

Pacific Journal of Mathematics

**PTOLEMY'S INEQUALITY, CHORDAL METRIC,
MULTIPLICATIVE METRIC**

M. S. KLAMKIN AND A. MEIR

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Ptolemy's inequality in R^2 states: If A, B, C, D are vertices of a quadrilateral, then

$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD$$

with equality only $ABCD$ is a convex cyclic quadrilateral. A real normed linear vector space is called *ptolemaic* if

$$\|x - y\| \|z\| + \|y - z\| \|x\| \geq \|z - x\| \|y\|$$

for all x, y and z in the space and it is called *symmetric* if

$$\|\lambda x - y\| = \|x - \lambda y\|$$

for all unit vectors x, y and real λ . The equivalence of these two properties of a normed linear space is established and related results concerning distance functions in such spaces are proven.

Although Ptolemy's inequality is a useful tool and has often been applied (e.g., see [7]) it does not seem to be as widely known as would be desirable. Recently Apostol [1] gave an elegant proof of this inequality using complex numbers in the plane (see also [2], [4] and [5]) and extended the inequality to R^3 thereafter. Apostol used Ptolemy's inequality to show that the chordal distance

$$\chi(a, b) = \frac{|a - b|}{\sqrt{1 + |a|^2} \sqrt{1 + |b|^2}},$$

defined for pairs of complex numbers, satisfies the triangle inequality $\chi(a, b) + \chi(b, c) \geq \chi(a, c)$. In an earlier paper, Schoenberg [9], answering a problem raised by Blumenthal, proved the following: If S is a real, seminormed space which is ptolemaic then the seminorm is a norm which springs from an inner product. In this note we wish to treat these results from a different point of view. We provide simpler proofs for some of the earlier results and extend a recent result of Schattschneider [6], [8].

2. DEFINITION 2. Let X be real normed linear space with norm $\|\cdot\|$.

(i) X is called *ptolemaic* if for every $x, y, z \in X$ we have

$$(2.1) \quad \|x - y\| \cdot \|z\| + \|y - z\| \cdot \|x\| \geq \|x - z\| \cdot \|y\|.$$

(ii) X is called *symmetric* if for every $x, y \in X$ with $\|x\| =$

$\|y\| = 1$ and for all real λ we have

$$(2.2) \quad \|\lambda x - y\| = \|x - \lambda y\|.$$

3. **THEOREM 1.** *Let $(X, \|\cdot\|)$ be normed linear space. Then X is ptolemaic if and only if X is symmetric.*

Proof. Suppose X is symmetric. Let $x, y, z \in X$; we wish to prove (2.1). Clearly we may assume without loss of generality that $\|x\| > 0, \|y\| > 0, \|z\| > 0$. Now, by (2.2),

$$(3.1) \quad \|x - y\| = \left\| \frac{x}{\|x\|} \|y\| - \frac{y}{\|y\|} \|x\| \right\| = \|x\| \cdot \|y\| \left\| \frac{x}{\|x\|^2} - \frac{y}{\|y\|^2} \right\|$$

and similar relations hold for the pair of vectors x and z and for y and z . Thus (2.1) is equivalent to the triangle inequality for the vectors $x/\|x\|^2, y/\|y\|^2$ and $z/\|z\|^2$ in X . Conversely, if X is ptolemaic, then by [9], X is a real inner product space. (2.2) is then immediate, i.e., X is symmetric.

COROLLARIES. (i) R_n ($n = 1, 2, \dots$) is ptolemaic, for, it is clearly symmetric.

(ii) If X is a symmetric normed linear space, then the distance function

$$(3.2) \quad d(x, y) = \frac{\|x - y\|}{\|x\| \cdot \|y\|}$$

defined for $\|x\|, \|y\| > 0$, satisfies the triangle inequality. For, by (3.1), the triangle inequality for $d(x, y)$ follows from the triangle inequality in X .

We note that the proof of Ptolemy's inequality using the symmetry condition is, in R^n , equivalent to using inversion.

4. **The chordal metric.** We shall establish the following extension of Apostol's result mentioned in our introduction.

