+\frac{aq(1-aqc^{-1})(1-a^2)(1-q^{-N})}{(1+a^2qc^{-1})(1-a^2qc^{-1})(1-c^2q^{1-N})}{}_3\phi_2\left[\begin{matrix}a^2q,\ cq,\ q^{1-N};\ q;\ q\\ a^2q^2/c,\ c^2q^{2-N}\end{matrix}\right],$$

(the last two series may be summed by q-analogue of Saalschütz summation theorem), it is not difficult to show the equivalence of the summation theorems (1.5) and (1.7). However, we prefer to stick to the form (1.5) as it has the added advantage that it gives the q-analogue of (1.4) as well as of (1.6) in an straight forward form.

The summation (1.5) has been further employed for obtaining two transformations connecting a terminating nearly-poised Saalschützian ${}_6\phi_5$ into a terminating well-poised ${}_{12}\phi_{11}$. It may be remarked that Bailey in his book [7] has mentioned four known transformations of nearly-poised hypergeometric series [7; 4.5 (3–6)]. The q-analogues of two of these [7; 4.5(3) and 4.5(6)] only were obtained by Bailey [8]. The above two transformations deduced by us are q-analogues of the remaining two transformations 4.5(4) and 4.5(5) given in Bailey's Tract [7]. We conclude the paper by obtaining the summation formula

$$(1.9)\quad {}_{10}\phi_9\left[\begin{matrix}a,\ q\sqrt{a},\ -q\sqrt{a},\ \dfrac{a}{d},\ \dfrac{a^2}{d}q^{1+N},\ \sqrt{d},\ -\sqrt{d},\ \sqrt{dq},\ -\sqrt{dq},\ q^{-N};\ q;\ q\\ \sqrt{a},\ -\sqrt{a},\ dq,\ \dfrac{d}{a}q^{-N},\ \dfrac{aq}{\sqrt{d}},\ -\dfrac{aq}{\sqrt{d}},\ a\sqrt{\dfrac{q}{d}},\ -a\sqrt{\dfrac{q}{d}},\ aq^{1+N}\end{matrix}\right]$$
$$=\frac{[aq;q]_N[a^2qd^{-2};q]_N}{[aqd^{-1};q]_N[a^2qd^{-1};q]_N},$$

which is a q-analogue of a summation theorem for well-poised ${}_7F_6(1)$ (different from the Dougall's theorem) due to Bailey [7; Ex. 8, p. 98] (see also [6]).

2. Definitions and notations.

If we let,

$$|q|<1,\ [a;q]_n=(1-a)(1-aq)\cdots(1-aq^{n-1}),\ [a;q]_0=1$$

and

$$[a;q]_\infty=\prod_{r=0}^{\infty}(1-aq^r),$$

then we may define the basic hypergeometric series as

$${}_{p+1}\phi_{p+r}\left[\begin{matrix}a_1,\ a_2,\ \cdots,\ a_{p+1};\ q;\ x\\ b_1,\ b_2,\ \cdots,\ b_{p+r}\end{matrix}\right]$$
$$=\sum_{n=0}^{\infty}\frac{[a_1;q]_n\cdots[a_{p+1};q]_n(-)^{rn}x^nq^{(r/2)n(n-1)}}{[q;q]_n[b_1;q]_n\cdots[b_{p+r};q]_n},$$

where the series ${}_{p+1}\phi_{p+r}(x)$ converges for all positive integral values of r and for all x, except when $r = 0$, it converges only for $|x|<1$.

Further, we shall denote by

$$\Pi\begin{bmatrix} a_1, a_2, \cdots, a_r; q \\ b_1, b_2, \cdots, b_s \end{bmatrix}, \text{ the infinite product } \prod_{j=0}^{\infty} \frac{(1-a_1q^j)\cdots(1-a_rq^j)}{(1-b_1q^j)\cdots(1-b_sq^j)} .$$

3. For obtaining the transformations used by Singh [13] to obtain the q-analogues of identities of Cayley-Orr, we begin by setting $b = ak$, $c = -a^2b^2k^2q^N$ and $d = q^{-N}$ in (1.1) to obtain

$$(3.1)\quad {}_4\phi_3\begin{bmatrix} a^2, a^2k^2, -a^2b^2k^2q^N, q^{-N}; q; q \\ a^2k\sqrt{q}, -a^2k\sqrt{q}, a^2b^2k^2 \end{bmatrix} = {}_4\phi_3\begin{bmatrix} a^2, a^2k^2, a^4b^4k^4q^{2N}, q^{-2N}; q^2; q^2 \\ (abk)^2q, (abk)^2, a^4k^2q \end{bmatrix}.$$

Using the transformation [12; 8.3]

$$(3.2)\quad {}_4\phi_3\begin{bmatrix} a, b, c, q^{-N}; q; q \\ e, g, h \end{bmatrix} = \frac{\left[\frac{g}{c}; q\right]_N \left[\frac{eg}{ab}; q\right]_N}{[g; q]_N \left[\frac{eg}{cab}; q\right]_N} {}_4\phi_3\begin{bmatrix} \frac{e}{a}, \frac{e}{b}, c, q^{-N}; q; q \\ e, \frac{c}{g}q^{1-N}, \frac{c}{h}q^{1-N} \end{bmatrix},$$

(where $abc = eghq^{N-1}$) on both the sides of (3.2) (in the left hand side with $a \to a^2k^2$, $b \to -a^2b^2k^2q^N$, $c \to a^2$, $e \to (abk)^2$, $g \to a^2k\sqrt{q}$, $h \to -a^2k\sqrt{q}$ and on the right hand side with $q \to q^2$, $a \to a^2$, $b \to (abk)^4q^{2N}$, $c \to a^2k^2$, $e \to (abk)^2q$, $g \to (abk)^2$, $h \to a^4k^2q$), we get

$$(3.3)\quad \begin{aligned} {}_4\phi_3\begin{bmatrix} a^2, b^2, -q^{-N}, q^{-N}; q; q \\ (abk)^2, \frac{1}{k}q^{-N+\frac{1}{2}}, -\frac{1}{k}q^{-N+\frac{1}{2}} \end{bmatrix} &= \frac{[a^2q; q^2]_N[b^2; q^2]_N k^{2N}}{[a^2b^2k^2; q^2]_N[k^2q; q^2]_N} \\ &\times {}_4\phi_3\begin{bmatrix} a^2k^2, b^2k^2q, (abk)^{-2}q^{1-2N}, q^{-2N}; q^2; q^2 \\ a^2b^2k^2q, \frac{1}{b^2}q^{?-2N}, \frac{1}{a^2}q^{1-2N} \end{bmatrix}. \end{aligned}$$

