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Let X denote a compact Riemann surface of genus g and suppose $P \in X$. The Weierstrass nongaps at P are those positive integers n such that there exists a meromorphic function on X which has a pole of order n at P and is holomorphic everywhere else. The Weierstrass semigroup at P, which we denote by $\Gamma(P)$, is the additive semigroup consisting of 0 and the nongaps. A point P is a Weierstrass point if there exists a nongap less than $g+1$ at P and is a normal Weierstrass point if $\Gamma(P)=\{0, g, g+2, g+3, g+4, \cdots\}$. We consider here the following problem: Given a collection $P_1, \cdots, P_n$ of points on X, describe the infinitesimal variations of complex structure on X which preserve the Weierstrass semigroups at $P_1, \cdots, P_n$. Our main result says, roughly speaking, that normal Weierstrass points deform as independently of each other as possible.

1. Let T_g denote the Teichmuller space for Teichmuller surfaces of genus $g > 1$ and let $\pi\colon V \to T_g$ denote the universal curve of genus g. Let $\mathscr{W}_k^r$, for $k = 2, \cdots, 2g-2$ and $r = 1, 2, \cdots$, denote the closed complex subspaces of V of Weierstrass points of the universal curve which were defined in [1], [2]. These spaces may be described set-theoretically as follows:

1) for $k \leqq g$, then
$|\mathscr{W}_k^r| = \{(t, P) \in V$: in the Weierstrass gap sequence at $P \in V_t$, there are at least r nongaps $\leqq k\}$.

2) For $k \geqq g$, then
$|\mathscr{W}_k^r| = \{(t, P) \in V$: in the Weierstrass gap sequence at $P \in V_t$, there are at least r gaps $> k\}$.

If P is a point on a compact Riemann surface, let $\Gamma(P)$ denote the semigroup of Weierstrass nongaps at P. Let Γ be an additive subsemigroup of the nonnegative integers and suppose Γ has g gaps.

DEFINITION. Put $\mathscr{W}(\Gamma) = \{(t, P) \in V\colon \Gamma(P) = \Gamma\}$.

It is not hard to see that $\mathscr{W}(\Gamma)$ is a (possibly empty) open subset of a finite intersection of $\mathscr{W}_k^r$'s. Thus $\mathscr{W}(\Gamma)$ has the structure of a complex analytic subspace of V.

Suppose $(t, P) \in V$ and put $X = V_t$. Let $c_1, \cdots, c_{3g-3}$ denote Patt's local coordinates [3] on T_g centered at t. These coordinates arise as variation parameters used in performing local variations of structure around $3g-3$ generally chosen points on X. Let z be a local coordinate on X centered at P. Let $1<\gamma_2<\cdots<\gamma_g$ denote the Weierstrass gaps at P and choose a basis $d\zeta_1, \cdots, d\zeta_g$ of holomorphic differentials on X such that $\mathrm{ord}_P\, d\zeta_j=\gamma_j-1$, $j=1, \cdots, g$. Write $d\zeta_j=f_j(z)dz$.

We next describe coordinates for $T_{(t,P)}(V)$, the tangent space to V at (t, P). We may view a tangent vector ξ to V at (t, P) as a C-homomorphism of local rings, $\xi\colon \mathcal{O}_{V,(t,P)}\to C[\varepsilon]/(\varepsilon^2)$. Then ξ is determined by its values on a set of local parameters of $\mathcal{O}_{V,(t,P)}$. So, if $\xi(t)=u\varepsilon$, $\xi(c_m)=b_m\varepsilon$, $m=1, \cdots, 3g-3$, then $(u, b_1, \cdots, b_{3g-3})$ serve as coordinates for $T_{(t,P)}(V)$. From [1], we have

PROPOSITION 1. *Suppose* $(t, P)\in \mathscr{W}_k^r$. *Then* $\xi\in T_{(t,P)}(\mathscr{W}_k^r)$ *if and only if all minors of order* $\rho-r+1$, *where* $\rho=\min(k, g)$, *of the matrix*

$$M(t, P)=\left[\begin{array}{c|c} f_j^{(i)}(0) & \varepsilon\Big(u\cdot f_j^{(i+1)}(0)+(i!)\sum_{m=1}^{3g-3} b_m\tau'_{P,i}(Q_m)\zeta'_j(Q_m)\Big) \\ i=0, \cdots, k-1 & i=0, \cdots, k-1 \\ j=1, \cdots, \rho-r & j=\rho-r+1, \cdots, g \end{array}\right]$$

vanish, where $\tau_{P,\ell}$ *is an elementary integral of the second kind on* X *with pole of order* $\ell+1$ *at* P *and where* $(Q_1, \cdots, Q_{3g-3})$ *is any point chosen from an open subset of* X^{3g-3}.

