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ON COMPACTIFICATIONS OF METRIC SPACES WITH TRANSFINITE DIMENSIONS

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In this paper we prove that every separable metric space X with transfinite dimension $\operatorname{Ind} X$ has metric compactification cX such that

$$\operatorname{Ind} cX = \operatorname{Ind} X, \quad \operatorname{ind} cX = \operatorname{ind} X, \quad D(cX) = D(X),$$

where $\operatorname{ind} X(\operatorname{Ind} X)$ denotes small (large) inductive transfinite dimension, and $D(X)$ denotes the transfinite D-dimension. More generally, let T be a set of invariants (ind, Ind, D). We consider the following problem:

Let $R \subseteqq T$ and X be a metric space. Does there exist a bicompactum (complete space) $cX \supset X$ such that

$$\mu(X) = \mu(cX) \quad \text{for} \quad \mu \in R.$$

When it is not so, we give counterexamples. We give also necessary and sufficient conditions of the existence of transfinite dimensions of separable metric space in terms of compactifications.

0. Introduction. In this paper we consider three transfinite invariants: $\operatorname{ind} X$, $\operatorname{Ind} X$, $D(X)$ where $\operatorname{ind} X$ (respectively, $\operatorname{Ind} X$) is small (respectively large) transfinite inductive dimension and $D(X)$ is D-dimension, see [3], Henderson.

DEFINITION 0.1. (a) $\operatorname{ind} X = -1 \Leftrightarrow X = \varnothing$.

(b) We assume that for every ordinal number $\alpha < \beta$ the class of spaces X with $\operatorname{ind} X \leqq \alpha$ is defined. Then $\operatorname{ind} X \leqq \beta$ if for every point $x \in X$ and a closed subset F, $x \notin F \subset X$, there exists a neighborhood O_x of x such that

$$O_x \subset X \backslash F$$
$$\operatorname{ind} FrO_x \leqq \alpha < \beta_1 .$$

We put $\operatorname{ind} X = \min\{\beta: \operatorname{ind} X \leqq \beta\}$.

(c) The dimension $\operatorname{ind}_x X$ of a space X in a point $x \in X \leqq \beta$ if there exists a base $\{O_\lambda, \lambda\}$ in this point, such that

$$\operatorname{ind} FrO_\lambda < \beta .$$

We put $\operatorname{ind}_x X = \min\{\beta: \operatorname{ind}_x X \leqq \beta\}$.

[1] FrA denotes the boundary of A.

DEFINITION 0.2. (a) $\operatorname{Ind} X = -1 \Leftrightarrow X = \varnothing$.

(b) Let for every ordinal number $\alpha < \beta$ the class of spaces X with $\operatorname{Ind} X \leqq \alpha$ is defined. Then, $\operatorname{Ind} X \leqq \beta$ if for every pair of disjoint closed subsets F and G there exists a partition[2] C between F and G such that

$$\operatorname{Ind} C \leqq \alpha < \beta .$$

We put

$$\operatorname{Ind} X = \min\{\beta\colon \operatorname{Ind} X \leqq \beta\} .$$

We note that the dimension ind we can also introduce using partitions, because if $x \in \bar{V} \subset X \backslash F$ and V is open, then $Fr\, V$ is a partition between x and F.

Let us introduce some notations. For every ordinal number β the equality $\beta = \alpha + n$ holds, where α is a limit number or 0, and $n = 0, 1, 2, \cdots$. Then we set

$$K(\beta) = n\ , \quad J(\beta) = \alpha\ .$$

DEFINITION 0.3. See [3], Henderson. We put $D(\phi) = -1$. If $X \neq \varnothing$ then $D(X)$ is the smallest ordinal number β such that there exists a collection of sets

$$\{A_\xi\colon 0 \leqq \xi \leqq \gamma\}$$

satisfying the following conditions:

(a) $X = \cup\{A_\xi\colon 0 \leqq \xi \leqq \gamma\}$.

(b) Every set A_ξ is closed and finite dimensional.

(c) For any $\delta \leqq \gamma$ the set

$$\cup\{A_\alpha\colon \delta \leqq \alpha \leqq \gamma\} \quad \text{is closed in} \quad X\ .$$

(d) $J(\beta) = \gamma$, $\operatorname{Ind} A_\gamma \leqq K(\beta)$.

(e) For any point $x \in X$ there exists the greatest number $\delta \leqq \gamma$ such that $x \in A_\delta$.

If there is no such number β we put $D(X) = \Delta$, where Δ is an abstract symbol such that $\Delta > \beta$ for any ordinal number β and $\Delta + \beta = \beta + \Delta = \beta \times \Delta = \Delta \times \beta = \Delta$.

If conditions (a)–(e) hold then equality (a) is called a β-D-representation of a space X.

Hence, for any space X the dimension $D(X)$ is either an ordinal number or the symbol Δ.

For any compact metric space X having dimension $\operatorname{Ind} X$

[2] A partition C in X between sets A and B is a closed set in X such that $X\backslash C = U \cup V$, $U \cap V = \varnothing$, $A \subset U$, $B \subset V$, for some open in X sets U and V.

$$\operatorname{Ind} X \leqq D(X)$$

(see [3], Henderson). For further results concerning D-dimension see [11], Luxemburg. Inequalities

$$\operatorname{ind} X \leqq \operatorname{Ind} X$$

and

(1) $$D(X) \leqq D(Y), \operatorname{ind} X \leqq \operatorname{ind} Y \quad \text{for} \quad X \subset Y$$

are evident for all topological spaces. However, the dimensions $\operatorname{ind} X$ and $\operatorname{Ind} X$ are not defined for every space. For example Hilbert cube I^ω has no inductive transfinite dimensions [5], p. 51 (Hurewicz and Wallman). Let

$$Z = \bigcup_{n=1}^{\infty} I^n$$

be the discrete union of Euclidean cubes I^n. Then obviously, $\operatorname{ind} Z = \omega_0$. However the dimension $\operatorname{Ind} X$ does not exist. But obviously

(2) if for a space X the dimension $\operatorname{Ind} X$ exists, then the dimension $\operatorname{ind} X$ also exists[3].

There are compact metric spaces X such that $\operatorname{ind} X < \operatorname{Ind} X < D(X)$, see [12], [13], Luxemburg.

In what follows all spaces are assumed to be metric and all mappings to be continuous if otherwise is not stated.

A space X we call finite dimensional if $\dim X \leqq n$ for some $n = -1, 0, 1, 2, \cdots$, where $\dim X$ is the covering dimension[4]. For finite dimensional metric space X

(3) $$D(X) = \operatorname{Ind} X = \dim X$$

and

(4) $$D(X) = \operatorname{Ind} X = \operatorname{ind} X = \dim X$$

if X is separable.

DEFINITION 0.4. For any space Z we denote by $P(Z)$ a closed subset such that $Z \backslash P(Z)$ is the union of all finite dimensional sets, open in X. For spaces X and Y with $\operatorname{diam} Y < \infty$ we denote by

[3] For compact metric space X the dimension $\operatorname{ind} X$ exists if and only if the dimension $\operatorname{Ind} X$ exists (see [22], Smirnov).

[4] $\dim X \leqq n$ if for any open covering U of a space X there exists an open refinement V of U having the order $\leqq n+1$, i.e., the intersection of any $(n+2)$ elements of V is empty.

$C(X, Y)$ the space of all continuous mappings $f: X \to Y$ with the metric $d(f_1, f_2) = \sup\{\delta(f_1(x), f_2(x)): x \in X\}$, where δ is the metric on Y. We note that the space $C(X, Y)$ is complete if Y is complete. We consider a Hilbert cube I^ω to be the set of all sequences $\{x_i\}$, $0 \leqq x_i \leqq 1$, with the metric, defined by the equality

$$d(\{x_i\}, \{y_i\}) = \sum_{i=1}^{\infty} |x_i - y_i| \cdot 2^{-i}, \quad i = 1, 2, \cdots .$$

The number x_i is called the i-coordinate of a point $x = \{x_i\} \in I^\omega$.

1. On homeomorph mappings to the Hilbert cube.

THEOREM 1.1. (*The Compactification Theorem*). *Let X be a separable space and a fixed countable system of closed sets L_i, ($i = 1, 2, \cdots$) such that dimensions* $\operatorname{Ind} L_i$ *exist.*

Then the set Ψ of all homeomorphisms $f: X \to I^\omega$ of the space X to the Hilbert cube I^ω such that for each i the equalities

(a) $\operatorname{Ind} L_i = \operatorname{Ind} \overline{f(L_i)}$

(b) $\operatorname{ind} L_i = \operatorname{ind} \overline{f(L_i)}$

(c) $D(L_i) = D(\overline{f(L_i)})$

(d) $f(P(X)) = \overline{Pf(X)}$

are satisfied contains an everywhere dense set of type G_δ in the space $C(X, I^\omega)$.

In the case when the dimensions $\operatorname{Ind} C_i$ are finite, this theorem has been proved in [8], Kuratowski. See also [14], Luxemburg for infinite dimensional case.

COROLLARY 1.1. *For any separable space X having dimension* $\operatorname{Ind} X$ *there exists a separable compactification X such that*

$$\operatorname{Ind} cX = \operatorname{Ind} X, \operatorname{ind} cX = \operatorname{ind} X, D(cX) = D(X), P(cX) = P(X) .$$

Proof. It is sufficient to put in Theorem 1.1

$$L_i = X , \quad cX = \overline{f(L_i)} .$$

For the proof of this theorem we need some preliminary lemmas.

LEMMA 1.1. *Let A and B be two fixed closed sets in the space X and put*

$$C = A \cap B .$$

Then for any $n = 1, 2, \cdots$ the set ϕ_n of mappings f from the space

X to a compact space Z such that

$$\overline{f(A)} \cap \overline{f(B)} \backslash O_{1/n}\overline{f(C)} = \varnothing^{5}$$

is open in $C(X, Z)$.

Proof. We suppose that the set ϕ_n is not open in $C(X, Z)$. Then for some $f \in \phi_n$ there exists a sequence of mappings $g_k \in C(X, Z)\backslash\phi_n$, $k = 1, 2, \cdots$, such that

$$\lim_{k\to\infty} g_k = f \tag{2}$$

and

$$\overline{g_k(A)} \cap \overline{g_k(B)} \backslash O_{1/n}\overline{g_k(C)} \neq \varnothing .$$

Therefore for any k there eixsts a pair of points $a_k \in A$, $b_k \in B$, such that

$$\begin{aligned} &\delta(g_k(a_k), g_k(b_k)) < \frac{1}{k}, \ \delta(g_k(a_k), \overline{g_k(C)}) \geqq \frac{1}{n} - \frac{1}{k} \\ &\delta(g_k(b_k), \overline{g_k(C)} \geqq \frac{1}{n} - \frac{1}{k} \end{aligned} \tag{3}$$

where δ denotes a metric on Z. Since Z is a compactum there exists a point $p \in Z$ such that

$$p = \lim_{k_i\to\infty} g_{k_i}(a_{k_i}) = \lim_{k_i\to\infty} g_{k_i}(b_{k_i}) \tag{4}$$

for some subsequence of integers k_i. Since $\lim_{k_i\to\infty} g_{k_i} = f$ we have

$$p = \lim_{k_i\to\infty} f(a_{k_i}) = \lim_{k_1\to\infty} f(b_{k_i}) . \tag{5}$$

Consequently

$$p \in \overline{f(A)} \cap \overline{f(B)} . \tag{6}$$

Let us show that

$$\delta(p, \overline{f(C)}) \geqq \frac{1}{n} \tag{7}$$

where $\delta(p, \overline{f(C)} = \inf\{\delta(p, x) : x \in \overline{f(C)}\}$.

We suppose on the contrary. Then there eixsts a point $q \in C$, satisfying the condition:

$$\delta(f(q), p) < \frac{1}{n} - \varepsilon \tag{8}$$

[5] $O_\varepsilon \mathscr{M}$ denotes ε-neighborhood of the set $\mathscr{M}$.

for some $\varepsilon > 0$. By virtue of (2), (5) we can find an integer k_i such that

$$(9) \qquad d(g_{k_i}, f) < \frac{\varepsilon}{4}, \delta(p, g_{k_i}(a_{k_i})) < \frac{\varepsilon}{4}, \frac{1}{k_i} < \frac{\varepsilon}{2}.$$

From (3), (8), (9) it follows that

$$\begin{aligned}\frac{1}{n} - \frac{1}{k_i} &\leqq \delta(g_{k_i}(a_{k_i}), g_{k_i}(q)) \\ &\leqq \delta(g_{k_i}(a_{k_i}), p) + \delta(p, f(q)) + \delta(f(q), g_{k_i}(q)) \\ &< \frac{\varepsilon}{4} + \frac{1}{n} - \varepsilon + \frac{\varepsilon}{4} \\ &= \frac{1}{n} - \frac{\varepsilon}{2}.\end{aligned}$$

Consequently $1/k_i > \varepsilon/2$, which contradicts the condition (9). Thus, condition (7) holds. Consequently from conditions (6), (7) it follows that

$$p \in \overline{f(A)} \cap \overline{f(B)} \backslash O_{1/n}(\overline{f(C)}),$$

which contradicts condition (1). Hence the set ϕ_n is open in $C(X, Z)$. Since obviously

$$(10) \qquad \phi = \bigcap_{n=1}^{\infty} \phi_n$$

we obtain that ϕ has type G_δ in the space $C(X, Z)$. □

LEMMA 1.2. *If in Lemma 1.1 we take $Z = I^\omega$, then the set ϕ of all mappings $f: X \to I^\omega$ such that*

$$\overline{f(A)} \cap \overline{f(B)} = \overline{f(C)}$$

is everywhere dense G_δ-set in $C(X, I^\omega)$.