Again, using the transformation (3.2) on the right hand side of (3.3) (with $q \to q^2$, $a \to a^2k^2$, $b \to 1/(abk)^2\, q^{1-2N}$, $c \to b^2k^2q$, $e \to b^{-2}q^{2-2N}$, $g \to a^{-2}q^{1-2N}$, $h \to a^2b^2k^2q$), we have

$$(3.4)\quad \begin{aligned} {}_4\phi_3\begin{bmatrix} a^2, b^2, -q^{-N}, q^{-N}; q; q \\ a^2b^2k^2, k^{-1}q^{-N+\frac{1}{2}}, -k^{-1}q^{-N+\frac{1}{2}} \end{bmatrix} &= \frac{[a^2; q^2]_N[b^2; q^2]_N[(abkq)^2; q^2]_N k^{2N}}{[a^2b^2k^2; q]_{2N}[k^2q; q^2]_N} \\ &\times {}_4\phi_3\begin{bmatrix} b^2k^2q, a^2k^2q, (abk)^{-2}q^{2-2N}, q^{-2N}; q^2; q^2 \\ b^{-2}q^{2-2N}, a^{-2}q^{2-2N}, (abkq)^2 \end{bmatrix}. \end{aligned}$$

Once again using the transformation (3.2) on the right hand side of (3.4) (with $q \to q^2$, $a \to b^2k^2q$, $b \to a^2k^2q$, $c \to (abk)^2q^{-2-2N}$, $e \to (abkq)^2$, $g \to b^{-2}q^{2-2N}$, $h \to a^{-2}q^{2-2N}$), we get

(3.5)
$$
{}_4\phi_3\begin{bmatrix} a^2, b^2, -q^{-N}, q^{-N}; q; q \\ a^2b^2k^2, k^{-1}q^{-N+\frac{1}{2}}, -k^{-1}q^{-N+\frac{1}{2}} \end{bmatrix}
= \frac{[a^2k^2; q^2]_N[b^2k^2; q^2]_N[(abkq)^2; q^2]_N}{[(abk)^2; q]_{2N}[k^2q; q^2]_N}
{}_4\phi_3\begin{bmatrix} a^2q, b^2q, (abk)^{-2}q^{2-2N}, q^{-2N}; q^2; q^2 \\ (abkq)^2, b^{-2}k^{-2}q^{2-2N}, a^{-2}k^{-2}q^{2-2N} \end{bmatrix}.
$$

(3.5) is one of the results proved by Singh [13]. All the other results due to Singh [13] may be deduced by applying the transformation (3.2) to (3.1) and (3.3) (see [1] for details).

Next, for proving (1.3), we start with the Watson's transformation [14; 3.4.1.5]:

(3.6)
$$
{}_8\phi_7\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, c, d, e, f, q^{-n}; q; \dfrac{a^2q^{2+n}}{cdef} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{c}, \dfrac{aq}{d}, \dfrac{aq}{e}, \dfrac{aq}{f}, aq^{1+n} \end{bmatrix}
= \frac{[-aq; q]_n\left[\dfrac{aq}{ef}; q\right]_n}{\left[\dfrac{aq}{e}; q\right]_n\left[\dfrac{aq}{f}; q\right]_n}
{}_4\phi_3\begin{bmatrix} \dfrac{aq}{cd}, e, f, q^{-n}; q; q \\ \dfrac{ef}{a}q^{-n}, \dfrac{aq}{c}, \dfrac{aq}{d} \end{bmatrix}.
$$

Reversing the order of the series on the right hand side of (3.6), we obtain (on setting $f = -e$, $d = -q^{-n}$):

$$
{}_8\phi_7\begin{bmatrix} a, q\sqrt{a}, -q\sqrt{a}, c, e, -e, -q^{-n}, q^{-n}; q; \dfrac{a^2q^{2+2n}}{ce^2} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{c}, \dfrac{aq}{e}, -\dfrac{aq}{e}, -aq^{1+n}, aq^{1+n} \end{bmatrix}
= \frac{(-aq)^n[aq; q]_n[e^2; q^2]_n\left[-\dfrac{a}{c}q^{1+n}; q\right]_n}{e^{2n}[-aq^{1+n}; q]_n\left[\dfrac{aq}{c}; q\right]_n\left[\dfrac{a^2q^2}{e^2}; q^2\right]_n}
$$
$$
\times {}_4\phi_3\begin{bmatrix} -\dfrac{aq}{e^2}, -\dfrac{q^{-2n}}{a}, \dfrac{cq^{-n}}{a}, q^{-n}; q; q \\ \dfrac{q^{1-n}}{e}, -\dfrac{q^{1-n}}{e}, -\dfrac{cq^{-2n}}{a} \end{bmatrix}
$$

$$= \frac{[e^2; q^2]_n[a^2q^2; q^2]_n\left[-\frac{aq}{c}; q\right]_{2n}(-aq)^n}{[-aq; q]_{2n}\left[\frac{a^2q^2}{c^2}; q^2\right]_n\left[\frac{a^2q^2}{e^2}; q^2\right]_n e^{2n}}$$

$$\times\ {}_4\phi_3\left[\begin{matrix}-\frac{aq}{e^2}, & -\frac{q^{-2n}}{a}, & -\frac{c^2}{a^2}q^{-2n}, q^{-2n}; q^2; q^2\\ \frac{q^{2-2n}}{e^2}, & -\frac{c}{a}q^{-2n}, & -\frac{c}{a}q^{1-2n}\end{matrix}\right],$$

(using (1.1) with $a^2 \to -aq/e^2$, $b^2 \to -\, q^{-2n}/a$, $c \to (c/a)q^{-n}$, $d \to q^{-n}$)

$$= \frac{[e^2; q^2]_n[a^2q^2; q^2]_n\left[\frac{a^2q^2}{e^2c}; q\right]_{2n}}{\left[\frac{a^2q^2}{c^2}; q^2\right]_n\left[\frac{a^2q^2}{e^2}; q^2\right]_n[-aq; q]_{2n}}$$

$$\times\ {}_4\phi_3\left[\begin{matrix}-\frac{aq}{e^2}, & -\frac{aq^2}{e^2}, & \frac{a^2q^2}{e^2c^2}, & q^{-2n}; q^2; q^2\\ \frac{q^{2-2n}}{e^2}, & \frac{a^2q^3}{e^2c}, & \frac{a^2q^2}{e^2c}\end{matrix}\right],$$

(using (3.2) with $q \to q^2$, $a \to -a^{-1}q^{-2n}$, $b \to c^2a^{-2}q^{-2n}$, $c \to -aqe^{-2}$, $e \to e^{-2}q^{2-2n}$, $g \to -ca^{-1}q^{-2n}$, $h \to -ca^{-1}q^{1-2n}$).