The vanishing of each minor of order $\rho-r+1$ of the above matrix gives a linear equation in $u, b_1, \cdots, b_{3g-3}$ and these equations cut out $T_{(t,P)}(\mathscr{W}_k^r)$ as a linear subspace of $T_{(t,P)}(V)$. These equations are of the form $E_n\colon a_n u+\sum_{m=1}^{3g-3}\beta_n(Q_m)b_m=0$, where the β_n are quadratic differentials, possibly with a pole at P, on X (arising from the products $d\tau_{P,\ell}d\zeta_j$ which occur in M) and where by $\beta_n(Q_m)$ we mean $\beta_n/(dz_m)^2(0)$ with z_m being a local coordinate on X centered at Q_m.

We note that if $(t, P)\in \mathscr{W}_k^r-\mathscr{W}_k^{r+1}$, then M will have a nonzero minor of order $\rho-r$, call it μ, which comes from the first $\rho-r$ columns. In order that all minors of order $\rho-r+1$ vanish, it is sufficient that only those minors of order $\rho-r+1$ which contain μ should vanish. This gives rise to $r(|k-g|+r)$ linear equations in $u, b_1, \cdots, b_{3g-3}$.

Now, since $\mathscr{W}(\Gamma(P))$ is an open subspace of an intersection of $\mathscr{W}_k^r$'s, the tangent space to $\mathscr{W}(\Gamma(P))$ at (t, P) is cut out as a linear subspace of $T_{(t,P)}(V)$ by a collection of linear equations of the above form. Put

$$Y(P)=T_{(t,P)}(\mathscr{W}(\Gamma(P)))\cap\{u=0\}\,.$$

We may view $Y(P)$ as a linear subspace of $T_t(T_g)$ via identifying the subspace $\{u = 0\}$ of $T_{(t,P)}(V)$ with $T_t(T_g)$. One may think of $Y(P)$ as being the space of infinitesimal deformations of X which leave the semigroup of Weierstrass nongaps at P unchanged.

In some special cases, we know the dimension of $Y(P)$. For example, the next proposition follows from the description of $T_{(t,p)}(\mathscr{W}^1_{g+\ell-1})$ in [2].

PROPOSITION 2. *Suppose that the Weierstrass gap sequence at $P \in X$ is $1, 2, \cdots, g-1, g+\ell$, where $1 \leqq \ell \leqq g-1$. Then $\dim Y(P) = 3g-3-\ell$. Thus, the family of infinitesimal deformations of X which leave the Weierstrass semigroup at P unchanged has codimension ℓ in the family of all infinitesimal deformations of X.*

DEFINITION. Suppose $P_1, \cdots, P_n$ are Weierstrass points on X. We will say that $P_1, \cdots, P_n$ are *infinitesimally in general position* if $Y(P_1), \cdots, Y(P_n)$ are in general position as linear subspaces of $T_t(T_g)$. (By this we mean that if $\{Y(P_i)\}_{i \in I}$ is any subset of $\{Y(P_1), \cdots, Y(P_n)\}$, then these linear subspaces of $T_t(T_g)$ may be cut out by a collection of linear equations such that any subset of these equations has maximum rank.)

To illustrate this concept, we will consider hyperelliptic curves of genera 2 and 3 in the next section.

2.

PROPOSITION 3. *The Weierstrass points of a compact Riemann surface of genus 2 are infinitesimally in general position.*

Proof. Suppose $X = V_t$ is the compact Riemann surface defined by the equation $y^2 = \prod_{i=1}^6 (x - a_i)$ and let P_i denote the branch point over a_i, $i = 1, \cdots, 6$. It is well-known that $P_1, \cdots, P_6$ are the Weirstrass points of x.