Proof. First we show that the set ϕ_n is everywhere dense in $C(X, I^\omega)$. Let $f: X \to I^\omega$ be an arbitrary mapping and $\varepsilon > 0$. We can find an integer k such that

$$(11) \qquad 2^{-k} < \frac{1}{4n}, \quad 2^{-k} < \varepsilon.$$

Let us define a mapping $g: X \to I^\omega$, such that

$$(12) \qquad \delta(f, g) < \min\left[\frac{1}{4n}, \varepsilon\right].$$

We put

$$U = f^{-1}(O_{1/4n}\overline{f(C)}) . \tag{13}$$

Then the set U is a neighborhood of the set C in the space X. Therefore,

$$(A \backslash U) \cap (B \backslash U) = \varnothing .$$

Consequently, there eixsts a function $\varphi\colon X \to [0, 1]$ such that

$$\varphi(A \backslash U) = 0 , \quad \varphi(B \backslash U) = 1 . \tag{14}$$

We will define a mapping $g\colon X \to I^{\omega}$ by the equalities

$$\begin{aligned} &g_i(x) = f_i(x) \quad \text{for} \quad i \leqq k , \quad g_{k+1}(x) = \varphi(x) , \\ &g_i(x) = 0 \quad \text{for} \quad i > k + 1 . \end{aligned} \tag{15}$$

where g_i (respectively f_i) is i-coordinate of g (respectively f). Inequality (12) follows from (11), (15). Let us show that

$$\overline{g(A)} \cap \overline{g(B)} \backslash O_{1/n}\overline{g(C)} = \varnothing . \tag{16}$$

We assume on the contrary. Then there exist two points $a \in A$ and $b \in B$ such that

$$\delta(g(a), \overline{g(C)}) \geqq \frac{1}{n} , \quad \delta(g(b), \overline{g(C)}) \geqq \frac{1}{n} , \quad \delta(g(a), g(b)) < 2^{-(k+1)} \tag{17}$$

where δ is a metric in I^{ω}. From condition (11), (15) it follows that

$$\delta(\overline{g(A \backslash U)}, \overline{g(B \backslash U)}) \geqq 2^{-(k+1)} .$$

Consequently

$$\text{either} \quad g(a) \notin g\overline{(A \backslash U)} \quad \text{or} \quad g(b) \notin g\overline{(B \backslash U)} .$$

Let, for example $g(a) \notin \overline{g(A \backslash U)}$, then obviously $a \notin (A \backslash U)$ and, consequently $a \in U$. From condition (13) it follows that

$$\delta(f(a), \overline{f(C)}) < \frac{1}{4n} .$$

Consequently, for some point $c \in C$

$$\delta(f(a), f(c)) < \frac{1}{4n} . \tag{18}$$

By virtue of (12), (18)

$$\begin{aligned} \delta(g(a), \overline{g(C)}) &\leqq \delta(g(a), g(c)) \\ &\leqq \delta(g(a), f(a)) + \delta(f(a), f(c)) + \delta(f(c), g(c)) \\ &< \frac{3}{4n} < \frac{1}{n} \end{aligned}$$

which contradicts the condition (17). Consequently equality (16) holds and $g \in \phi_n$. We have proved that the set ϕ_n is everywhere dense in $C(X, I^\omega)$. By Lemma 1.1 ϕ_n is open in $C(X, I^\omega)$. Since, obviously

$$\phi = \bigcap_{n=1}^{\infty} \phi_n$$

then by Baire's theorem ϕ is an everywhere dense G_δ-set. □

Let X be a space, then we denote by G_X the set of all subsets $A \subset C(X, I^\omega)$ such that A contains an everywhere dense G_δ-set in $C(X, I^\omega)$. We note that from Baire's theorem it follows that

(19) $$\text{if } A_i \in G_X (i = 1, 2, \cdots) , \quad \text{then} \quad \bigcap_{i=1}^{\infty} A_i \in G_X .$$

In what follows we shall use the assertion (see [5], Hurewicz and Wallman).

(20) If X is a separable space, then the set of all homeomorphisms $f: X \to I^\omega$ contains an everywhere dense G_δ-set in $C(X, I^\omega)$.

The following two lemmas can be proved by well known standard methods. However we shall prove them for the completeness.

We remind the reader that $g: X \to Y$ is an ε-mapping if diameter $g^{-1}(y) < \varepsilon$ for any point $y \in Y$.

LEMMA 1.3. *Let $K \subset X$ be a compactum in a space X, then the set of all mappings $f: X \to I^\omega$ such that the restriction of f to K is a homeomorphism is an everywhere dense G_δ-set in $C(X, I^\omega)$.*

Proof. Let ϕ_ε be a set of all mappings $f: X \to I^\omega$ such that the restriction f_K of a mapping f to K is an ε-mapping, and ψ_ε be a set of all ε-mappings $g: K \to I^\omega$. Then ψ_ε is open and everywhere dense in $C(K, I^\omega)$, see [5]. Consequently ϕ_ε is open in $C(X, I^\omega)$. Let us show that ϕ_ε is everywhere dense. Let $f: X \to I^\omega$ be a mapping and $\delta > 0$. Since ψ_ε is everywhere dense there exists an ε-mapping $g: K \to I^\omega$ such that

$$d(f_K, g) \leqq \frac{\delta}{2} .$$

Since $I^\omega \in AR$ there is an extension $\bar{g}: X \to I^\omega$ of a mapping g. We put

$$h(x) = \begin{cases} \bar{g}(x) & \text{for} \quad |(f - \bar{g})(x)| \leqq \dfrac{\delta}{2} \\ f(x) + \dfrac{\delta}{2} \dfrac{(\bar{g} - f)(x)}{|(\bar{g} - f)(x)|} & \text{for} \quad |f - \bar{g}| > \dfrac{\delta}{2} . \end{cases}$$

Then, obviously $d(h, f) \leqq \delta/2 < \delta$ and since $g(x) = (x)$ for $x \in K$ we have $h \in \phi_\varepsilon$. Therefore, the set ϕ_ε is open and everywhere dense in $C(X, I^\omega)$. Our lemma now follows from (19) and the equality

$$\phi = \bigcap_{n=1}^{\infty} \phi_{1/n} . \qquad \square$$

LEMMA 1.4. *Let A be a closed subset of a space X and* $\dim A \leqq n$, *then the set ϕ of all mappings $f\colon X \to I^\omega$ such that*

$$\dim \overline{f(A)} \leqq n$$

is an everywhere dense G_δ-set in $C(X, I^\omega)$.

Proof. Let K_ε be a set of all mappings $f\colon X \to I^\omega$ such that

$$d_{n+1}(\overline{f(A)}) < \varepsilon$$

where $d_{n+1}(\overline{f(A)})$ is an $(n + 1)$-coefficient of Urysohn, i.e., the inf of $\varepsilon > 0$ such that there is a covering of $\overline{f(A)}$ with open sets of diameter $<\varepsilon$ and of order $\leqq n + 1$.

Then, clearly

$$\phi = \bigcap_{n=1}^{\infty} K_{1/n} .$$

Therefore by virtue of (19) it is sufficient to prove that $K_{1/n}$ is open and everywher dense set. Let $f \in K_{1/n}$ then there exists a finite collection of open in I^ω sets $V = \{V_1, \cdots, V_s\}$ such that

$$\overline{f(A)} \subset \bigcup_{i=1}^{s} V_i, \ \operatorname{diam} V_i \leqq \frac{1}{n} , \quad \text{order} \quad V \leqq n + 1 . \tag{21}$$

It is evident that the set 0 of all mappings $f\colon X \to I^\omega$ satisfying the condition (21) is a neighborhood of f in $C(X, I^\omega)$. Thus, $K_{1/n}$ is open. Let us prove that $K_{1/n}$ is everywhere dense. Let $g\colon X \to I^\omega$ be a mapping. Since $\dim A \leqq n$ we can construct by Kuratowski method [8] a mapping $f\colon X \to I^\omega$, such that

$$d_X(f, g) < \varepsilon$$

and $f(A)$ is contained in n-dimensional polyhedra. Therefore $\dim \overline{f(A)} \leqq n$ and $d_{n+1}(\overline{f(A)}) = 0$. Consequently, $f \in K_{1/p}$. Thus the set $K_{1/p}$ is everywhere dense in $C(X, I^\omega)$. $\square$

2. Further lemmas for the compactification theorem.

DEFINITION 2.1. A space X is called weakly infinite dimensional if for any countable sequence of pairs $\{F_i, G_i\}$ of closed sets in X $(i = 1, 2, \cdots)$, $F_i \cap G_i = \emptyset$, there exists for every i a partition C_i between F_i and G_i such that

$$\bigcap_{i=1}^{k} C_i = \emptyset ,$$

for some $k = 1, 2, \cdots$.

Every finite dimensional space is clearly weakly infinite dimensional.

DEFINITION 2.2. A countable system of open sets U_n, $n = 1, 2, \cdots$, in a space X is called convergent if for any discrete in X sequence of points $\{x_i\}$, $i = 1, 2, \cdots$, there exist numbers p and $n, p, n = 1, 2, \cdots$ such that $x_i \in U_n$ for $i \geqq p$.

If the system $\{U_n\}$ is convergent then the set $X \backslash \bigcup_{n=1}^{\infty} U_n$ is called the limit of this system. Let a space X be weakly infinite dimensional and

$$(1) \qquad \begin{aligned} U_n = \{x\colon x \in X, \quad &\text{there exists a neighborhood} \quad O_x \ni x \\ &\text{such that} \quad \dim O_x \leqq n\} . \end{aligned}$$

Then we have, see [19], Sklyarenko:

(S1) The system of sets $\{U_n\}$ is convergent and has a compact limit.

COROLLARY 2.1. *If a space X is infinite dimensional and weakly infinite dimensional then*

(i) *the set $P(X)$ is a nonempty compactum.*

(ii) $\operatorname{Ind}(X \backslash OP(X)) < \omega_0$ *for any neighborhood $OP(X)$ of a compactum $P(X)$.*

Proof. The compactness of $P(X)$ follows the equality

$$(2) \qquad P(X) = X \Big\backslash \bigcup_{n=1}^{\infty} U_n$$

and Theorem (S1). Let $P(X) = \emptyset$. Since X is infinite dimensional, there exists a sequence of points x_i in X such that

$$(3) \qquad x_i \in U_{n_i} \backslash \cup \{U_j\colon j < n_i\} , \quad n_{i+1} > n_i,\ n_i = 1, 2, \cdots .$$

From condition (3) it follows that a sequence $\{x_i\}$ is discrete in X but by Theorem (S1) the system $\{U_n\}$ is convergent. We obtain the contradiction. Consequently $P(X) \neq \emptyset$.

If condition (ii) does not hold then there exists a sequence of points $\{x_i\}$ with property (3) and such that $x_i \in X\backslash OP(X)$, consequently a sequence $\{x_i\}$ is discrete in X and we again obtain the contradiction. Therefore the property (ii) holds. □

We shall use the following theorem, see [21]; Smirnov.

(SM1) If the space X has dimension $\operatorname{Ind} X$ then X is weakly infinite dimensional.

LEMMA 2.1. *If sets A, B are closed in $X = A \cup B$ and the set B is finite dimensional, $\alpha \geqq \omega_0$ and*

$$\operatorname{Ind} A < \alpha \quad (\operatorname{ind} A < \alpha)$$

then

$$\operatorname{Ind}(A \cup B) < \alpha \quad (\operatorname{ind}(A \cup B) < \alpha) .$$

This lemma directly follows from [10] (Levshenko), Theorems 1, 1′, p. 257. □

LEMMA 2.2. *If B is a bicompactum (not necessarily metrizable) and $C \subset B\backslash P(B)$ is a closed subset, then*

$$\dim C < \infty .$$

Proof. From Defininition 0.4 it follows that every point $x \in C$ has a closed finite dimensional neighborhood $V(x)$. The lemma now follows from the compactness of C and the sum theorem for dimension dim.