Reversing the order of the series ${}_4\phi_3$ in the right hand side of the above expression, we get (1.3).

Furthermore, using (1.3) we prove the following three transformations. These transformations on specialization yield identities of Rogers-Ramanujan type related to the moduli 11 and 13:

(3.7)
$$[a^4q^4; q^4]_\infty \sum_{n=0}^{\infty} \frac{[aq; q]_{4n}[-a^2q^2c^{-2}; q^2]_{2n}(-)^n a^{4n}q^{2n(n-2p)}}{[q^4; q^4]_n[a^4q^4; q^4]_{2n}[a^4q^4c^{-4}; q^4]_n}$$
$$\times\ {}_4\phi_3\left[\begin{matrix}-ca^{-2}q^{-4n-1}, -ca^{-2}q^{-4n}, -q^{-2n}, q^{-2n}; q^2; q^2\\ -c^2a^{-2}q^{-4n}, a^{-1}q^{-4n}, a^{-1}q^{1-4n}\end{matrix}\right]$$
$$= \sum_{s=0}^{p} \frac{[q^{-4p}; q^4]_s(-)^s a^{4s}q^{2s(s+1)}}{[q^4; q^4]_s}$$
$$\times \sum_{n=0}^{\infty} \frac{[a; q]_n(1-aq^{2n})[c; q]_n a^{6n}q^{2n(3n-2p+4s)}}{[q; q]_n(1-a)\left[\frac{aq}{c}; q\right]_n c^n},$$

(3.8)
$$[a^4q^4; q^4]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[aq; q]_{4n+2r}(-)^n a^{4n+4r}q^{2n^2+3r^2+4nr+4n+3r}}{[q^2; q^2]_r[q^4; q^4]_n[a^4q^4; q^4]_{2n+2r}(1+a^2q^{4n+4r+2})}$$
$$= \sum_{n=0}^{\infty} \frac{[aq; q]_n(1-a^2q^{4n+2})(-)^n a^{6n}q^{(1/2)n(13n+9)}}{[q; q]_n}$$

and

$$(3.9)\qquad \begin{aligned}&[a^4q^4;q^4]_\infty \sum_{r=0}^{\infty}\sum_{n=0}^{\infty} \frac{[aq;q]_{4n+2r}(-)^r a^{4n+6r} q^{4n^2+8r^2+12nr+4n+5r}}{[q^2;q^2]_r[q^4;q^4]_n[a^4q^4;q^4]_{2n+2r}(1+a^2q^{4n+4r+2})}\\ &= \sum_{n=0}^{\infty}\frac{[aq;q]_n(1-a^2q^{4n+2})(-)^n a^{5n}q^{(1/2)n(11n+7)}}{[q;q]_n}.\end{aligned}$$

Proof of (3.7). Setting $e = iq^{-n}$ in (1.3), we get

$$(3.10)\qquad \begin{aligned}&\sum_{r=0}^{n}\frac{[a;q]_r(1-aq^{2r})[c;q]_r a^{2r}q^{2r^2}}{[q;q]_r(1-a)\left[\frac{aq}{c};q\right]_r[a^4q^4;q^4]_{n+r}[q^4;q^4]_{n-r}c^r}\\ &= \frac{[aq;q]_{4n}[-a^2q^2c^{-2};q^2]_{2n}(-)^n q^{-2n^2}}{[a^4q^4;q^4]_{2n}[a^4q^4c^{-4};q^4]_n[q^4;q^4]_n}\\ &\quad\times {}_4\phi_3\left[\begin{matrix}-ca^{-2}q^{-4n-1},\ -ca^{-2}q^{-4n},\ -q^{-2n},\ q^{-2n};\ q^2;\ q^2\\ -c^2a^{-2}q^{-4n},\ a^{-1}q^{-4n},\ a^{-1}q^{1-4n}\end{matrix}\right].\end{aligned}$$

Now, in Bailey's transformation [14] choosing

$$u_s = \frac{1}{[q^4;q^4]_s},\quad v_s = \frac{1}{[a^4q^4;q^4]_s},\quad \alpha_s = \frac{[a;q]_s(1-aq^{2s})[c;q]_s a^{2s}q^{2s^2}}{[q;q]_s(1-a)\left[\frac{aq}{c};q\right]_s c^s}$$

and

$$\delta_s = \frac{[x;q^4]_s[y;q^4]_s a^{4s}q^{4s(1-p)}}{x^s y^s}$$

and evaluating $\langle\beta_n\rangle$, $\langle\gamma_n\rangle$ by using (3.10) and following formula [15]

$$(3.11)\qquad {}_2\phi_1\left[\begin{matrix}a,\ b;\ q;\ \frac{ec}{ab}\\ e\end{matrix}\right] = \Pi\left[\begin{matrix}\frac{e}{a},\ \frac{e}{b};q\\ e,\ \frac{e}{ab}\end{matrix}\right]{}_3\phi_2\left[\begin{matrix}a,\ b,\ c;\ q;\ q\\ \frac{abq}{e},\ 0\end{matrix}\right],$$

(where, either a, b, or c is of the form q^{-p}, p a nonnegative integer. In case only c is of the form q^{-p} then (3.8) is valid only if $|ec/ab|<1$), we get (3.7) on letting $x, y \to \infty$.

Proof of (3.8). In (3.10), letting $c \to \infty$, we have

$$\begin{aligned}&\frac{[aq;q]_{4n}(-)^n q^{-2n^2}}{[a^4q^4;q^4]_{2n}[q^4;q^4]_n}\sum_{r=0}^{n}\frac{[q^{-4n};q^4]_r q^{r(r-4n)}}{[q^2;q^2]_r[a^{-1}q^{-4n};q]_{2r}a^{2r}}\\ &= \sum_{r=0}^{n}\frac{[a;q]_r(1-aq^{2r})(-)^r a^{2r}q^{(1/2)r(5r-1)}}{[q;q]_r(1-a)[a^4q^4;q^4]_{n+r}[q^4;q^4]_{n-r}}\\ &= \sum_{r=0}^{n}\frac{[aq;q]_r(-)^r a^{2r}q^{(1/2)r(5r+1)}}{[q;q]_r[a^4q^4;q^4]_{n+r}[q^4;q^4]_{n-r}}\\ &\quad + \sum_{r=1}^{n}\frac{[aq;q]_{r-1}(1-q^r)(-)^r a^{2r}q^{(1/2)r(5r-1)}}{[q;q]_r[a^4q^4;q^4]_{n+r}[q^4;q^4]_{n-r}}\end{aligned}$$

$$= \sum_{r=0}^{n} \frac{[aq;q]_r(-)^r a^{2r} q^{(1/2)r(5r+1)}\{(1-a^4q^{4n+4r+4}) - a^2q^{4r+2}(1-q^{4n-4r})\}}{[q;q]_r[a^4q^4;q^4]_{n+r+1}[q^4;q^4]_{n-r}}$$

or,

(3.12)
$$\sum_{r=0}^{n} \frac{[aq;q]_r(1-a^2q^{4r+2})(-)^r a^{2r} q^{(1/2)r(5r+1)}}{[q;q]_r[a^4q^8;q^4]_{n+r}[q^4;q^4]_{n-r}} = \frac{(1-a^4q^4)(-)^n q^{-2n^2}}{(1+a^2q^{4n+2})[a^4q^4;q^4]_{2n}} \sum_{r=0}^{n} \frac{[aq;q]_{4n-2r}(-)^r q^{r(r-1)}}{[q^2;q^2]_r[q^4;q^4]_{n-r}} .$$