THEOREM 2. *Let $(X, \|\cdot\|)$ be a normed linear space. If X is symmetric, then the chordal distance given by*

$$(4.1) \quad \chi(x, y) = \frac{\|x - y\|}{(\alpha + \beta \|x\|^p)^{1/p} \cdot (\alpha + \beta \|y\|^p)^{1/p}}$$

is a metric for every $\alpha > 0, \beta \geq 0, p \geq 1$.

Proof. We only have to prove that χ satisfies the triangle inequality. Let x, y, z be arbitrary vectors in X . Then by the triangle inequality

$$(4.2) \quad \alpha \cdot (\|x - y\| + \|y - z\|)^p \geq \alpha \cdot \|x - z\|^p,$$

and since X is ptolemaic,

$$(4.3) \quad \beta \cdot (\|z\| \cdot \|x - y\| + \|x\| \cdot \|y - z\|)^p \geq \beta \cdot (\|y\| \cdot \|x - z\|)^p.$$

Adding (4.2) and (4.3) and using Minkowski's inequality, we get

$$\begin{aligned} & \|x - y\| \cdot (\alpha + \beta \|z\|^p)^{1/p} + \|y - z\| (\alpha + \beta \|x\|^p)^{1/p} \\ & \geq \|x - z\| (\alpha + \beta \|y\|^p)^{1/p} \end{aligned}$$

which proves that χ in (4.1) satisfies the triangle inequality.

5. A multiplicative metric. We shall establish the following extension of Schattschneider's result [8].

THEOREM 3. *Let $(X, \|\cdot\|)$ be a normed linear vector space. If X is symmetric, then the distance function defined by*

$$(5.1) \quad d(x, y) = \frac{\|x - y\|}{(\|x\|^p + \|y\|^p)^{1/p}}, \quad \text{if } \|x\| + \|y\| > 0$$

$$= 0, \quad \text{if } \|x\| + \|y\| = 0$$

is a metric for every $p \geq 1$.

Proof. Denote, for brevity, $\|x - y\| = a$, $(\|x\|^p + \|y\|^p)^{1/p} = a'$, $\|y - z\| = b$, $(\|y\|^p + \|z\|^p)^{1/p} = b'$ and $\|z - x\| = c$, $(\|z\|^p + \|x\|^p)^{1/p} = c'$. We only need to prove the triangle inequality for $d(x, y)$, i.e., with the above notation, that

$$(5.2) \quad \frac{a}{a'} + \frac{b}{b'} \geq \frac{c}{c'}.$$

By the triangle inequality of the norm,

$$(5.3) \quad a + b \geq c,$$

and by Ptolemy's inequality,

$$(5.4) \quad a\|z\| + b\|x\| \geq c\|y\|.$$

If $c' \geq a'$ and $c' \geq b'$, then (5.2) follows from (5.3). If $c' \leq a'$ and $c' \leq b'$, then, one sees easily, $\|y\|c' \geq \|z\|a'$ and $\|y\|c' \geq \|x\|b'$. Hence, (5.2) follows from (5.4). In the remaining case, c' is between a' and b' , say $a' < c' < b'$ or equivalently $\|x\| < \|y\| < \|z\|$. Now, using the inequality $u^p + v^p \geq 2^{1-p}(u + v)^p$ and then (5.3) and (5.4), we obtain

$$ab' + ba' \geq 2^{(1-p)/p}(a\|y\| + a\|z\| + b\|x\| + b\|y\|) \geq 2^{1/p} \cdot c \cdot \|y\|.$$

A simple calculation shows that, because of $\|x\| < \|y\| < \|z\|$, we have

$$2^{1/p} \cdot \|y\| \geq \frac{a'b'}{c'}.$$

Whence,

$$ab' + ba' \geq a'b' \frac{c}{c'}.$$

This proves (5.1) in the last case.

COROLLARY. *The multiplicative distance defined by (5.1) is a metric in \mathbf{R}^n ($n = 1, 2, \dots$) and, in fact, in any inner product space. (Schattschneider's metric corresponds to the special case $p = 1$ in \mathbf{R}^n .)*

We do not know whether or not $d(x, y)$ of (5.1) is a metric for every $p \geq 1/2$. We can prove that the triangle inequality holds if $p = 1/2$ and fails if $p = 1/4$.

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Received May 2, 1980.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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