LEMMA 2.3. *Let X be an arbitrary normal space (not necessarily metrizable) and (A, B) be a pair of two closed disjoint sets in X. Let also $P \subset X$ be a closed set and C be a partition in X between $A_1 = A \cap P$ and $B_1 = B \cap P$. If the set C has a type G_δ in X then there exists a partition C_0 in X between A and B such that*

$$C_0 = C_1 \cup C_2 ,$$

where sets C_1 and C_2 are closed in X, $C_1 \subset C$, $C_2 \subset X\backslash OP$ for some neighborhood OP of the set P. (We do not suppose here that $A_1 \neq \varnothing$ and $B_1 \neq \varnothing$.)

Proof. Since C is a partition between A_1 and B_1 there are such open sets U, V of that

$$U \cap V = \varnothing, X\backslash(U \cup V) = C, A_1 \subset U, B_1 \subset V . \tag{4}$$

Consequently

$$(A\backslash U) \cap P = (B\backslash V) \cap P = \varnothing .$$

Since X is a normal space, there exists a closed neighborhood W of the set P such that

$$(5) \qquad W \cap ((A\backslash U) \cup (B\backslash V)) = \varnothing$$

and W has a type G_δ. We put

$$(6) \quad U_1 = (W \cup A \cup B)\backslash(B \cup (W\backslash U)),\ V_1 = (W \cup A \cup B)\backslash(A \cup (W\backslash V)) .$$

Then sets U_1, V_1 are open in $W \cup A \cup B$. Since $A \cap B = \varnothing$, by virtue of (4), (5), (6)

$$U_1 \cap V_1 = W \cup A \cup B\backslash A \cup B \cup (W\backslash U) \cup (W\backslash V_|) = \varnothing ,$$
$$A \subset U_1 , \quad B \subset V_1.$$

Consequently the set

$$C_1 = (W \cup A \cup B)\backslash(U_1 \cup V_1) = (A \cup (W\backslash V)) \cap (B \cup (W\backslash U))$$
$$= W \cap C \subset C$$

is a partition in $W \cup A \cup B$ between A and B. Since the sets W and C are G_δ, the set $C_1 = W \cap C$ is also a G_δ-set. Consequently there exists a continuous function $\varphi: W \cup A \cup B \to [-1, 1]$ such that

$$\varphi^{-1}(0) = C_1 , \quad \varphi(A) = -1 , \quad \varphi(B) = 1 .$$

Let $\varphi: X \to [-1, 1]$ be any continuous extension of φ. We put

$$C_0 = \varphi^{-1}(0) , \quad C_2 = \overline{C_0\backslash C_1} .$$

Then obviously

$$C_0 = C_1 \cup C_2 ,$$

since $C_0\backslash C_1 \subset X\backslash W$ and W is a closed neighborhood of P, there exists a neighborhood $OP \subset W$ of the set P such that

$$C_0\backslash C_1 = C_2 \subset X\backslash OP . \qquad \square$$

DEFINITION 2.3. Let $\mathscr{F} = \{F_i: i=1, 2, \cdots\}$ be a countable system of sets in a space X and the set $U \subset X$ be open. Then the system $\mathscr{F}$ is called simple with respect to U if

$$(7) \qquad U = \bigcup_{i=1}^{\infty} F_i$$

$$(8) \qquad F_i \cap F_j = \varnothing \quad \text{for} \quad |i - j| > 1$$

(9) The system F is locally finite on U and sets F_i are closed in X.

LEMMA 2.4. *Let B be a bicompactum not necessarily metrizable and $f: B \to K$ be a zero dimensional mapping*[6] *in a compactum K.*[7] *If there is a closed set $P \subset B$ and a simple, with respect to $(B \backslash P)$, system $\{B_i\}$ in B such that:*

(10) $$f(P) \cap f(B_i) = \varnothing \quad \textit{for every} \quad i = 1, 2, \cdots$$

(11) $$\dim f(B_i) < \infty$$

(12) *the restriction of f to P is a homeomorphism*

and

(13) $$\operatorname{Ind} K \leqq \alpha$$

then $\operatorname{Ind} B \leqq \alpha$.

Proof. We shall prove this lemma by induction on α. If $\alpha < \omega_0$ then for any bicompactum B having zero-dimensional mapping in α dimensional compactum K we have

$$\operatorname{Ind} B \leqq \operatorname{Ind} K \leqq \alpha .$$

(See [10], Pasynkov.) Let $\alpha \geqq \omega_0$ and for all $\alpha' < \alpha$ our lemma is proved. Let F, G be a pair of two disjoint closed sets in B. By virtue of (12)

$$f(F \cap P) \cap f(G \cap P) = \varnothing .$$

Then by virtue of (13) there exists a partition D between $f(F \cap P)$ and $f(G \cap P)$ such that

(14) $$\operatorname{Ind} D \leqq \beta < \alpha .$$

We put

(15) $$C = f^{-1}(D) .$$

Then, since K is metrizable and $D \subset K$ is closed, D is a G_δ-set. Consequently, C is also a G_δ-set. Moreover the set C is a partition between $F \cap P$ and $G \cap P$ in B. By virtue of Lemma 2.3 there exists a partition C_0 between F and G such that:

(16) $$C_0 = C_1 \cup C_2 , \quad C_1 \subset C , \quad C_2 \subset X \backslash OP$$

for some closed sets C_1 and C_2 and a neighborhood OP of a set P.

[6] A mapping f is zero dimensional if $\dim f^{-1}(x) \leqq 0$ for any point x in image f.

[7] In our terminology a compactum is a metrizable bicompactum.

We note that if $B' \subset B$ is a closed set, $B'_i = B_i \cap B'$, $P' = P \cap B$, $K' = f(B')$ then all conditions of Lemma 2.4 are fulfilled for bicompactum B' (we have only to change notation). Consequently, if $\operatorname{Ind} f(B') < \alpha$ then by inductive assumption $\operatorname{Ind} B' \leqq \operatorname{Ind} f(B') < \alpha$. Therefore for proving inequality

$$\operatorname{Ind} C_0 < \alpha \tag{17}$$

we have only to prove that

$$\operatorname{Ind} f(C_2) < \infty . \tag{18}$$

Indeed, as it was mentioned above we have only to prove that

$$\operatorname{Ind} f(C_0) = \operatorname{Ind} f(C_1) \cup f(C_2) < \alpha .$$

By virtue of (14), (15), (16) we obtain the inequality

$$\operatorname{Ind} f(C_1) \leqq \operatorname{Ind} D \leqq \beta < \alpha . \tag{19}$$

By Lemma 2.1 and (18), (19)

$$\operatorname{Ind} (f(C_1) \cup f(C_2)) < \alpha .$$

Let us prove inequality (18). By conditions of the lemma we have

$$B \backslash P = \bigcup_{i=1}^{\infty} B_i$$

and the system $\{B_i, i = 1, 2, \cdots\}$ is locally finite on $B \backslash P$. Since $C_2 \subset B \backslash P$ and C_2 is a subbicompactum of B then for some finite collection $B_{i_1}, \cdots, B_{i_k}$ we have

$$C_2 \subset \bigcup_{s=1}^{k} B_{i_s} .$$

Consequently, by virtue of (11)

$$\dim f(C_2) \leqq \max\{\dim f(B_{i_s}) \colon s = 1, \cdots, k\} < \infty .$$

Since $f(C_2) \subset K$ and K is a metrizable space, $\dim f(C_2) = \operatorname{Ind} f(C_2)$. Thus inequality (18) and consequently (17) holds. Therefore $\operatorname{Ind} B \leqq \alpha$.

LEMMA 2.5. *If for any two closed disjoint sets F, G in weakly infinite dimensional space Z such that*

$$F \cup G \subseteqq P(Z)$$

there exists a partition C having the dimension $\operatorname{Ind} C < \alpha (\alpha \geqq \omega_0)$ *then* $\operatorname{Ind} Z \leqq \alpha$.

Proof. Let (A, B) be an arbitrary pair of disjoint closed sets in Z. If either $A \cap P(Z) = \varnothing$ or $B \cap P(Z) = \varnothing$ then either the set A or B belongs to the set $Z \backslash OP(Z)$ for some neighborhood $OP(Z)$ of the set $P(Z)$. By virtue of Corollary 2.1 there exists a finite dimensional partition C between A and B; consequently

$$\operatorname{Ind} C < \omega_0 \leqq \alpha .$$

Let $A_1 = A \cap P(Z) \neq \varnothing$ and $B_1 = B \cap P(Z) \neq \varnothing$. By the given condition there exists a partition C between A_1 and B_1 such that

$$\operatorname{Ind} C < \alpha . \tag{20}$$

Since Z is metrizable, C is a G_δ-set and by virtue of Lemma 2.3 there exists a partition C_0 between A and B such that

$$C_1 \subset C , \quad C_2 \subset X \backslash OP , \quad C_0 = C_1 \cup C_2 \tag{21}$$

for some closed sets C_1, C_2 and a neighborhood OP of the set P. By virtue of Corollary 2.1

$$\operatorname{Ind} C_2 < \infty .$$

From (20), (21) it follows that

$$\operatorname{Ind} C_1 < \alpha .$$

The inequality $\operatorname{Ind} C_0 < \alpha$ now follows from Lemma 2.1 and (21). □

LEMMA 2.6. *If for any point $p \in P(Z)$ in a weakly infinite dimensional space Z and for any closed set $F \subset P(Z)$, $F \not\ni p$, there exists a partition C in Z such that*

$$\operatorname{ind} C < \alpha (\alpha \geqq \omega_0)$$

then $\operatorname{ind} Z \leqq \alpha$.

The proof is similar to the proof of Lemma 2.5.

COROLLARY 2.2. *For any weakly infinite dimensional space Z the following assertions hold:*

(a) $\operatorname{Ind} Z \leqq \alpha \Leftrightarrow$ *for any closed sets F, G, $F \cup G \subset P(Z)$, $F \cap G = \varnothing$ there is a partition C with* $\operatorname{Ind} C < \alpha$. $\alpha \geqq \omega_0$.

(b) $\operatorname{ind} Z \leqq \alpha \Leftrightarrow$ *for any closed set $F \subset P(Z)$ and a point $x \in P(Z) \backslash F$ there is a partition C with* $\operatorname{ind} C < \alpha$. $(\alpha \geqq \omega_0)$

COROLLARY 2.3. *If X has the dimension* $\operatorname{Ind} X$ *and $Y \subset X$ is weakly infinite dimensional then*

$$\operatorname{Ind} Y \leqq \operatorname{Ind} X . \tag{22}$$

We shall prove this corollary by induction on $\alpha = \operatorname{Ind} X$. If $\alpha < \omega_0$ our assertion is known. Let $\alpha \geqq \omega_0$ and F, G be a pair of two disjoint closed sets in $P(Y)$. Since $Y \subset X$ we have

$$P(Y) \subset P(X) .$$

Since by Corollary 2.1 $P(Y)$ is a compactum, the sets F, G are closed in $P(X)$ and by inductive assumption there exists a partition C between F and G in X such that

$$\operatorname{Ind} C < \alpha .$$

Then $C \cap Y$ is a partition between F and G in Y and since $C \cap Y$ is closed in Y it is weakly infinite dimensional space. Consequently by inductive assumption

$$\operatorname{Ind}(C \cap Y) \leqq \operatorname{Ind} C < \alpha .$$

Inequality (22) now follows from Corollary 2.2. □

LEMMA 2.7. *Let $\mathscr{U} = \{U_i : i = 1, 2, \cdots\}$ be a system of open sets in a space X, α a limit ordinal number $<\omega_1$ and*

$$\begin{gathered} D(U_i) < \alpha \\ U = \bigcup_{i=1}^{\infty} U_i . \end{gathered} \tag{23}$$

Then for any sequence of ordinal numbers γ_i such that

$$\gamma_{i+1} > \gamma_i , \quad \sup \gamma_i = \alpha \quad i = 1, 2, \cdots \tag{24}$$

there exists a simple, with respect to U, system $\mathscr{F} = \{F_i\}$ such that

$$D(F_i) \leqq \gamma_i .$$

Proof. We take a system of open sets $\mathscr{V} = \{V_i\}$ such that

$$\bar{V}_i \subset V_{i+1} \subseteqq U , \quad \bigcup_{i=1}^{\infty} V_i = U \quad (i = 1, 2, \cdots) . \tag{24}$$

Since X is metrizable, there exists an open covering $\{D_i\}$ of the set U such that the covering $\{\bar{D}_i\}$ consisting of closures $\bar{D}_i$ is a refinement of $\mathscr{V}$ and $\mathscr{U}$. Since coverings $\mathscr{V}$ and $\mathscr{U}$ are countable, we can require a covering $\{D_i\}$ to be countable. Then there exists a closed covering $\mathscr{H} = \{H_i\}$ of U such that

$$H_i = \bar{H}_i \subset D_i \quad (i = 1, 2, \cdots) . \tag{26}$$

By virtue of (25) the sets H_i, $\bar{D}_i$ are closed in X. From condition (23) it follows that

(27) $$D(H_i) \leqq D(D_i) \leqq D(\bar{D}_i) \leqq \beta_i < \alpha .$$

We can now construct by induction open sets W_i such that:

(28) $$\bigcup_{k=1}^{i} H_k \subset W_i \subset \bar{W}_i \subset \bigcup_{i=1}^{k} D_i \subset U$$

(29) $$\bar{W}_i \subset W_{i+1} .$$

By (27), (28) and the sum theorem for union of finite number of closed sets with D-dimension [3], Henderson, we have

(30) $$D(\bar{W}_i) \leqq \max \{D(\bar{D}_i) : i = 1, \cdots, k\} = \delta_i < \alpha .$$

Since $\mathscr{H}$ is a covering of U, we have, by virture of (28)

(31) $$\bigcup_{i=1}^{\infty} W_i = U .$$

From conditions (24), (30) it follows that there exists such subsequence $\{\gamma_{n(i)}\} i = 1, 2, \cdots$ of a sequence $\{\gamma_i\}$ so that

(32) $$D(\bar{W}_i) \leqq \gamma_{n(i-1)} \quad n(i+1) > n(i) > 0 , \quad i = 1, 2, \cdots .$$

Since the space X is normal, from condition (29), it follows that for any i there exist such open sets

$$V_{n(i)}, \cdots, V_{n(i+1)} \quad (i = 0, 1, \cdots)$$

so that

(33) $$V_{n(i)} = W_i , \quad V_{n(i+1)} = W_{i+1} , \quad V_i = \varnothing \quad \text{for} \quad 0 \leqq i < n(1) .$$

(34) $$V_{n(i)+k} \subset \bar{V}_{n(i)+k} \subset V_{n(i)+k+1} \quad k = 0, \cdots, n(i+1) - n(i) .$$

By virtue of (11), (32), (33), (34), (31) we have

(35) $$D(\bar{V}_j) \leqq D(\bar{W}_{i+1}) \leqq \gamma_{n(i)} \leqq \gamma_j \quad \text{for} \quad n(i) \leqq j \leqq n(i+1)$$

(36) $$D(\bar{V}_j) = -1 \leqq \gamma_j \quad \text{for} \quad 0 \leqq j < n(1)$$

(37) $$\bar{V}_i \subset V_{i+1} \subset U, \bigcup_{i=1}^{\infty} V_i = U .$$

We put

(38) $$F_i = \bar{V}_i \setminus V_{i-1} \quad \text{for} \quad i > 0 , \quad F_0 = \varnothing .$$

Then for $j > i$, $V_i \cap F_j = \varnothing$. Consequently, by virtue of (37) the system $\{F_i\}$ is locally finite. Conditions (7), (8) follow from (37), (38). Inequality $D(F_i) \leqq \gamma_i$ follows from (35), (36), (38). □

COROLLARY 2.4. *For any space X there exists a simple with respect to $X \setminus P(X)$ system $\{C_i\}$ such that*

$$\dim C_i \leqq i \quad (i = 1, 2, \cdots) . \tag{39}$$

Proof. Let U_n be the set defined by equality (1), then obviously

$$\dim U_n \leqq n$$

and by definition

$$X \setminus P(X) = \bigcup_{n=1}^{\infty} U_n .$$

Hence by Lemma 2.7 there is a simple, with respect to $X \setminus P(X)$, system $\{C_i\}$ such that

$$\dim C_i = D(C_i) \leqq i = \gamma(i) . \qquad \square$$

LEMMA 2.8. *If $D(X) \geqq \omega_0$ then $D(X) = \omega_0 + D(P(X))$.*

Proof. If $D(X) = \Delta$ then our lemma is trivial. Let $D(X) = \beta < \Delta$, $D(P(X)) = \alpha$ and the equality

$$P(X) = \cup \{A_\xi : \xi \leqq J(\alpha)\}$$

be the α-D-representation of $P(X)$. By virtue of Corollary 2.4, we have the representation

$$X = P(X) \cup \bigcup_{i=1}^{\infty} C_i$$

for simple, with respect to $X \setminus P(X)$, system $\{C_i\}$ such that property (39) holds. Let us put

$$B_{\omega_0 + \xi} = A_\xi , \quad B_i = C_i , \quad \text{for} \quad i < \omega_0 .$$

Then clearly the equality

$$X = \cup \{B_\xi : \xi \leqq J(\omega_0 + \alpha) = \omega_0 + J(\alpha)\}$$

is a $(\omega_0 + \alpha)$-D-representation of X. Therefore

$$\beta \leqq \omega_0 + \alpha .$$

Let the equality

$$X = \{A_\xi = \xi \leqq J(\beta)\}$$

be a β-D-representation of X and $\beta = \omega_0 + \delta$. Let us put

$$B_\xi = A_{\omega_0 + \xi}$$

and

$$T = \cup \{B_\xi : \xi \leqq J(\delta)\} . \tag{40}$$

Then (40) is clearly a δ-D-representation of T. From conditions (b), (c), (e) of Definition 0.3 it follows that the set $X \backslash T$ is a union of open finite dimensional sets. Consequently, $P(X) \subset T$ and

$$D(P(X)) \leqq D(T) \leqq \delta .$$

Hence $\omega_0 + \delta \geqq \omega_0 + \alpha$. □

3. Standard representations and standard mappings.

DEFINITION 3.1. Let

$$X = P(X) \cup \bigcup_{i=0}^{\infty} C_i . \tag{1}$$

Then equality (1) is called a standard representation of a space X if

(a) The system $\{C_i\}$ is simple with respect to $X \backslash P(X)$. (In particular $C_i \cap P(X) = C_i \cap C_j = \varnothing$ for $|i - j| > 1$.)

(b) $\dim C_i \leqq n(i)$.

(c) For any $x \in C_i$, $\delta(x, y) < 1/i$ for some $y \in P(X)$, $i > 0$, where δ is a metric on X.

LEMMA 3.1. *Let X be a weakly infinite dimensional space. Then there exists a standard representation of X.*

Proof. We put

$$C_i = \left\{x: \frac{1}{i+2} \leqq \delta(x, P(X)) \leqq \frac{1}{i+1}\right\} \quad \text{for} \quad i \geqq 1 ,$$

$$C_0 = \left\{x: \delta(x, P(X)) \geqq \frac{1}{2}\right\} .$$

Then, clearly the equality (1) and properties (a), (c) hold. Property (b) follows from Corollary 2.1. □

DEFINITION 3.2. Let (1) be a standard representation of X and $f: X \to I^\omega$ be a mapping such that:

(a) f is a homeomorphism on $P(X)$.

(b) $\overline{f(C_i)} \cap f(P(X)) = \overline{f(C_j)} \cap \overline{f(C_i)} = \varnothing$ for $|i - j| > 1$.

(c) $\dim \overline{f(C_i)} \leqq \dim C_i \leqq n_i < \infty$.

Then f is called a standard mapping.

LEMMA 3.2. *Let $f: X \to I^\omega$ be a standard mapping of a weakly* *infinite dimensional and infinite dimensional space X. Then*

(2) $$\overline{f(X)} = f(P(X)) \cup \bigcup_{i=1}^{\infty} \overline{f(C_i)} .$$

(3) *the system* $\{\overline{f(C_i)}\}$ *is locally finite on* $\overline{f(X)} \backslash f(P(X))$.

(4) $$f^{-1}P\overline{f(X)} \subset P(X) .$$

(5) $$D\overline{f(X)} \leqq D(X) .$$

Proof. Let us show that

(6) for any sequence $\{y_i\}$, such that $y_i \in \overline{f(X)}$, $y_i \notin \overline{f(C_j)}$ for $j < i$ we have

(7) $\lim_{i\to\infty} y_{k(i)} = a$ for some point $a \in f(P(X))$, and some subsequence $\{y_{k(i)}\}$ of sequence $\{y_i\}$.

Indeed, we can take for any i a point $f(x_i)$ such that

(8) $$\delta(f(x_i), y_i) < \frac{1}{i} , \quad f(x_i) \notin \overline{f(C_j)} \quad \text{for} \quad j < i .$$

Consequently $x_i \in P(x) \cup \cup\{C_j: j \geqq i\}$ and by condition (c) of Definition 3.1 we have

(9) $$\delta(x_i, P(X)) < 1/i .$$

Since X is weakly infinite dimensional, $P(X)$ is compact and from (9) it follows that

(10) $\lim_{i=\infty} x_{n(i)} = b$ for some point $b \in P(X)$ and some subsequence $\{x_{n(i)}\}$ of sequence $\{x_i\}$.

Let $f(b) = a$, then property (7) follows from (8), (10). Let us prove (2). Let $x \in \overline{f(X)} \backslash f(X)$, $x \notin \overline{f(C_i)}$ for any i, and $x \notin f(P(X))$, then clearly we can construct a sequence $\{y_i\}$ such that

$$\lim_{i\to\infty} y_i = x \quad \text{and} \quad y_i \notin \overline{f(C_j)} \quad \text{for} \quad j < i .$$

From condition (7) it follows that $x \in f(P(X))$ and we obtain the contradiction. Hence property (2) holds. Similarly we can prove (3). Indeed, if $x \in \overline{f(X)} \backslash f(P(X))$ and any neighborhood Ox of x contains points of infinitely many sets $\overline{f(C_i)}$, we can construct a sequence $\{y_i\}$ satisfying (6) and such that

$$\lim_{i\to\infty} y_i = x .$$

By virtue of (7) $x \in f(P(X))$ and we again get the contradiction.

Therefore the system $\{\overline{f(C_i)}\}$ is locally finite on $\overline{f(X)}\backslash f(P(X))$. Let us prove (4). Since $P(X)$ is compact, the set $f(P(X))$ is closed and the set $\overline{f(X)}\backslash f(P(X)$ is open in $\overline{f(X)}$. By virtue of condition (2) and (b) of Definition 3.2 we have

$$\begin{aligned} &\overline{f(X)}\backslash f(P(X)) = \bigcup_{i=1}^{\infty} \overline{f(C_i)} \\ &f^{-1}f(P(X)) = P(X) . \end{aligned} \tag{11}$$

From conditions (3) and (c) of Definition 3.2 it follows that

$$P\overline{f(X)} \subset f(P(X)) . \tag{12}$$

Consequently, condition (4) follows from (11), (12). Property (5) follows from (12). Indeed, by Lemma 2.8

$$\begin{aligned} D(\overline{f(X)}) &= \omega_0 + D(P\overline{f(X)}) \leqq \omega_0 + Df(P(X)) \\ &= \omega_0 + D(P(X)) = D(X) . \end{aligned}$$

We used here the equality $Df(P(X)) = D(P(X)$ which follows from condition (a) of Definition 3.2. □

LEMMA 3.3. *Let U be an open set in some closed subset A of a space X and ϕ the set of all mappings $f: X \to I^\omega$ such that*

$$f(U) \cap \overline{f(A\backslash \bar{U})} = \varnothing$$

then

$$\phi \in G_X .$$

Proof. Let $\{F_i: i = 1, 2, \cdots\}$ be a collection of closed sets in a space X such that $\bigcup_{i=1}^{\infty} F_i = U$. Then by virtue of Lemma 1.2 and (19) §1 the set ψ of all mappings $f: X \to I^\omega$ such that

$$\overline{f(F_i)} \cap \overline{f(A\backslash \bar{U})} = \varnothing$$

belongs to G_X. Since clearly $\phi \supset \psi$, we have also $\phi \in G_X$. □

LEMMA 3.4. *Let X be a weakly infinite dimensional closed subset of a space Y and ϕ be a set of all mappings $f: Y \to I^\omega$ such that f is standard on X. Then $\phi \in G_Y$.*

Proof. By Lemma 3.1 there exists a standard representation (1) of a space X. Since the system $\{C_i\}$ is countable, our lemma follows from Lemmas 1.2, 1.3, 1.4 and (19) §1. □

LEMMA 3.5. *Let U be an open set in a space X and $V = \operatorname{Int} \bar{U}$*

(where $\operatorname{Int}\bar{U}$ *is an interior of* $\bar{U}$*). Then*

$$Fr\,V \subset Fr\,U\,, \quad V = \operatorname{Int}\bar{V}\,, \quad U \subset V$$

and the set $X \backslash (Fr\,V \cup V)$ *is everywhere dense in* $X \backslash V$. *The lemma is trivial.*

An open set V *such that* $V = \operatorname{Int}\bar{V}$ *is called canonic.*

LEMMA 3.6. *Let* $\operatorname{Ind} X = \alpha$, $\operatorname{ind} X = \beta (\alpha \geqq \beta \geqq \omega_0)$ *and* $\lambda(\mu) = \{U_\gamma^\lambda : \gamma \in \Gamma\} \mu = a, b$ *be two system of open canonic in* X *sets such that for all* $\gamma \in \Gamma$:

(k1) $\operatorname{Ind} Fr\,U_\gamma^a < \alpha$

(k2) $\operatorname{ind} Fr\,U_\gamma^b < \beta$

(k3) $\lambda(a)$ *forms a large base in* $P(X)$[8] *and* $\lambda(b)$ *forms a base in* $P(X)$.