Next, in Bailey's transformation [14] choosing

$$u_s = \frac{1}{[q^4;q^4]_s},\ v_s = \frac{1}{[a^4q^8;q^4]_s},\ \alpha_s = \frac{[aq;q]_s(1-a^2q^{4s+2})(-)^s q^{\frac{1}{2}(5s+1)} a^{2s}}{[q;q]_s}$$

$\delta_s = [x;q^4]_s[y;q^4]_s a^{4s}q^{8s}/x^sy^s$ and evaluating $\langle\beta_n\rangle$, $\langle\gamma_n\rangle$ by using (3.12) and the q-analogue of Gauss'summation theorem [14; 3.3.2.5], we get (3.8) on letting $x, y \to \infty$.

Proof of (3.9). In (3.10), setting $c \to 0$, we get

(3.13)
$$\frac{[aq;q]_{4n}}{[q^4;q^4]_n[a^4q^4;q^4]_{2n}} \sum_{r=0}^{n} \frac{[q^{-4n};q^4]_r q^{2r}}{[q^2;q^2]_r[a^{-1}q^{-4n};q]_{2r}} = \sum_{r=0}^{n} \frac{[a;q]_r(1-aq^{2r})(-)^r a^r q^{(1/2)r(3r-1)}}{[q;q]_r(1-a)[a^4q^4;q^4]_{n+r}[q^4;q^4]_{n-r}}$$

(3.13) may be rewritten in the following form (its proof follows on the lines of the proof of (3.12))

(3.14)
$$\sum_{r=0}^{n} \frac{[aq;q]_r(1-a^2q^{4r+2})(-a)^r q^{(1/2)r(3r-1)}}{[q;q]_r[a^4q^8;q^4]_{n+r}[q^4;q^4]_{n-r}} = \frac{(1-a^4q^4)}{(1+a^2q^{4n+2})[a^4q^4;q^4]_{2n}} \sum_{r=0}^{n} \frac{[aq;q]_{4n-2r}(-)^r a^{2r} q^{r(4n+1)}}{[q^2;q^2]_r[q^4;q^4]_{n-r}} .$$

However, in Bailey's transformation, choosing

$$u_s = \frac{1}{[q^4;q^4]_s},\ v_s = \frac{1}{[a^4q^8;q^4]_s},\ \alpha_s = \frac{[aq;q]_s(1-a^2q^{4s+2})(-a)^s q^{(1/2)s(3s-1)}}{[q;q]_s},$$

$$\delta_s = \frac{[x;q^4]_s[y;q^4]_s a^{4s} q^{8s}}{x^s y^s}$$

and evaluating $\langle\beta_n\rangle$, $\langle\gamma_n\rangle$ by using (3.14) and the q-analogue of Gauss' summation theorem [14; 3.3.2.5], we get (3.9) on letting $x, y \to \infty$.

Identities of Rogers-Ramanujan type related to the modulus 13. (3.7) for $c \to \infty$, $a = 1$ and $p = 0$ yields

$$(3.15)\qquad \frac{[q^4;q^4]_\infty}{[q;q]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q;q]_{4n+2r}(-)^n q^{2n^2+3r^2+4nr-r}}{[q^2;q^2]_r[q^4;q^4]_n[q^4;q^4]_{2n+2r}} = \prod_{n\neq 0,6,7\ (\mathrm{mod}\ 13)}(1-q^n)^{-1}.$$

But, (3.7) for $c\to\infty$, $a=1$ and $p=1$, gives

$$(3.16)\qquad \frac{[q^4;q^4]_\infty}{[q;q]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q;q]_{4n+2r}(-)^n q^{2n^2+3r^2+4nr-4n-5r}}{[q^2;q^2]_r[q^4;q^4]_n[q^4;q^4]_{2n+2r}} = \prod_{n\neq 0,2,11\ (\mathrm{mod}\ 13)}(1-q^n)^{-1}+\prod_{n\neq 0,3,10\ (\mathrm{mod}\ 13)}(1-q^n)^{-1}.$$

On the other hand, (3.7) for $c\to\infty$, $a=q$ and $p=0$, reduces to

$$(3.17)\qquad \frac{[q^4,q^4]_\infty}{[q;q]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q;q]_{4n+2r+1}(-)^n q^{2n^2+3r^2+4nr+4n+3r}}{[q^2;q^2]_r[q^4;q^4]_n[q^4;q^4]_{2n+2r+1}} = \prod_{n\neq 0,1,12\ (\mathrm{mod}\ 13)}(1-q^n)^{-1}.$$

Next, on setting $a=1$, (3.8) yields

$$(3.18)\qquad \frac{[q^4;q^4]_\infty}{[q;q]_\infty}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q;q]_{4n+2r}(-)^n q^{2n^2+3r^2+4nr+4n+3r}}{[q^2;q^2]_r[q^4;q^4]_n[q^4;q^4]_{2n+2r}(1+q^{4n+4r+2})} = \prod_{n\neq 0,2,11\ (\mathrm{mod}\ 13)}(1-q^n)^{-1}$$

Whereas, in (3.8) setting $a=q^{-1}$ and using (3.15), we get

$$(3.19)\qquad \frac{[q^4;q^4]_\infty}{[q;q]_\infty}\left\{1+\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q;q]_{4n+2r+1}(-)^n q^{2n^2+3r^2+4nr+8n+7r+4}}{[q^2;q^2]_r[q^4;q^4]_n[q^4;q^4]_{2n+2r+2}}\right\} = \prod_{n\neq 0,5,8\ (\mathrm{mod}\ 13)}(1-q^n)^{-1}.$$

Lastly, in (3.8) setting $a=q$ and using (3.19), we have

$$(3.20)\qquad \frac{[q^4;q^4]_\infty}{[q;q]_\infty}\left\{1+\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{[q;q]_{4n+2r+1}(-)^n q^{2n^2+3r^2+4nr+12n+11r+8}}{[q^2;q^2]_r[q^4;q^4]_n[q^4;q^4]_{2n+2r+2}}\right. = \prod_{n\neq 0,4,9\ (\mathrm{mod}\ 13)}(1-q^n)^{-1}.$$

Similarly the five identities of Rogers-Ramanujan type related to the modulus 11 due to Andrews [2] may be obtain from (3.7) and (3.9).