Note that $\mathscr{W}(\Gamma(P_i)) = \mathscr{W}^1_2$. We compute $T_{(t,P_i)}(\mathscr{W}^1_2)$. As basis for the holomorphic differentials on X, we take $d\zeta_1 = y^{-1}dx$, $d\zeta_2 = (x-a_i)y^{-1}dx$. Let $d\tau_{i,\ell}$ denote the normalized elementary differential of the second kind on X with pole of order $\ell + 2$ at P_i. Then the matrix $M(t, P_i)$ of § 1 is

$$M(t, P_i) = \begin{bmatrix} d_1 & \varepsilon \sum_{m=1}^{3} b_m \tau'_{i,0}(Q_m)\zeta'_2(Q_m) \\ 0 & \varepsilon\left(d_2 u + \sum_{m=1}^{3} b_m \tau'_{i,1}(Q_m)\zeta'_2(Q_m)\right) \end{bmatrix}$$

where d_1, d_2 are nonzero complex numbers. Then it is easy to see that $Y(P_i) = \{(b_1, b_2, b_3): \sum_{m=1}^{3} \beta_i(Q_m)b_m = 0\}$ where $\beta_i = d\tau_{i,1}d\zeta_2$. Now, since β_i has a pole at P_i and is finite at P_j, $i \neq j$, we have that $\beta_1, \cdots, \beta_6$ are linearly independent. Then, since (Q_1, Q_2, Q_3) is any point from an open subset of X^3, it is not hard to see that the hyperplanes $\sum_{m=1}^{3} \beta(Q_i)b_m = 0$, $i = 1, \cdots, 6$, will be in general position in $T_t(T_2)$.

The proposition is intuitively clear if one views deformations of a Riemann surface of genus 2 as arising from variations of the branch points of the two-sheeted convering of the Riemann sphere. One can certainly vary these branch points independently (although there is only a three-dimensional family of independent variations) and the Weierstrass points are just these branch points.

Now suppose X is a hyperelliptic Riemann surface of genus 3 and let $P_1, \cdots, P_8$ denote the Weierstrass points on X. By the above reasoning, one should expect that $Y(P_1), \cdots, Y(P_8)$ will be in general position as subspaces of the space of infinitesimal deformations of X *which preserve hyperellipticity*, and so these spaces will not be in general position as subspaces of the tangent space to T_3 at the point corresponding to X. Intuitively, if P_1 is to remain a hyperelliptic Weierstrass point, then a condition is imposed on $P_2, \cdots, P_8$, namely that they cannot deform to normal Weierstrass points.

We can indicate what should be occurring in the matrices $M(t, P_i)$ at two points on a hyperelliptic Riemann surface of genus 3. Suppose $X = V_t$ is defined by the equation $y^2 = \prod_{i=1}^{8} (x - a_i)$ and that P_1 and P_2 are the branch points over a_1 and a_2 respectively. Let $d\tau_{i,\ell}$ denote the normalized elementary differential of the second kind with pole of order $\ell + 2$ at P_1.

To compute $T_{(t,P_1)}(\mathscr{W}_2^1)$, take $d\zeta_{1,1} = y^{-1}dx$, $d\zeta_{1,2} = (x - a_1)y^{-1}dx$, $d\zeta_{1,3} = (x - a_1)^2y^{-1}dx$. Then the matrix $M(t, P_1)$ will be

$$\begin{bmatrix} d_1 & \varepsilon\Big(\sum_{m=1}^{6} b_m\tau'_{1,0}(Q_m)\zeta'_{1,2}(Q_m)\Big) & \varepsilon\Big(\sum_{m=1}^{6} b_m\tau'_{1,0}(Q_m)\zeta'_{1,3}(Q_m)\Big) \\ 0 & \varepsilon\Big(d_2u + \sum_{m=1}^{6} b_m\tau'_{1,1}(Q_m)\zeta'_{1,2}(Q_m)\Big) & \varepsilon\Big(\sum_{m=1}^{6} b_m\tau'_{1,1}(Q_m)\zeta'_{1,3}(Q_m)\Big) \end{bmatrix}$$

where d_1, d_2 are nonzero complex numbers. Hence $Y(P_1)$ will be $\{(b_1, \cdots, b_6): \sum_{m=1}^{6} \beta_{1,1}(Q_m)b_m = 0$ and $\sum_{m=1}^{6} \beta_{1,2}(Q_m)b_m = 0\}$, where $\beta_{1,1} = d\tau_{1,1}d\zeta_{1,2}$ and $\beta_{1,2} = d\tau_{1,1}d\zeta_{1,3}$.