Then if $f: X \to I^\omega$ *is a standard mapping such that:*

(k4) $\operatorname{Ind} Fr\,U_\gamma^a \geqq \operatorname{Ind} \overline{f(Fr\,U_\gamma^a)}$

(k5) $\operatorname{ind} Fr\,U_\gamma^b \geqq \operatorname{ind} \overline{f(Fr\,U_\gamma^b)}$

(k6) $\overline{f(\bar{U}_\gamma^\mu)} \cap \overline{f(X \backslash U_\gamma^\mu)} = \overline{f(Fr\,U_\gamma^\mu)}$

(k7) *If* $\bar{U}_{\gamma_1}^\mu \subset U_{\gamma_2}^\mu$ *then* $\overline{f(\bar{U}_{\gamma_1}^\mu)} \cap \overline{f(X \backslash U_{\gamma_2}^\mu)} = \varnothing$

(k8) $\overline{f(Fr\,U_\gamma^\mu)} \cap f(X \backslash Fr\,U_\gamma^\mu) = \varnothing$.

For all $\gamma \in \Gamma$ *and* $\mu = a, b$ *then*

(a) $\operatorname{Ind} \overline{f(X)} \leqq \operatorname{Ind} X$

(b) $\operatorname{ind} \overline{f(X)} \leqq \operatorname{ind} X$

(c) $D\overline{f(X)} \leqq D(X)$.

Moreover, for proving condition (a) *it is sufficient to assume that conditions* k1, k3, k4, k6, k7, k8 *hold and for proving condition* (b) *it is sufficient to assume that conditions* k2, k3, k5, k6, k7, k8 *hold, and condition* (c) *follows from the standardness of mapping* f.

Proof. Condition (c) follows from Lemma 3.2, property (5). Let us prove (a). Let F, G be a pair of disjoint closed sets in $\overline{Pf(X)}$. We put

$$F_1 = f^{-1}(F)\,, \quad G_1 = f^{-1}(G)\,. \tag{13}$$

Since f is a standard mapping, by Lemma 3.2(4) we have

$$F_1 \cup F_2 \subseteqq P(X) \tag{14}$$

and clearly

$$F_1 \cap F_2 = \varnothing\,.$$

[8] This means that for any pair (F, G) of closed disjoint sets in (X) there is a set U_γ^a, open in X, such that $F \subset U_\gamma^a \subset X/G$.

By virtue of k1, k3 there exist such sets $U^{\alpha}_{\gamma_1}$, $U^{\alpha}_{\gamma_2}$, $U^{\alpha}_{\gamma_3} \in \lambda(\alpha)$ so that

(15) $$F_1 \subset U^{\alpha}_{\gamma_1} \subset \bar{U}^{\alpha}_{\gamma_1} \subset U^{\alpha}_{\gamma_1} \subset \bar{U}^{\alpha}_{\gamma_2} \subset U^{\alpha}_{\gamma_3} \subset X \setminus G_1 ,$$

(16) $$\operatorname{Ind} Fr\, U^{\alpha}_{\gamma_2} < \alpha .$$

We put

(17) $$U = U^{\alpha}_{\gamma_2},\ V = \operatorname{Int} \overline{f(\bar{U}^{\alpha}_{\gamma_2})} = \overline{f(\bar{U}^{\alpha}_{\gamma_2})} \setminus Fr\, \overline{f(\bar{U}^{\alpha}_{\gamma_2})},\ A = \overline{\overline{f(X)} \setminus \overline{f(\bar{U}^{\alpha}_{\gamma_2})}} .$$

Since f is a continuous mapping,

(18) $$\overline{f(\bar{U})} = \overline{f(U)} .$$

Let us show that

(19) $$A = \overline{f(X \setminus U)} .$$

By virtue of (k6), (k8), (18)

(20) $$\overline{f(U)} \cap \overline{f(X \setminus U)} \cap f(X \setminus Fr\, U) = \varnothing .$$

Since U is canonic set, from Lemma 3.5 it follows that the set $\overline{f(X \setminus U)} \cap f(X \setminus Fr\, U)$ is everywhere dense in $\overline{f(X \setminus U)}$, consequently

(21) $$\overline{f(X \setminus U)} = \overline{\overline{f(X \setminus U)} \cap f(X \setminus Fr\, U)} .$$

From (18), (20), (21) it follows that

(22) $$\overline{f(X \setminus U)} = \overline{\overline{f(X \setminus U)} \cap \overline{f(X \setminus Fr\, U)}} \subset \overline{\overline{f(X)} \setminus \overline{f(\bar{U})}} = A .$$

Since

$$\overline{f(X)} \setminus \overline{f(U)} = (\overline{f(X \setminus U)} \cup \overline{f(U)}) \setminus \overline{f(U)} \subset \overline{f(X \setminus U)}$$

and the set $\overline{f(X \setminus U)}$ is closed, we have

(23) $$A = \overline{\overline{f(X)} \setminus \overline{f(U)}} \subset \overline{f(X \setminus U)} .$$

The condition (19) follows from (22), (23). By virtue of (17), (18), (19) we have

(24) $$V = \overline{f(U)} \setminus (\overline{f(U)} \cap (\overline{\overline{f(X)} \setminus \overline{f(U)}})) = \overline{f(\bar{U})} \setminus (\overline{f(U)} \cap \overline{f(X \setminus U)}) .$$

From conditions (13), (15), (17), k7 it follows that

$$F \subset \overline{f(U^{\alpha}_{\gamma_1})},\ \overline{f(U^{\alpha}_{\gamma_1})} \cap \overline{f(X \setminus U)} = \varnothing .$$

Consequently,

(25) $$F \subset V .$$

On the other hand, by virtue of k7, (15), (24), (13)

$$V \subseteqq \overline{f(U)},\ \overline{f(U)} \cap \overline{f(X\setminus U^a_{\gamma_3})} = \varnothing\,, \quad G \subset \overline{f(X\setminus U^a_{\gamma_3})}\,.$$

Consequently

$$\bar{V} \subset \overline{f(X)}\setminus G\,. \tag{26}$$

Since the set V is open (condition 17), it follows from (25) and (26) that $Fr\,V$ is a partition between F and G in $\overline{f(X)}$. Since by (24) $\bar{V} \subset \overline{f(U)}$ and V is open, we have

$$Fr\,V = \bar{V}\setminus V \subset \overline{f(U)}\setminus V = \overline{f(U)} \cap \overline{f(X\setminus U)}\,. \tag{27}$$

But by virtue of (19), (17)

$$Fr\,\overline{f(U)} = \overline{f(U)} \cap \overline{(\overline{f(X)}\setminus\overline{f(U)})} = \overline{f(U)} \cap \overline{f(X\setminus U)}\,. \tag{28}$$

From (k6), (27), (28) it follows that

$$\overline{f(Fr\,U)} = Fr\,\overline{f(U)} \supset Fr\,V$$
$$\overline{f(Fr\,U)} = Fr\,\overline{f(U)} \supset Fr\,V\,.$$

By virtue of (k4), (16), (17)

$$\operatorname{Ind} \overline{f(Fr\,U)} \leqq \operatorname{Ind} Fr\,U < \alpha\,.$$

Consequently $\operatorname{Ind} Fr\,V < \alpha$. Inequality (a) now follows from Corollary 2.2(a). Similarly one can prove inequality (b). □

4. Proof of the compactification theorem. First we shall prove some general theorems.

THEOREM 4.1. *Let X be a closed subset of a space Y and dimension* $\operatorname{Ind} X$ *exists. Let ϕ be the set of all standard on X mappings $f: Y \to I^\omega$ such that*

$$\operatorname{Ind} \overline{f(X)} \leqq \operatorname{Ind} X$$
$$D\overline{f(X)} \leqq D(X)\,.$$

Then $\phi \in G_Y$.

LEMMA 4.1. *Let X be a closed subset of a space Y and dimension* $\operatorname{Ind} X$ *exists. (We note that by virtue of* (2) *(introduction) the dimension* $\operatorname{ind} X$ *also exists.) Let* $\operatorname{Ind} X = \alpha$, $\operatorname{ind} X = \beta$ *and* $\beta \geqq \omega_0$. *Then there exist two countable systems $\lambda(\mu) = \{U^\mu_\gamma : \gamma \in \Gamma\}\mu = a, b$ of open in X canonic sets such that for all $\gamma \in \Gamma$ conditions* k1, k2, k3 *of Lemma* 3.6 *are fulfilled.*

Proof. The existence of systems $\lambda(\mu)$ with properties k1, k2, k3 is evident. Since $\operatorname{Ind} X$ exists by (SM1) §2 and Corollary 2.1, $P(X)$ is compact. Consequently we can consider systems $\lambda(\mu)$ to be countable. Indeed, from any base (large base) of compact metric space we can select a countable subsystem which is also a base (large base). By virtue of Lemma 3.5 all elements of these bases we can consider to be canonic. □

LEMMA 4.2. *Let in Lemma* 4.1 ϕ_1 *be the set of all mappings* $f: Y \to I^\omega$ *such that* f *is standard on* X *and conditions* k6, k7, k8 *of Lemma* 3.6 *are fulfilled. Then* $\phi_1 \in G_Y$.

Proof. Let ϕ_i be the set of all mappings $f: Y \to I^\omega$ such that the condition $k_i (i = 6, 7, 8)$ of Lemma 3.6 holds. Since

$$Fr\, U_\gamma^\mu = \bar{U}_\gamma^\mu \cap X \backslash U_\gamma^\mu$$

and the systems $\lambda(\mu)$ are countable, from Lemma 1.2 and (19) §1 it follows that

$$\phi_6 \in G_X .$$

Analogously $\phi_7 \in G_X$. Inclusion $\phi_8 \in G_X$ follows from (19) §1 and Lemma 3.3. Let ψ be the set of all mappings $f: Y \to I^\omega$ such that f is standard on X. Then $\psi \in G_X$ by Lemma 3.4. Since obviously

$$\phi_1 = \psi \cap \phi_6 \cap \phi_7 \cap \phi_8 ,$$

our lemma follows from (19) §1.

Proof of Theorem 4.1. We shall prove the theorem by induction on $\alpha = \operatorname{Ind} X$. Let $\alpha < \omega_0$ then by virtue of (3) (introduction)

$$\operatorname{Ind} X = \dim X = D(X)$$

and our theorem follows from Lemma 1.4. Let $\alpha \geqq \omega_0$ then by Lemma 4.1 there exists a countable system $\lambda(\alpha)$ of open canonic in X sets satisfying conditions k1, k3 of Lemma 3.6. Let ψ be the set of all mappings $f: Y \to I^\omega$ such that the condition (k4) is fulfilled. Then from inductive assumption and (19) §1 it follows that

$$\psi \in G_Y .$$

Consequently,

$$\phi_0 = \psi \cap \phi_1 \in G_Y$$

where ϕ_1 is the set defined in Lemma 4.2. Let $f \in \phi_0$, then f is standard on A mapping. Morerver the conditions (k1), (k3), (k4), (k6), (k7), (k8) are fulfilled. Consequently, by Lemma 3.6 $f \in \phi$, where ϕ

is defined in Theorem 4.1. Hence $\phi_0 \subset \phi$. Consequently $\phi \in G_Y$. □

THEOREM 4.2. *Let A be a closed subset of a space X and dimension* Ind A *exists. Let A be strongly metrizable*[9] *and ϕ be the set of all standard on A mappings $f: X \to I^\omega$ such that*

$$(1) \qquad \begin{aligned} \operatorname{Ind} \overline{f(A)} &\leqq \operatorname{Ind}(A) \\ D\overline{f(A)} &\leqq D(A) \\ \operatorname{ind} \overline{f(A)} &\leqq \operatorname{ind} A . \end{aligned}$$

Then $\phi \subset G_X$.

Proof. We shall prove the theorem by induction on $\beta = \operatorname{ind} A$. By virtue of Theorem 4.1 it is sufficient to prove that the set ψ of all mappings satisfying condition (1) belongs to G_X. Let $\beta < \omega_0$, then

$$\operatorname{ind} A = \operatorname{Ind} A = \dim A ,$$

see [23], Zarelua. Consequently, our theorem now follows from Lemma 1.4. Let $\beta \geqq \omega_0$. In this case our proof is completely similar to the proof of Theorem 4.1.

COROLLARY 4.1. *Let X be a space and $\{L_i\}$ be a countable system of closed subsets such that the dimension* Ind L_i *exists for every i. Let Φ be the set of all mappings $f: X \to I^\omega$ such that*

(a) $\operatorname{Ind} \overline{f(L_i)} \leqq \operatorname{Ind} L_i \quad (i = 1, 2, \cdots)$.
(b) $D\overline{f(L_i)} \leqq D(L_i)$.
(c) *if L_j is strongly metrizable for some j then* $\operatorname{ind} \overline{f(L_j)} \leqq \operatorname{ind} L_j$.
(d) *f is a standard mapping on L_i.*
(e) $f^{-1}(P\overline{f(L_i)}) \subset P(L_i)$

for any $i = 1, 2, \cdots$. Then $\Phi \in G_X$.