In view of the above applications of (1.1), it may be of interest to record a generalization of (1.1). In fact we prove that if a, b, e, f is of the form q^{-N}, then

$$(3.21)\qquad {}_4\phi_3\begin{bmatrix} a^2, b^2, e^2, f^2; q^2; \dfrac{c^2q}{a^2b^2} \\ c^2, -ef, -efq \end{bmatrix} = \sum_{n\geq 0}\frac{[a^2;q^2]_n[b^2;q^2]_n\left[\dfrac{a^2b^2q}{c^2};q^2\right]_n}{[q^2;q^2]_n[c^2;q^2]_{2n}} \times\frac{[e;q]_{2n}[f;q]_{2n}c^{4n}q^{n(1-2n)}}{[-ef;q]_{2n}(ab)^{4n}}\,{}_4\phi_3\begin{bmatrix} a^2q^{2n}, b^2q^{2n}, eq^{2n}, fq^{2n}; q; \dfrac{c^2q^{-2n}}{a^2b^2} \\ cq^{2n}, -cq^{2n}, -efq^{2n} \end{bmatrix}.$$

(3.21) reduces to (1.1) for $c = ab\sqrt{q}$.

We complete the proof of (3.21) by evaluating

$$S = \sum_{r\geq 0} \frac{[a^2; q^2]_r[b^2; q^2]_r[e; q]_r[f; q]_r c^{2r} q^{-(1/2)r(r-1)}}{[q; q]_r[c^2; q^2]_r[-ef; q]_r(ab)^{2r}} \times {}_3\phi_2\left[\begin{matrix} q^{-r}, q^{1-r}, \dfrac{q^{2-2r}}{c^2}; q^2; q^2 \\ a^{-2}q^{2-2r}, b^{-2}q^{2-2r} \end{matrix}\right], \tag{3.22}$$

in two different ways. Firstly, if we substitute the series definition of ${}_3\phi_2$, change the order of summations and then diagonalize the two series, we get

$$S = \sum_{r\geq 0} \frac{[a^2; q^2]_r[b^2; q^2]_r[e; q]_r[f; q]_r c^{2r} q^{-(1/2)r(r-1)}}{[q; q]_r[c^2; q^2]_r[-ef; q]_r(ab)^{2r}} {}_3\phi_2\left[\begin{matrix} q^{-r}, eq^r, fq^r; q; q \\ -q, -efq^r \end{matrix}\right].$$

Summing the inner ${}_3\phi_2$ by the q-analogue of Saalschütz summation theorem [14; 3.3.2.2], we get the left hand side of (3.21).

Secondly, we may rewrite (3.22) as

$$\begin{aligned} S = &\sum_{r\geq 0} \frac{[a^2; q^2]_{2r}[b^2; q^2]_{2r}[e; q]_{2r}[f; q]_{2r} c^{4r} q^{r(1-2r)}}{[q; q]_{2r}[c^2; q^2]_{2r}[-ef; q]_{2r}(ab)^{4r}} \\ &\times {}_3\phi_2\left[\begin{matrix} q^{-2r}, q^{1-2r}, \dfrac{q^{2-4r}}{c^2}; q^2; q^2 \\ \dfrac{q^{2-4r}}{a^2}, \dfrac{q^{2-4r}}{b^2} \end{matrix}\right] \\ &+ \sum_{r\geq 0} \frac{[a^2; q^2]_{2r+1}[b^2; q^2]_{2r+1}[e; q]_{2r+1}[f; q]_{2r+1} c^{4r+2} q^{-r(1+2r)}}{[q; q]_{2r+1}[c^2; q^2]_{2r+1}[-ef; q]_{2r+1}(ab)^{4r+2}} \\ &\times {}_3\phi_2\left[\begin{matrix} q^{-2r}, q^{-2r-1}, \dfrac{q^{-4r}}{c^2}; q^2; q^2 \\ \dfrac{q^{-4r}}{a^2}, \dfrac{q^{-4r}}{b^2} \end{matrix}\right]. \end{aligned} \tag{3.23}$$

In the transformation

$${}_3\phi_2\left[\begin{matrix} b, c, q^{-N}; q; \dfrac{egq^N}{bc} \\ e, g \end{matrix}\right] = \frac{\left[\dfrac{g}{c}; q\right]_N}{[g; q]_N} {}_3\phi_2\left[\begin{matrix} \dfrac{e}{b}, c, q^{-N}; q; q \\ e, \dfrac{c}{g}q^{1-N} \end{matrix}\right], \tag{3.24}$$

(which is obtained from (3.2) by substituting for h and then letting $a \to \infty$), transforming the ${}_3\phi_2$ on the left hand side by the same formula (3.24) (with e replaced by g), we get

$${}_3\phi_2\left[\begin{matrix} \dfrac{e}{b}, c, q^{-N}; q; q \\ e, \dfrac{c}{g}q^{1-N} \end{matrix}\right] = \frac{[g; q]_N\left[\dfrac{e}{c}; q\right]_N}{\left[\dfrac{g}{c}; q\right]_N[e; q]_N} {}_3\phi_2\left[\begin{matrix} \dfrac{g}{b}, c, q^{-N}; q; q \\ g, \dfrac{c}{e}q^{1-N} \end{matrix}\right]. \tag{3.25}$$

Now, using (3.25) for transforming the two ${}_3\phi_2$ series in (3.23) [to transform the first of the two ${}_3\phi_2$ in (3.23), we use (3.25) with $q \to q^2$, $N = r$, $e \to q^{2-4r}/a^2$, $e/b \to q^{2-4r}/c^2$, $c \to q^{1-2r}$, $c/g \to q^{-2r}/b^2$ and for transforming the second ${}_3\phi_2$ in (3.23), we use (3.25) with $q \to q^2$, $N = r$, $e \to q^{-4r}/a^2$, $e/b \to q^{-4r}/c^2$, $c \to q^{-2r-1}$, $c/g \to q^{-2r-2}/b^2$], we get

$$S = \sum_{r\geq 0} \frac{[a^2; q]_{2r}[b^2; q]_{2r}[e; q]_{2r}[f; q]_{2r}c^{4r}}{[q; q]_{2r}[c^2; q^2]_{2r}[-ef; q]_{2r}(ab)^{4r}} {}_3\phi_2\left[\begin{matrix}\frac{a^2b^2q}{c^2}, q^{1-2r}, q^{-2r}; q^2; q^2 \\ b^2q, a^2q\end{matrix}\right]$$