Similarly, taking $d\zeta_{2,1} = y^{-1}dx$, $d\zeta_{2,2} = (x - a_2)y^{-1}dx$, $d\zeta_{2,3} = (x - a_2)^2y^{-1}dx$ and computing $M(t, P_2)$, one finds that $Y(P_2) = \{(b_1, \cdots, b_6): \sum_{m=1}^{6} \beta_{2,1}(Q_m)b_m = 0$ and $\sum_{m=1}^{6} \beta_{2,2}(Q_m)b_m = 0\}$, where $\beta_{2,1} = d\tau_{2,1}d\zeta_{2,2}$ and $\beta_{2,2} = d\tau_{2,1}d\zeta_{2,3}$.

Now $\beta_{1,1}$ has a simple pole at P_1 and $\beta_{2,1}$ has a simple pole at P_2, but $\beta_{1,2}$ and $\beta_{2,2}$ are finite everywhere. Indeed, *if* $(x - a_i)^{-2}dx$

were the *normalized* elementary differential of the second kind with pole of order 3 at P_i, then we would have $\beta_{1,2} = \beta_{2,2} = y^{-1}(dx)^2$ and the two 4-planes $Y(P_1)$ and $Y(P_2)$ would intersect in a 3-plane. Unfortunately, it seems that one must know these normalized elementary differentials of the second kind explicitly to be able to perfect this argument.

3. We now prove our main result, which generalizes Proposition 3.

THEOREM. *Let X be a compact Riemann surface of genus $g>1$. Suppose $P_1, \cdots, P_n$ are normal Weierstrass points on X and Q is any other Weierstrass point on X. Then $P_1, \cdots, P_n, Q$ are infinitesimally in general position.*

Proof. Let P denote a normal Weierstrass point on X. Then $\mathscr{W}(\Gamma(P)) = \mathscr{W}_g^1 - (\mathscr{W}_{g-1}^1 \cup \mathscr{W}_{g+1}^1)$. We compute $T_{(t,P)}(\mathscr{W}_g^1)$, where $X = V_t$.

With notation as in § 1, the matrix $M(t, P)$ will be

$$\left[\begin{array}{c|c} f_j^{(i)}(0) & \varepsilon\left(u\cdot f_g^{(i+1)}(0)+(i!)\sum_{m=1}^{3g-3} b_m \tau'_{P,i}(Q_m)\zeta'_g(Q_m)\right) \\ i = 0, \cdots, g-1 & i = 0, \cdots, g-1 \\ j = 1, \cdots, g-1 & \end{array}\right].$$

Let $\beta_P(Q_m)$ denote the coefficient of b_m in the equation $\det M(t, P) = 0$. Expanding $\det M(t, P)$ by its last row, it is easy to see that β_P will have a nonzero term in $d\tau_{P,g-1}d\zeta_g$ and thus the order of β_P at P will be $-(g-1+2)+g=-1$. Performing this analysis at $P_1, \cdots, P_n$, we see that if $\beta_{P_i}(Q_m)$ denotes the coefficient of b_m in the equation $\det M(t, P_i) = 0$, then β_{P_i} will have a pole at P_i and be finite elsewhere.

Now suppose that $Y(Q)$ is cut out as a subspace of $T_t(T_g)$ by linearly independent equations $E_1, \cdots, E_k$, where E_j is of the form $\sum_{m=1}^{3g-3} \alpha_j(Q_m)b_m = 0$, as in § 1. Then $\alpha_1, \cdots, \alpha_k$ will be linearly independent and will be finite at $P_1, \cdots, P_n$. Hence, $\beta_{P_1}, \cdots, \beta_{P_n}, \alpha_1, \cdots, \alpha_k$ are linearly independent quadratic differentials and $Y(P_1), \cdots, Y(P_n)$, $Y(Q)$ will be in general position as subspaces of $T_t(T_g)$.

COROLLARY. *The normal Weierstrass points on a compact Riemann surface are infinitesimally in general position.*

The question of when nonnormal Weierstrass points are infinitesimally in general position remains open.

REFERENCES

1. R. F. Lax, *Weierstrass points of the universal curve*, Math. Ann., **216** (1975), 35-42.
2. ———, *Gap sequences and moduli in genus four*, Math. Z., **175** (1980), 67-75.
3. C. Patt, *Variations of Teichmuller and Torelli surfaces*, J. Analyse Math., **11** (1963), 221-247.

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