Proof. By Lemma 3.2 the condition (e) follows from (d). Our corollary now follows from Theorem 4.1, Theorem 4.2 and (19) §1.

Proof of Compactification Theorem (1.1). Let ψ_0 be the set of all homeomorphisms $f: X \to I^\omega$ of separable space X in Hilbert cube. Then, see [5], Hurewicz and Wallman,

$$(2) \qquad \psi_0 \in G_X .$$

Since for every i and $f \in \psi_0$

[9] The condition of strong metrizability of X is equivalent to the following one: There exists an imbedding $f: X \to I^\omega \times \prod_{i=1}^\infty B_i$, where $I^\omega \times \prod_{i=1}^\infty B_i$ is a product of Hilbert cube and a countable number of discrete spaces.

$$\operatorname{Ind} f(L_i) = \operatorname{Ind} L_i\,,\quad \operatorname{ind} f(L_i) = \operatorname{ind} L_i\,,\quad D(f(L_i)) = D(L_i)$$

and $f(L_i) \subset \overline{f(L_i)}$, we have

$$(3) \qquad \operatorname{ind} L_i \leqq \operatorname{ind} \overline{f(L_i)}\,,\quad D(L_i) \leqq D\overline{f(L_i)}$$

see [3] (introduction). Inequality

$$(4) \qquad \operatorname{Ind} L_i \leqq \operatorname{Ind} \overline{f(L_i)}$$

follows from Corollary 2.3 and SM1 §2. Let $\phi_0 = \phi \cap \psi_0$, where ϕ is defined in Corollary 4.1. Then from Corollary 4.1, (2) and (19) §1 it follows that

$$(5) \qquad \phi_0 \in G_X\,.$$

Since f is a homeomorphism, we have $f(P(X)) = P(f(X)) \subset P\overline{f(X)}$ Inclusion $P\overline{f(X)} \subset f(P(X))$ follows from condition (e) of Corollary 4.1. Consequently condition (d) of the Compactification Theorem also holds. We obtain that $\psi \supset \phi_0$, where the set ψ was defined in Theorem 1.1 and, by virtue of (5), $\psi \in G_X$. The theorem is proved. □

5. Uniformly zero dimensional mappings.

THEOREM 5.1. *Let X be a space and a fixed countable system of closed sets L_i, $i = 1, 2, \cdots$ such that dimensions* $\operatorname{Ind} L_i$ *exist. Then the set ψ of all uniformly zero dimensional mappings*[10] $f\colon X \to I^\omega$ *of the space X to the Hilbert cube I^ω such that for each i we have:*

(a) $\operatorname{Ind} L_i = \operatorname{Ind} \overline{f(L_i)}$

(b) $D(L_i) = D\overline{f(L_i)}$

(c) $\operatorname{ind} L_i \leqq \operatorname{ind} \overline{f(L_i)} \leqq \omega_0 + \operatorname{ind} L_i$

(d) $\operatorname{ind} L_i = \operatorname{ind} \overline{f(L_i)}$ *if L_i is strongly metrizable*

(e) *f is a standard mapping on L_i*

contains an everywhere dense set of type G_δ in the space $C(X, I^\omega)$.

To prove this theorem we need some preliminary lemmas.

LEMMA 5.1. *Let X be a bicompactum, not necessarily metrizable (respectively by a metric space) and $f\colon X \to Y$ be a zero dimensional mapping (respectively uniformly zero dimensional) in a compactum Y. Then*

(a) $\operatorname{ind} Y \geqq \operatorname{ind} X$

(b) $D(Y) \geqq D(X)$.

[10] We recall that $f\colon X \to Y$ is uniformly zero dimensional if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for every set $\mathscr{M} \subset Y$ of diameter $\mathscr{M} < \delta$ the set $f^{-1}(\mathscr{M})$ is a union of a discrete collection of sets, each of them having the diameter $< \varepsilon$.

Proof. Inequality (a) was proved in [23] (Theorem 1), Zarelua. Let $D(Y) = \alpha$ and

$$Y = \{Y_\gamma : \gamma \leqq J(\alpha)\}$$

be an α-D-representation of Y. Since zero dimensional mappings of bicympacta (respectively uniformly zero dimensional mappings of metric spaces) do not lower dimension Ind; see [17] (respectively [6]), it is easy to see that

$$X = \cup \{Y = f^{-1}(Y_\gamma) : \gamma \leqq J(\alpha)\}$$

is an α-D-representation of X. Consequently, $D(X) \leqq \alpha$. □

LEMMA 5.2. *Let X, Y be spaces with dimensions* $\operatorname{Ind} X$, $\operatorname{Ind} Y$ *and $f: X \to Y$ be a uniformly zero dimensional standard mapping. Then* $\operatorname{Ind} Y \geqq \operatorname{Ind} X$.

Proof. We shall prove this theorem by induction on $\operatorname{Ind} Y$. Let $\operatorname{Ind} Y = \alpha$ and $\alpha < \omega_0$. Then $\dim Y = D(Y) = \operatorname{Ind} Y$ ((3) introduction) and our lemma follows from Lemma 5.1(b). Let $\alpha \geqq \omega_0$ and F, G be a pair of closed disjoint subsets in X such that

$$F \cup G \subset P(X) .$$

By virtue of (a) definition 3.2 we have

$$f(F) \cap f(G) = \varnothing .$$

Since X is weakly infinite dimensional (SM1 §2), $P(X)$ is compact by Corollary 2.1 and consequently sets $f(F)$, $f(G)$ are closed. Let C be a partition between $f(F)$ and $f(G)$ such that

$$\operatorname{Ind} C < \alpha . \tag{1}$$

Then $f^{-1}(C)$ is a partition between F and G in X. Since clearly the restriction of any uniformly zero dimensional standard mapping to a closed subset is also zero dimensional standard mapping we have by inductive assumption and (1)

$$\operatorname{Ind} f^{-1}(C) \leqq \operatorname{Ind} C < \alpha .$$

Our lemma now follows from Corollary 2.2(a). □

LEMMA 5.3. *Let X be a space with dimension* $\operatorname{Ind} X$. *Then* $\operatorname{ind} X \leqq \omega_0 + \operatorname{ind} P(X)$.[11]

Proof. We will prove this lemma by induction on $\operatorname{ind} P(X)$. If

[11] We consider that $\omega_0 + (-1) = \omega_0$.

$\operatorname{ind} P(X) = 0$, then for any point $x \in P(X)$ and a closed set $F \subset P(X)\setminus\{x\}$ there exists a partition C in X between $\{x\}$ and F such that

$$C \cap P(X) = \varnothing .$$

Then by Theorem SM1 and Corollary 2.1, C is finite dimensional and $\operatorname{ind} C < \omega_0 + \operatorname{ind} P(X)$. Therefore $\operatorname{ind} X \leqq \omega_0 + \operatorname{ind} P(X)$ by Corollary 2.2(b). If $\operatorname{ind} P(X) \geqq 1$ then for any point $x \in P(X)$ and a closed set $F \subset P(X)\setminus\{x\}$ there exists a partition C in X between $\{x\}$ and F such that

$$\operatorname{ind}(C \cap P(X)) < \operatorname{ind} P(X) .$$

Since clearly $P(C) \subset C \cap P(X)$ we have by inductive assumption that

$$\operatorname{ind} C \leqq \omega_0 + \operatorname{ind} P(C) < \omega_0 + \operatorname{ind} P(X) .$$

Our lemma now follows from Corollary 2.2(b). □

Proof of Theorem 5.1. Let R be the set of all uniformly zero dimensional mappings $f: X \to I^\omega$. Then by virtue of [6], Theorem 2.15, p. 359 and (B), p. 354, Katetov

$$R \in G_X .$$

Let ϕ be the setdefined in Corollary 4.1. By virtue of this corollary

$$\phi \in G_X .$$

To prove our theorem it is sufficient to show that

$$\psi \supset \phi \cap R . \tag{3}$$

Let $f \in \phi \cap R$. Then (e) is evident and (a), (b), (d) follow from Corollary 4.1 and Lemmas 5.1, 5.2. Let us prove (c). Inequality

$$\operatorname{ind} L_i \leqq \operatorname{ind} f(L_i) \tag{4}$$

follows from Lemma 5.1. By virtue of Lemma 5.3

$$\operatorname{ind} \overline{f(L_i)} \leqq \omega_0 + \operatorname{ind} P\overline{f(L_i)} . \tag{5}$$

By Corollary 4.1(e)$f^{-1}Pf(L_i) \subset P(L_i)$ and since f is a homeomorphism on $P(L_i)$ (see Definition 3.2(a)), we have

$$\operatorname{ind} P\overline{f(L_i)} \leqq \operatorname{ind} P(L_i) \leqq \operatorname{ind} L_i . \tag{6}$$

Property (c) now follows from (4), (5), (6). Thus property (3) and Theorem 5.1 are proved.

6. On bicompactifications of metric spaces.

THEOREM 6.1. *For any space X with dimension* $\operatorname{Ind} X$ *there exists a bicompactification $bX \supset X$ such that*:

(a) $\operatorname{Ind} bX = \operatorname{Ind} X$

(b) $DbX = DX$

(c) $\operatorname{ind} X \leqq \operatorname{ind} bX \leqq \omega_0 + \operatorname{ind} X$

(d) $\operatorname{ind} bX = \operatorname{ind} X$ *if X is strongly metrizable*

(e) *weight bX = weight X.*

Property (a) *was also proved in* [17], *Pasynkov, for normal spaces. We note that for every metric space X there is not a bicompactum bX such that* $\operatorname{ind} bX = \operatorname{ind} X$, $\operatorname{Ind} bX = \operatorname{Ind} X$. *For example, if* $\operatorname{ind} X = 0$, $\operatorname{Ind} X = 1$ (*see* [18], *Roy*) *then for any bicompactum bX with* $\operatorname{ind} bX = 0$ *we have also* $\operatorname{Ind} bX = 0$.

We recall that a mapping $f: X \to Y$ is called scattering (see [23], Zarelua) if for any point $x \in X$ and its any neighborhood $U \ni x$ there is a neighborhood $V \ni f(x)$ such that for some open sets $W_1, W_2 \subset X$ we have:

$$f^{-1}(V) = W_1 \cup W_2 , \quad W_1 \cap W_2 = \emptyset , \quad x \in W_1 \subset U .$$

Every uniformly zero dimensional mapping is obviously scattering.

Proof of Theorem 6.1. Since for a space X with weight $X < \aleph_0$ the theorem is trivial, we suppose weight $X \geqq \aleph_0$. By virtue of Theorem 5.1 there exists a uniformly zero dimensional mapping $f: X \to K$ in compact $K \subset I^\omega$ such that we put $bX = K$ the conditions (a) – (d) hold and f is a standard mapping. Since f is scattering, there exists a bicompactification $bX \supset X$ and a scattering mapping

$$F: bX \longrightarrow K$$

such that the restriction of F to X is f and the condition (e) holds. This result was proved in [23]. Since $X \subset bX$ we have

$$(1) \qquad \operatorname{ind} X \leqq \operatorname{ind} bX$$

see (1) (introduction). Since every scattering mapping is obviously zero dimensional, we obtain by virtue of Lemma 5.1

$$(2) \qquad D(K) = D(X) \geqq D(bX)$$

$$(3) \qquad \omega_0 + \operatorname{ind} X \geqq \operatorname{ind} K \geqq \operatorname{ind} bX$$

$$(4) \qquad \operatorname{ind} X = \operatorname{ind} K \geqq \operatorname{ind} bX \quad \text{if} \quad X \quad \text{is strongly metrizable} .$$

Thus, conditions (b)-(e) hold. Moreover we obviously can consider

$$(5) \qquad bX = \bar{X}$$

where closures are taken in bX. Let us prove the inequality

$$(6) \qquad \operatorname{Ind} bX \leqq \operatorname{Ind} X .$$

Since $f\colon X \to K$ is a standard mapping there is a standard representation

$$(7) \qquad X = P(X) \cup \bigcup_{i=1}^{\infty} C_i$$

such that conditions (a), (b), (c) of Definition 3.2 hold. Let us prove the equality:

$$(8) \qquad bX = P(X) \cup \bigcup_{i=1}^{\infty} \bar{C}_i .$$

Let $x \in bX \backslash P(X)$ be any point. Then there are open in bX sets $V \supset P(X)$ and $W \ni x$ such that

$$(9) \qquad \bar{V} \cap \bar{W} = \varnothing .$$

We put

$$U = V \cap X .$$

Then U is a neighborhood of $P(X)$ in X. Since $P(X)$ is compact for some $\varepsilon > 0$, we have:

$$O_\varepsilon(P(X)) = \{x\colon \delta(x, P(X)) < \varepsilon\} \subset U .$$

Since (7) is a standard representation of X, by virute of (c) (Definition 3.1) we have:

$$C_i \subset U \quad \text{for} \quad 1/i < \varepsilon .$$

Consequently, by virtue of (9),

$$\bar{W} \cap \overline{\left(P(X) \cup \bigcup_{j=k}^{\infty} \bar{C}_j\right)} = \varnothing \quad \text{for any } k \text{ such that } 1/k < \varepsilon .$$

This proves the condition (8) and shows that

(10) the system $\{\bar{C}_i\}$ is locally finite on $bX \backslash P(X)$.