$$+ \sum_{r\geq 0} \frac{[a^2; q]_{2r+1}[b^2; q]_{2r+1}[e; q]_{2r+1}[f; q]_{2r+1}c^{4r+2}}{[q; q]_{2r+1}[c^2; q^2]_{2r+1}[-ef; q]_{2r+1}(ab)^{4r+2}}$$

$$\times {}_3\phi_2\left[\begin{matrix}\frac{a^2b^2q}{c^2}, q^{-2r-1}, q^{-2r}; q^2; q^2 \\ b^2q, a^2q\end{matrix}\right]$$

$$= \sum_{r\geq 0} \frac{[a^2; q]_r[b^2; q]_r[e; q]_r[f; q]_r c^{2r}}{[q; q]_r[c^2; q^2]_r[-ef; q]_r(ab)^{2r}} {}_3\phi_2\left[\begin{matrix}\frac{a^2b^2q}{c^2}, q^{1-r}, q^{-r}; q^2; q^2 \\ b^2q, a^2q\end{matrix}\right].$$

Writing the series definition for inner ${}_3\phi_2$ and then interchanging the order of summations of the two series, we get the right hand side of (3.21).

If a or b is of the form q^{-N}, $e = x$, $f = o$, then (3.21) yields

$$ {}_3\phi_2\left[\begin{matrix}a^2, b^2, x^2; q^2; \frac{c^2q}{a^2b^2} \\ c^2, o\end{matrix}\right] = \sum_{n\geq 0} \frac{[a^2; q^2]_n[b^2; q^2]_n\left[\frac{a^2b^2q}{c^2}; q^2\right]_n[x; q]_{2n}c^{4n}}{[q^2; q^2]_n[c^2; q^2]_{2n}(ab)^{4n}q^{n(2n-1)}}$$

$$\times {}_3\phi_2\left[\begin{matrix}a^2q^{2n}, b^2q^{2n}, xq^{2n}; q; \frac{c^2q^{-2n}}{a^2b^2} \\ cq^{2n}, -cq^{2n}\end{matrix}\right].$$

In which replacing $a, b, c,$ by q^{-N}, q^b, q^c respectively and letting $q \to 1$, we only get a terminating version of the following formula of Burchnell and Chaundy [9; 5.7] (with x replaced by $1 - 2x$):

$$ {}_2F_1\left[\begin{matrix}a, b; 4x(1-x) \\ c\end{matrix}\right]$$

$$(3.26) \qquad = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n\left(a+b-c+\frac{1}{2}\right)_n 4^n x^{2n}}{(1)_n(c)_{2n}} \times {}_2F_1\left[\begin{matrix}2a+2n, 2b+2n; x \\ c+2n\end{matrix}\right].$$

On the other hand to obtain the non-terminating version of (3.26), we start (3.21) by replacing e by $-e$, $f = q^{-N}$ and then replace a, b, c, e by q^a, q^b, q^c, q^e respectively and let $q \to 1$ to obtain

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix} a, b, e, -N; \\ c, \frac{1}{2}(e-N), \ \frac{1}{2}(e-N+1) \end{matrix}\right] \\
(3.27) \qquad &= \sum_{n=0}^{[N/2]} \frac{(a)_n(b)_n\left(a+b-c+\frac{1}{2}\right)_n(-N)_{2n}4n}{(1)_n(c)_{2n}(e-N)_{2n}} \\
&\quad \times {}_3F_2\left[\begin{matrix} 2a+2n, 2b+2n, -N+2n; \\ c+2n, e-N+2n \end{matrix}\right].
\end{aligned}
$$

In (3.27) on replacing e by $N(1-1/x)$ and letting $N \to \infty$, we get the non-terminating version of (3.26).

4. We begin this section by proving a q-analogue of the transformation due to Bailey [5; 2.5] in the form:

$$
\begin{aligned}
&{}_4\phi_3\left[\begin{matrix} \frac{a^2}{q}, a\sqrt{q}, -a\sqrt{q}, \frac{a}{b}\sqrt{q}; q; b^2z \\ \frac{a}{\sqrt{q}}, -\frac{a}{\sqrt{q}}, ab\sqrt{q} \end{matrix}\right] \\
(4.1) \qquad &= \frac{[a^2z; q]_\infty}{[b^2z; q]_\infty} {}_4\phi_3\left[\begin{matrix} \frac{b^2}{q}, b\sqrt{q}, -b\sqrt{q}, \frac{b}{a\sqrt{q}}; q; a^2z \\ \frac{b}{\sqrt{q}}, -\frac{b}{\sqrt{q}}, ab\sqrt{q} \end{matrix}\right]
\end{aligned}
$$

provided $|a^2z| < 1$, $|b^2z| < 1$.

Proof of (4.1). Using the q-analogue of a nearly-poised summation theorem due to Bailey [8; (3)] in the form

$$
\begin{aligned}
&{}_5\phi_4\left[\begin{matrix} \frac{b^2}{q}, b\sqrt{q}, -b\sqrt{q}, \frac{b}{a\sqrt{q}}, q^{-n}; q; q \\ \frac{b}{\sqrt{q}}, -\frac{b}{\sqrt{q}}, ab\sqrt{q}, \frac{b^2}{a^2}q^{1-n} \end{matrix}\right] \\
&= \frac{[a^2q^{-1}; q]_n[a^2q; q^2]_n\left[\frac{a}{b\sqrt{q}}; q\right]_n}{[a^2q^{-1}; q^2]_n[ab\sqrt{q}; q]_n[a^{+2}b^{-2}; q]_n},
\end{aligned}
$$

the left hand side of (4.1) may be rewritten as:

$$
\sum_{n=0}^{\infty} \frac{[a^2b^{-2}; q]_n b^{2n}z^n}{[q; q]_n} {}_5\phi_4\left[\begin{matrix} \frac{b^2}{q}, b\sqrt{q}, -b\sqrt{q}, \frac{b}{a\sqrt{q}}, q^{-n}; q; q \\ \frac{b}{\sqrt{q}}, -\frac{b}{\sqrt{q}}, ab\sqrt{q}, \frac{b^2}{a^2}q^{1-n} \end{matrix}\right]
$$

$$= \sum_{r=0}^{\infty} \frac{[b^2q^{-1}; q]_r[b^2q; q^2]_r \left[\frac{b}{a\sqrt{q}}; q\right]_r a^{2r}z^r}{[q; q]_r[b^2q^{-1}; q^2]_r[ab\sqrt{q}; q]_r} {}_1\phi_0\left[\begin{matrix}\frac{a^2}{b^2}; q; b^2z \\ -\end{matrix}\right].$$

Summing the ${}_1\phi_0$, we get the right hand side of (4.1).

Augmenting parameters on both sides of ϕ-series (4.1) by using q-beta transform [10], we get

$$(4.2)\quad {}_6\phi_5\left[\begin{matrix}\frac{a^2}{q},\ a\sqrt{q},\ -a\sqrt{q},\ \frac{a}{b\sqrt{q}},\ c,\ d;\ q;\ b^2z \\ \frac{a}{\sqrt{q}},\ -\frac{a}{\sqrt{q}},\ ab\sqrt{q},\ e,\ f\end{matrix}\right] = \sum_{n=0}^{\infty} \frac{[b^2q^{-1}; q]_n[b^2q; q^2]_n}{[q; q]_n[b^2q^{-1}; q^2]_n} \times \frac{\left[\frac{b}{a\sqrt{q}}; q\right]_n [c; q]_n[d; q]_n a^{2n}z^n}{[ab\sqrt{q}; q]_n[e; q]_n[f; q]_n} {}_3\phi_2\left[\begin{matrix}\frac{a^2}{b^2},\ cq^n,\ dq^n;\ q;\ b^2z \\ eq^n,\ fq^n\end{matrix}\right].$$

In (4.2) setting $d = q^{-N}$, $z = qb^{-2}$, $f = a^2cb^{-2}e^{-1}q^{1-N}$ and summing the inner ${}_3\phi_2$ on the right hand side by the q-analogue of Saalschütz summation theorem, we get

$$(4.3)\quad {}_6\phi_5\left[\begin{matrix}\frac{a^2}{q},\ a\sqrt{q},\ -a\sqrt{q},\ \frac{a}{b\sqrt{q}},\ c,\ q^{-N};\ q;\ q \\ \frac{a}{\sqrt{q}},\ -\frac{a}{\sqrt{q}},\ ab\sqrt{q},\ e,\ \frac{a^2cq^{1-N}}{eb^2}\end{matrix}\right] = \frac{[ec^{-1}; q]_N[eb^2a^{-2}; q]_N}{[e; q]_N[eb^2c^{-1}a^{-2}; q]_N} \times {}_6\phi_5\left[\begin{matrix}\frac{b^2}{q},\ b\sqrt{q},\ -b\sqrt{q},\ \frac{b}{a\sqrt{q}},\ c,\ q^{-N};\ q;\ q \\ \frac{b}{\sqrt{q}},\ -\frac{b}{\sqrt{q}},\ ab\sqrt{q},\ \frac{eb^2}{a^2},\ \frac{c}{e}q^{1-N}\end{matrix}\right].$$

(4.3) for $N\rightarrow\infty$ yields the q-analogue of a non-terminating version of a transformation due to Bailey [5; 2.51] in the form (with e replaced by a^2e):

$$(4.4)\quad \frac{[a^2e; q]_\infty}{[a^2ec^{-1}; q]_\infty} {}_5\phi_4\left[\begin{matrix}\frac{a^2}{q},\ a\sqrt{q},\ -a\sqrt{q},\ \frac{a}{b\sqrt{q}},\ c;\ q;\ \frac{b^2e}{c} \\ \frac{a}{\sqrt{q}},\ -\frac{a}{\sqrt{q}},\ ab\sqrt{q},\ a^2e\end{matrix}\right] = \frac{[b^2e; q]_\infty}{[b^2ec^{-1}; q]_\infty} {}_5\phi_4\left[\begin{matrix}\frac{b^2}{q},\ b\sqrt{q},\ -b\sqrt{q},\ \frac{b}{a\sqrt{q}},\ c;\ q;\ \frac{a^2e}{c} \\ \frac{b}{\sqrt{q}},\ -\frac{b}{\sqrt{q}},\ ab\sqrt{q},\ b^2e\end{matrix}\right].$$

On the other hand (4.3), for $b = -1$, reduces to the summation theorem:

$$
{}_4\phi_3\left[\begin{matrix}\dfrac{a^2}{q},\ a\sqrt{q},\ c,\ q^{-N};\ q;\ q\\[2mm] \dfrac{a}{\sqrt{q}},\ e,\ \dfrac{a^2c}{e}q^{1-N}\end{matrix}\right]
= \frac{\left[\dfrac{e}{c};\ q\right]_N [ea^{-2};\ q]_N}{[e;\ q]_N[ec^{-1}a^{-2};\ q]_N}\left\{1 + \frac{(1-c)(1-q^{-N})a\sqrt{q}}{(a^2-e)\left(1-\dfrac{c}{e}q^{1-N}\right)}\right\}. \tag{4.5}
$$

It may be worthwhile to remark that (4.5) could have been obtained directly by transforming the Saalschützian ${}_4\phi_3$ in (4.5) by using (3.2) with $a \to a^2q^{-1}$, $b \to a\sqrt{q}$, $e \to a/\sqrt{q}$, $g \to e$ and $h \to (c/e)a^2q^{1-N}$.

Now, if we specialize $e = a^2/c$ in (4.5), we get (1.5) (on replacing a by $a\sqrt{q}$).

Next, using the summation theorem (1.5), we can prove the q-analogue of a transformation of Bailey [7; 4.5 (4)] in the form if $k = a^2q/bcd$ then

$$
{}_6\phi_5\left[\begin{matrix}a,\ q\sqrt{a},\ b,\ c,\ d,\ q^{-N};\ q;\ q\\[1mm] \sqrt{a},\ \dfrac{aq}{b},\ \dfrac{aq}{c},\ \dfrac{aq}{d},\ \dfrac{a^2}{k^2}q^{1-N}\end{matrix}\right] = \frac{\left[\dfrac{k}{a};\ q\right]_N\left[\dfrac{k^2}{a};\ q\right]_N\left[-\dfrac{kq}{\sqrt{a}};\ q\right]_N}{[kq;\ q]_N[k^2a^{-2};\ q]_N\left[-\dfrac{k}{\sqrt{a}};\ q\right]_N}
$$

$$
\times\, {}_{12}\phi_{11}\left[\begin{matrix}k,\ q\sqrt{k},\ -q\sqrt{k},\ \dfrac{kb}{a},\ \dfrac{kc}{a},\ \dfrac{kd}{a},\ q\sqrt{a},\ -\sqrt{a},\ \sqrt{aq},\\ -\sqrt{aq},\ \dfrac{k^2}{a}q^N,\ q^{-N};\ q;\ q\\[2mm] \sqrt{k},\ -\sqrt{k},\ \dfrac{aq}{b},\ \dfrac{aq}{c},\ \dfrac{aq}{d},\ \dfrac{k}{\sqrt{a}},\ -\dfrac{kq}{\sqrt{a}},\ k\sqrt{\dfrac{q}{a}},\\ -k\sqrt{\dfrac{q}{a}},\ \dfrac{a}{k}q^{1-N},\ kq^{1+N}\end{matrix}\right]. \tag{4.6}
$$

Further replacing "$\sqrt{a}$" by "$-\sqrt{a}$" in (4.6), we get the q-analogue of another result of Bailey [7; 4.5(5)].