Moreover from condition (b) of Defininition 3.2 it follows that

$$(11) \qquad \overline{F(\bar{C}_i)} \cap F(P(X)) = \overline{F(\bar{C}_i)} \cap \overline{F(\bar{C}_j)} = \varnothing \quad \text{for} \quad |i - j| > 1$$

and consequently

$$(12) \qquad \bar{C}_i \cap \bar{C}_j = \varnothing = \bar{C}_i \cap P(X) \quad \text{for} \quad |i - j| > 1 .$$

From conditions (8), (10), (12) it follows that (see Definition 2.3),

(13) the system $\{\bar{C}_i\}$ is simple with respect to $X\backslash P(X)$. Since f is a standard mapping, we have:

$$\dim \overline{f(C_i)} = \dim \overline{F(\bar{C}_i)} < \infty , \tag{14}$$

and

(15) the restriction of $F = f$ to $P(X)$ is a homeomorphism.

From conditions (13), (11), (14), (15) it follows that conditions of Lemma 2.4 are fulfilled for $B = bX$, $B_i = \bar{C}_i$ and $P = P(X)$. Consequently by Lemma 2.4

$$\operatorname{Ind} bX \leqq \operatorname{Ind} K = \operatorname{Ind} X .$$

If $\operatorname{Ind} bX = X$ then our theorem is proved. Otherwise we can put

$$b_0X = bX$$

and consider the disjoint sum $bX = b_0X \cup K$ which obviously satisfies conditions (a)-(e). □

7. **Separable spaces which have no compactifications with the same dimension ind.** By Theorem 6.1 (or by Corollary 1.1) every separable space X with dimension $\operatorname{Ind} X$ has a compactification cX such that

$$\operatorname{ind} X = \operatorname{ind} cX .$$

In this section we construct examples of separable spaces which have no compactification with the same dimension ind. Similar examples for dimension D we shall give in the next section.

THEOREM 7.1. *For any limit ordinal number α, $\omega_0 \leqq \alpha < \omega_1$, there exists such complete*[12] *weakly-countable dimensional*[13] *separable space X_α with dimension* $\operatorname{ind} X_\alpha = \alpha$ *such that for any compactification $Y \supset X_\alpha$ we have*

$$\operatorname{ind} Y > \alpha = \operatorname{ind} X_\alpha .$$

We note that by Hurewicz's theorem [5] *every finite dimensional space X has a compactification $cX \supset X$ such that* $\operatorname{ind} cX = \operatorname{ind} X$. *It also follows from Corollary* 1.1 *because for finite dimensional space* X $\operatorname{Ind} X = \operatorname{ind} X$.

[12] We consider a space X to be complete if it is an absolute G_δ-set or equivalently if we can introduce on X a complete topology preserving metric.

[13] A space X is weakly countable dimensional if X is a union of countable number of finite dimensional closed subsets.

DEFINITION 7.1. *Let α be an ordinal number. We put*

$$\varphi(\alpha) = \alpha \quad for \quad \alpha \leqq \omega_0 .$$

If $\alpha > \omega_0$, then $\alpha = \omega_0 + \beta$ for some $\beta > 0$ and we put

$$\varphi(\alpha) = \omega_0 + \omega_0 \times \beta .$$

We will use the following results, see [13], §1, *Luxemburg.*

(1) *For any $\alpha < \omega_1$ there exists a weakly countable dimensional compactum Y_α such that:*

$$\operatorname{ind} Y_\alpha = \alpha \operatorname{Ind} Y_\alpha = \varphi(\alpha) .$$

(2) *For any compactum Y having a dimension* $\operatorname{ind} Y$ *we have*

$$\operatorname{ind} Y \leqq \operatorname{Ind} Y \leqq \varphi(\operatorname{ind} Y) .$$

From Definition 7.1 it follows that (see also [13, Lemma 1.1]).

(3) If $\alpha < \beta$ then $\varphi(\alpha) < \varphi(\beta)$.

(4) If $\alpha = \sup\{\beta_s\}$ then $\varphi(\alpha) = \sup\{\varphi(\beta_s)\}$.

Construction of the space X. Since α is a limit number there exists such a sequence of ordinal numbers $\{\gamma_i\}$ so that

$$(5) \quad \begin{cases} \gamma_i < \gamma_{i+1} < \alpha , \quad \gamma_i = \beta_i + 1 \\ \alpha = \sup\{\gamma_i : i = 1, 2, \cdots\} = \sup\{\beta_i : i = 1, 2, \cdots\} . \end{cases}$$

By virtue of (1) there exists a weakly countable dimensional compacta K_i such that

$$(6) \quad \operatorname{ind} K_i = \gamma_i , \quad \operatorname{Ind} K_i = \varphi(\gamma_i) .$$

Since by virtue of (3), (5), $\varphi(\beta_i) < \varphi(\gamma_i)$ we can take in every compactum K_i a pair of closed sets F_i, $G_i (F_i \cap G_i = \varnothing)$ such that

(7) Any partition C between F_i and G_i has the dimension $\operatorname{Ind} C \geqq \varphi(\beta_i)$.

Let K_i' be a compactum which we obtain by identification of all points of the F_i with some point $n_i \in F_i$ and by identification of all points of the set G_i with some point $v_i \in G_i$. Let

$$(8) \quad Z = \bigcup_{i=1}^{\infty} K_i' , \quad K_i^i \cap K_j' = \varnothing \quad \text{for} \quad i \neq j$$

be a discrete sum of compacta K_i'. We put

$$U = \bigcup_{n=1}^{\infty} \{u_n\}\ , \quad V = \bigcup_{n=1}^{\infty} \{v_n\}\ , \quad (v_n,\, u_n \in K_n')\ .$$

Then U and V are disjoint closed subsets in Z. We take a countable number of copies Z_p of a space Z. The set in Z_p which corresponds to a set $A \subset Z$ we shall denote by A_p. *In* the set $\bigcup_{p=1}^{\infty} Z_p$ with topology of a discrete union of copies Z_p we identify the point v_p^n with the point u_{p+1}^n for all n and p. Then we obtain the space

$$\tilde{X}_\alpha = \bigcup_{p=1}^{\infty} Z_p \tag{9}$$

such that

$$Z_p \cap Z_{p+1} = V_p = U_{p+1}\ .$$

We put

$$X_\alpha = \tilde{X}_\alpha \cup \{\delta\}$$

where in a point δ we define the topology by the open basis

$$G_{k+1} = X_\alpha \Big\backslash \bigcup_{p=1}^{k} Z_p$$

and in $\tilde{X}_\alpha$ the topology is preserved.

Proof. From condition (7) it follows that

(10) any partition C between $\{u_i\}$ and $\{v_i\}$ has the dimension $\operatorname{Ind} C \geqq \varphi(\beta_i)$.

Consequently $\operatorname{Ind} K_i' > \varphi(\beta_i)$ and by virtue of (2)

$$\operatorname{ind} K_i' \geqq \beta_i + 1 = \gamma_i\ . \tag{11}$$

Besides that

$$\operatorname{ind} K_i' \leqq \gamma_i + 1 < \alpha\ . \tag{12}$$

Indeed, from (6) it follows that for any point $x \in K_i'(\{v_i\} \cup \{u_i\})$

$$\operatorname{ind}_x K_i' \leqq \gamma_i\ .$$

Further if V is an arbitrary neighborhood of a point v_i in K_i' and $\bar{V} \not\ni u_i$ then

$$Fr\, V \subset K_i' \backslash (\{v_i\} \cup \{u_i\})$$

and consequently $\operatorname{ind} Fr\, V \leqq \gamma_i$. Similarly one can prove that $\operatorname{ind}_{v_i} K_i' \leqq \gamma_i$. Hence (12) holds. From conditions (5), (8), (9), (11), (12) it follows that

$$\operatorname{ind} \tilde{X}_\alpha = \alpha .$$

Since obviously $FrG_{k+1} = V_k = U_{k+1}$ and the set V_k is countable, we have $\operatorname{ind}_\delta X_\alpha \leqq 1$. Therefore

$$\operatorname{ind} X_\alpha = \alpha . \tag{13}$$

Since compacta K_i are weakly countable dimensional, compacta K_i' are also weakly countable dimensional. Consequently the space X_α is also weakly countable dimensional. Since X_α is a union of locally compact separable spaces $\tilde{X}_\alpha$ and a point δ, X_α is complete and separable. Let us prove that X_α is not contained in any compactum Y with

$$\operatorname{ind} Y = \alpha . \tag{14}$$

We suppose on the contrary. Then $Y \supset X_\alpha$ and (14) holds. By virtue of (2)

$$\operatorname{Ind} Y \leqq \varphi(\alpha) .$$

We consider a neighborhood O_δ in Y of the point δ such that

$$O_\delta \cap U_1 = \emptyset . \tag{15}$$

Since $\operatorname{Ind} Y \leqq \varphi(\alpha)$, it follows that there exist neighborhoods W_i, $i = 1, 2, \cdots$, of the point δ such that for all i

$$\bar{W}_i \subset O_\delta , \quad \bar{W}_i \subset W_{i+1} , \quad \operatorname{ind} Fr \bar{W}_i < \varphi(\alpha) . \tag{16}$$

Since sets G_k, $k = 2, 3, \cdots$, constitute a basis in X_α at the point δ, there is some integer m such that $\bar{G}_m \subset W_1$. Thus for $k \geqq m$ the sets U_m are contained in W_1. We shall show that $U_{m-1} \backslash \bar{W}_2$ contains only finitely many points. Indeed, if $u_{m-1}^{k_1}, u_{m-1}^{k_2}, \cdots$, is an infinite sequence of points which does not lie in $\bar{W}_2$, then since $u_m^{k_1}, u_m^{k_2}, \cdots \subset U_m \subset \bar{W}_m \subset W_2$, we have that the set $Fr \bar{W}_2$ separates points $u_{m-1}^{k_i}$ and $v_{m-1}^{k_i}$ and by virtue of (10)

$$\operatorname{Ind} Fr \bar{W}_2 \geqq \varphi(\beta_{k_i}) \quad \text{for any} \quad i . \tag{17}$$

By virtue of (5) $\sup\{\beta_{k_i}: i = 1, 2, \cdots\} = \alpha$ and consequently by virtue of (4) $\sup\{\varphi(\beta_{k_i}): i = 1, 2, \cdots\} = \varphi(\alpha)$. Therefore from (17) it follows that

$$\operatorname{Ind} Fr \bar{W}_2 \geqq \varphi(\alpha)$$

which contradicts the condition (16). Therefore, the set $U_{m-1} \backslash \bar{W}_2$ is finite. Analogously to the case $\ell = m - 1$ we can show by induction that for $\ell = 1, 2, \cdots, m - 1$ the set $U_\ell \backslash \bar{W}_{m+1-\ell}$ consists of only finitely many points. From this taking $\ell = 1$ we get that the set $U_1 \backslash \bar{W}_m$ is finite, but this contradicts conditions (15), (16). Thus, assumption of the existence of a compact space $Y \supset X_\alpha$ with $\operatorname{ind} Y = \alpha = \operatorname{ind} X_\alpha$ leads us to a contradiction. □

8. On compactifications of spaces with D-dimension. In [4], Henderson, it was proved that every separable weakly infinite dimensional space has a compactification of the same dimension D. This result also follows from Corollary 1.1. In that paper it was the conjecture which we prove in the following theorem:

THEOREM 8.1. *Every separable space X has a compactification $cX \supset X$ such that*

$$D(cX) \leqq D(X) + 1^{14} . \tag{1}$$

This result appeared also in [7] earlier than ours. In [4] Henderson gave an example of separable space $X(D(X) = \omega_0)$ which is not contained in any compactum with the same D-dimension. In the following theorem this result will be generalized for all $\alpha < \omega_1$, $\alpha \geqq \omega_0$.

THEOREM 8.2. *For any α, $\omega_0 \leqq \alpha < \omega_1$ there exists a separable space X_α such that $D(X_\alpha) = \alpha$ and X_α is not contained in any compactum with the same D-dimension.*

To prove Theorem 8.1 we need some preliminary lemmas.

LEMMA 8.1. *Let*

$$X = \cup \{A_\gamma : \gamma \leqq J(\beta)\} \tag{2}$$

be a β-D-representation of a separable space X. Then X is homeomorphic to a subset of a separable space Z such that for some β-D-representation of Z

$$Z = \cup \{B_\gamma : \gamma \leqq J(\beta)\} \tag{3}$$

the set $B_{J(\beta)}$ is compact.

Proof. By virtue of Lemmas 1.4 and 3.3 there exists a homeomorphism $f: X \to I^\omega$ such that:

$$\dim \overline{f(A_{J(\beta)})} \leqq \dim A_{J(\beta)}$$
$$\overline{f(A_{J(\beta)})} \cap f(X \backslash A_{J(\beta)}) = \varnothing .$$

[14] We will prove that it is possible to take such compactification cX that $cX \in AR$ and inequality (1) holds.