Proof of (4.6). Using the q-analogue of Dougall's theorem [14; 3.3.1.1] in the form

$$
{}_8\phi_7\left[\begin{matrix}k,\ q\sqrt{k},\ -q\sqrt{k},\ \dfrac{kb}{a},\ \dfrac{kc}{a},\ \dfrac{kd}{a},\ aq^n,\ q^{-n};\ q;\ q\\[2mm] \sqrt{k},\ -\sqrt{k},\ \dfrac{aq}{b},\ \dfrac{aq}{c},\ \dfrac{aq}{d},\ \dfrac{k}{a}q^{1-n},\ kq^{1+n}\end{matrix}\right]
$$

$$= \frac{[kq;q]_n[b;q]_n[c;q]_n[d;q]_n}{\left[\frac{a}{k};q\right]_n\left[\frac{aq}{b};q\right]_n\left[\frac{aq}{c};q\right]_n\left[\frac{aq}{d};q\right]_n},$$

we may rewrite the left hand side of (4.6) (denoted by S) in the form:

$$S = \sum_{n=0}^{N} \frac{[a;q]_n[q\sqrt{a};q]_n\left[\frac{a}{k};q\right]_n[q^{-N};q]_n q^n}{[q;q]_n[\sqrt{a};q]_n[a^2k^{-2}q^{1-N};q]_n[kq;q]_n}$$
$$\times {}_8\phi_7\left[\begin{matrix} k, q\sqrt{k}, -q\sqrt{k}, \frac{kb}{a}, \frac{kc}{a}, \frac{kd}{a}aq^n, q^{-n}; q; q \\ \sqrt{k}, -\sqrt{k}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{d}, \frac{k}{a}, q^{1-n}, kq^{1+n} \end{matrix}\right]$$
$$= \sum_{r=0}^{N} \frac{[k;q]_r[kq^2;q^2]_r\left[\frac{kb}{a};q\right]_r\left[\frac{kc}{a};q\right]_r\left[\frac{kd}{a};q\right]_r[a;q]_{2r}[q\sqrt{a};q]_r}{[q;q]_r[k;q^2]_r\left[\frac{aq}{b};q\right]_r\left[\frac{aq}{c};q\right]_r\left[\frac{aq}{d};q\right]_r[kq;q]_{2r}[\sqrt{a};q]_r}$$
$$\times \frac{[q^{-N}q]_r a^r q^r}{\left[\frac{a^2}{k^2}q^{1-N};q\right]_r k^r} \cdot {}_4\phi_3\left[\begin{matrix} aq^{2r}, \sqrt{a}\,q^{1+r}, \frac{a}{k}, q^{-N+r}; q; q \\ \sqrt{a}\,q^r, kq^{1+2r}, a^2k^{-2}q^{1-N+r} \end{matrix}\right],$$

summing the inner ${}_4\phi_3$ by (1.5), we get the desired result.

Lastly, we prove the formula (1.9).

Proof of (1.9). In view of the q-analogue of Dougall's theorem in the form:

$${}_8\phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, c, dq^n, eq^{-n}, kq^n, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{c}, \frac{a}{d}q^{1-n}, \frac{a}{e}q^{1+n}, \frac{a}{k}q^{1-n}, aq^{1+n} \end{matrix}\right]$$
$$= \frac{[aq;q]_n\left[\frac{aq}{ce};q\right]_{2n}\left[\frac{aq}{de};q\right]_n\left[\frac{aq}{e};q\right]_n\left[\frac{cd}{a};q\right]_n}{\left[\frac{aq}{c};q\right]_n\left[\frac{aq}{e};q\right]_{2n}\left[\frac{aq}{cde};q\right]_n\left[\frac{aq}{ce};q\right]_n\left[\frac{d}{a};q\right]_n c^n}$$

(where $k = a^2q(cde)^{-1}$), we have

$$\sum_{n=0}^{N} \frac{\left[\frac{cd}{a};q\right]_n\left[\frac{aq}{de};q\right]_n\left[\frac{aq}{ce};q\right]_{2n}[d;q]_n[k;q]_n[q^{-N};q]_n q^n}{[q;q]_n\left[\frac{q}{e};q\right]_n\left[\frac{aq}{c};q\right]_n\left[\frac{aq}{ce};q\right]_n[d^2a^{-2}q^{-N};q]_n\left[\frac{aq}{e};q\right]_{2n}c^n}$$

$$= \sum_{n=0}^{N} \frac{[d;q]_n[k;q]_n\left[\frac{aq}{cde};q\right]_n\left[\frac{d}{a};q\right]_n[q^{-N};q]_nq^n}{[q;q]_n\left[\frac{q}{e};q\right]_n[aq;q]_n\left[\frac{aq}{e};q\right]_n[d^2a^{-2}q^{-N};q]_n}$$

$$\times\ {}_8\phi_7\left[\begin{matrix}a,\ q\sqrt{a},\ -q\sqrt{a},\ c,\ dq^n,\ eq^{-n},\ kq^n,\ q^{-n};\ q;\ q\\ \sqrt{a},\ -\sqrt{a},\ \frac{aq}{c},\ \frac{a}{d}q^{1-n},\ \frac{a}{e}q^{1+n},\ \frac{a}{k}q^{1-n},\ aq^{1+n}\end{matrix}\right]$$

$$= \sum_{r=0}^{N} \frac{[a;q]_r[aq^2;q^2]_r[c;q]_r[d;q]_{2r}[k;q]_{2r}[q^{-N};q]_rq^r}{[q;q]_r[a;q^2]_r\left[\frac{aq}{c};q\right]_r\left[\frac{aq}{e};q\right]_{2r}[d^2a^{-2}q^{-N};q]_r[aq;q]_{2r}c^r}$$

$$(4.7)\qquad \times\ {}_5\phi_4\left[\begin{matrix}dq^{2r},\ kq^{2r},\ \frac{d}{a},\ \frac{k}{a},\ q^{-N+r};\ q;\ q\\ \frac{a}{e}q^{1+2r},\ aq^{1+2r},\ \frac{q}{e},\ d^2a^{-2}q^{-N+r}\end{matrix}\right].$$

In (4.7) taking $c = a/d$, $k = aq/e$ and then summing the inner ${}_3\phi_2$ on the right hand side by the q-analogue of Saalschütz summation theorem [14; 3.3.2.2], we get (1.9).

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