We put $Z = f\overline{(A_{J(\beta)})} \cup f(X)$, $B_\gamma = \overline{f(A_\gamma)} \cap Z$, $(\gamma \leqq J(\beta))$. Then clearly $B_{J(\beta)} = \overline{f(A_{J(\beta)}) \text{ and the}}$ equality (3) is a needed β-D-representation of Z. □

LEMMA 8.2. *Let (2) be a β-D-represntation of a space X, $\alpha = J(\beta) \geqq \omega_0$, then for any sequence of ordinal numbers $\{\gamma_i\}$ such that*

$$\gamma_{i+1} > \gamma_i\ , \quad \sup\{\gamma_i : i = 1, 2, \cdots\} = \alpha$$

there exists a simple with respect to $U = X \backslash A_{J(\beta)}$ system of sets $\mathscr{F} = \{F_i\}$ such that

$$D(F_i) \leqq \gamma_i\ .$$

Proof. Let $\delta < \alpha$ then we put

$$U = X \backslash \{A_\gamma : \delta \leqq \gamma \leqq J(\beta)\}\ .$$

Then by Lemma 8.3 in [13], Luxemburg, $D(U_\delta) < \alpha$. From Definition 0.3 it follows that the sets U_δ are open and since the system $(U_{\gamma_i} : i = 1, 2, \cdots\}$ is countable, our lemma follows. □

LEMMA 8.3. *Let P_i, $i = 1, 2, 3$, be clssed subsets of a space $P = P_1 \cup P_2 \cup P_3$ and $P_1 \cap P_3 = \varnothing$. Let $f: P \to K$ be a homeomorphism. Then there exists a homeomorphism $g: P \to K \times I$, where $I = [0, 1]$, such that $\overline{g(P_1)} \cap \overline{g(P_3)} = \varnothing$.*

Proof. Let $\varphi: P \to [0, 1]$ be a continuous function such that $\varphi(P_1) = 1$, $\varphi(P_3) = 0$. Then we put

$$g(x) = \{f(x), \varphi(x)\}\ , \quad x \in P\ .$$

Then, clearly, $g: P \to K \times I$ is a suitable mapping. □

DEFINITION 8.1. Let spaces P, $P_i i = 1, 2, 3$, satisfy the condition of Lemma 8.3 and $h: P_1 \cup P_2 \to K$ is a homeomorphism in a compactum K such that

$$\overline{h(P_1)} \cap \overline{h(P_2 \cap P_3)} = \varnothing\ .$$

We put $T = \overline{h(P_1)} \cup h(P_2) \subset K$ and in disjoint sum $T \cup P_3$ we identify every point $h(x) \in T$ with a point x for $x \in P_2 \cap P_3 \subset P_3$. Then we get a factor space, which we denote by $\mu(h, K, P_1, P_2, P_3)$.

LEMMA 8.4. *There exists an imbedding $\pi: P \to \mu(h, K, P_1, P_2, P_3) = \mu$. Moreover, the space μ is separable and metrizable if P is separable and metrizable, $\pi(P)$ is everywhere dense in μ,*

$$\mu = \overline{\pi(P_1)} \cup \pi(P_2) \cup \pi(P_3)$$

the set $\pi(P_3)$ *is closed in* μ, $\pi(P_1)$ *is compact,* $\overline{\pi(P_1)} \cup \pi(P_2)$ *is closed in* μ *and can be imbedded in* K *and* $\overline{\pi(P_1)} \cup \pi(P_3) = \varnothing$. *The lemma is evident.*

We will consider the following conjecture:

$\mathscr{C}(\alpha)$. For any separable space X with $D(X) < \alpha$ there exists a compactification $K \supset X$ with $D(K) < \alpha$.

DEFINITION 8.2. Let $X = \cup \{X_i : i \in I\}$ be a union of spaces X_i then X is called an inductive limit $X = \mathrm{Lim}\{X_i\}$ if a set $U \subset X$ is open in $X \Leftrightarrow U \cap X_i$ is open in X_i.

LEMMA 8.5. *Let* α *be a limit ordinal number and conjecture* $C(\alpha)$ *is ture. Let the system* $\{F_i\}(i = 1, 2, \cdots)$ *in a separable space* X *be simple with respect to* X *and* $D(F_i) < \alpha$. *Then there exists a locally compact separable space* G, *the system of compacta* $\{G_i\}$ *in* G *and imbedding* $f: X \to G$ *such that*:

(a) $\overline{f(F_i)} = G_i$.

(b) G_i *is a compactum.*

(c) *The system* G_i *is simple with respect to* G.

(d) $D(G_i) < \alpha$, $D(G) \leqq \alpha$.

Proof. We will define by induction spaces $Y_i \supset X$, $i = 0, 1, 2, \cdots$ such that

$$Y_i = P_0 \cup \cdots \cup P_i \cup F_{i+1} \cup F_{i+2} \cup \cdots \tag{4}$$

$$\bar{F}_j = P_j \quad \text{and} \quad P_j \text{ is compact} \quad j = 0, 1, \cdots, i\,. \tag{5}$$

where $\bar{F}_j$ is a closure of F_j in Y_i.

$$\bar{Q}_1^i \cap \bar{Q}_3^i = \varnothing\,, \quad Q_1^i = \bar{Q}_1^i\,, \quad Q_3^i = \bar{Q}_3^i\,, \quad F_{i+2} = \bar{F}_{i+2} \tag{6}$$

where $Q_1^i = \bigcup_{k=0}^i P_k \cup F_{i+1}$, $Q_3^i = \bigcup_{j=i+3}^\infty F_j$, $Q_2^i = F_{i+2}$.

$$P_k \cap P_\ell = P_k \cap F_\ell = \varnothing \quad \text{if} \quad |k - \ell| > 1\,. \tag{7}$$

$$D(Q_1^i) < \alpha\,. \tag{8}$$

We put $P_0 = F_0 = \varnothing$, $Y_0 = X$. Let a space Y_i for $i = i_0$ be defined. We put $Q_2^i = F_{i+2}$. Then from (6) it follows that sets Q_1^i, Q_2^i, Q_3^i are closed in Y_i. Since $D(F_k) < \alpha$, from condition (8) and the sum theorem for D-dimension (see [3], Henderson), it follows that $D(Q_1^i \cup Q <$ α. Consequently, by $C(\alpha)$ there exists an imbedding $f: Q_1^i \cup Q_2^i \to K$ in a compactum K with $D(K) < \alpha$. From Lemma 3.3 it follows that there exists an imbedding $\psi: Q_1^i \cup Q_2^i \to K \times I$ such that $\overline{\psi(Q_1^i)} \cap \overline{\psi(Q_2^i \cap Q_3^i)} = \emptyset$. We put $Y_{i+1} = \mu(\psi, K \times I, Q_1^i, Q_2^i, Q_3^i)$, $P_{i+1} = [F_{i+1}]$, where $[F_{i+1}]$ is a closure of F_{i+1} in Y_{i+1}. Conditions (4), (5), (6), (7), for $i_0 + 1$ follow from Lemma 8.4. Since by Lemma 8.4 $Q_1^{i_0+1} = Q_1^{i_0+1} \cup Q_2^{i_0}$ can be embedded in $K \times I$, we have $D(Q_1^{i_0+1}) < D(K \times I)$. Since α is a limit number, property (8) follows from the inequality $D(K \times I) \leqq D(K) + 1$ (see [3], Henderson). Hence, space Y_i are constructed. Let us put

$$G = \lim\{Y_i\}\,, \quad G_i = \pi_i(P_i)$$

where $\pi_i: Y_i \to G$ are inclusions. From conditions (6), (7) it follows that for every point $x \in G$ there exists i and an open in Y_i set U, $x \in U \subset P_j \cup P_{j+1} (j + 1 \leqq i)$, such that $\pi_i(U)$ is a neighborhood of x in G. Consequently, by virtue of (7) we obtain the condition (c). Conditions (a), (b) follow from (5). Since π_i is a homeomorphism, inequality $D(G_i) < \alpha$ follows from inclusion $P_j \subset Q_1^i (j < i)$ and (8). Since collection of compacta $\{G_i\}$ is locally finite, the inequality $D(G) \leqq \alpha$ follows from sum theorem for D-dimension (see [3], Henderson). Hence property (d) and Lemma 8.5 are proved. □

NOTATION. Let $K \subset Y$, $X \subset Y$, K is compact and $f: K \to R$ is a mapping in a compact space R, such that if $x \in K \cap X$ then for any $y \in K$, $y \neq x$, we have $f(x) \neq f(y)$. We will consider points $x, y \in Y$ to be equivalent if either $x = y \in Y \backslash K$ or $f(x) = f(y)$ and $x, y \in K$. We get a factor space which we will denote by $F = F(K, X, Y, f)$ and a factor mapping $\pi: Y \to F$.

LEMMA 8.6. *If a space Y is metrizable and separable then F is also metrizable and separable. Moreover, the restriction of π to X is a homeomorphism.*

The lemma follows from well known theorems on factor mappings.

DEFINITION 8.2. The equality

$$Y = K \cup \bigcup_{i=1}^{\infty} F_i \tag{9}$$

we will call canonic representation of a space Y if

(a) the system $\{F_i\}$ is simple with respect to $Y \backslash K$.

(b) any sequence of points $\{x_i\}$ such that $x_i \in F_{i(k)}$, $i(k+1) > i(k)$, has an accumulation point $y \in K$.

LEMMA 8.7. *Let X be a separable space and* (2) *be a β-D-representation of $X (\beta \geqq \omega_0)$. Then there exists a separable space $Y \supset X$ such that Y has a canonic representation* (9) *and*

(i) *K is compact.*
(ii) $D(F_i) < \alpha = J(\beta)$.
(iii) $K \cap X = A_{J(\beta)}$.

Proof. By Lemma 8.2 there exists a simple with respect to $V = X \setminus A_{J(\beta)}$ system $\{F'_i\}$ in a space X such that

$$D(F'_i) < \alpha . \tag{10}$$

Since $\{F'_i\}$ is a simple system, sets F'_i and $P_i = A_{J(\beta)} \cup \bigcup\{F'_k : |k - i| > 1\}$ are closed and disjoint. Since the set of all homeomorphisms $\varphi: X \to I^\omega$ contains an everywhere dense G_δ-set in $C(X, I^\omega)$, we get by Lemma 1.2 and (19)§1 a homeomorphism $f: X \to I^\omega$ such that

$$\overline{f(P_i)} \cap \overline{f(F_i^1)} = \varnothing \quad \text{for any} \quad i . \tag{11}$$

We put

$$K_i = \overline{\bigcup_{j=1}^{\infty} f(F_j) \cup A_{J(\beta)}} \subset I^\omega , \quad K = \bigcap_{i=1}^{\infty} K_i , \quad F_i = f(F'_i) \tag{12}$$

$$Y = K \cup \bigcup_{i=1}^{\infty} F_i . \tag{13}$$

Since f is a homeomorphism, $Y \supset f(X)$, we can consider that $X = f(X)$. Then conditions (i)–(iii) follow from (10), (11), (12), (13). Let us prove that representation (13) is canonic. Property (a) of Definition 8.2 follows from (11). Let $\{x_i\}$ be such a sequence as in (b) of Definition 8.2. Then $x_i \subset F_{i(k)} \subset K_{i(k)}$. Since $K_{i+1} \subset K_i$ and K_i are compact, there is an accumulation point $y \in \bigcap_{i=1}^{\infty} K_i = K$. The lemma is proved. □

NOTATION. It is known (see [1], Bothe) that every separable n-dimensional space is contained in compact separable $(n+1)$-dimensional AR-space R^{n+1}. In particular R^{n+1} contains a universal n-dimensional compact space $A_n \subset R^{n+1}$.

LEMMA 8.8. *In Lemma 8.7 we can also require that $K = R^{n+1}$, $R^{n+1} \cap X = A_{J(\beta)} \subset A_n$, where $n = K(\beta)$.*

Proof. By Lemma 8.1 we can consider the set $A_{J(\beta)}$ to be a compactum. Let Y be a space from Lemma 8.7. Since $\dim A_{J(\beta)} \leq K(\beta) = n$ (see Definition 0.3 (d)), there is a homeomorphism $g: A_{J(\beta)} \to A_n \subset R_{n+1}$. Let $f: K \to R_{n+1}$ be an extension of g. We put $Y' = F(K, X, Y, f)$. Then by Lemma 8.6 we can consider that $X \subset Y'$, and the equality

$$Y' = R^{n+1} \cup \bigcup_{i=1}^{\infty} F_i \tag{14}$$

is clearly a canonic representation of Y'. Condition

$$R_{n+1} \cap X = A_{J(\beta)}$$

is evident. □

LEMMA 8.9. *Let X be a separable space, $D(X) \leqq \beta$, $\beta = \alpha + n$, $\alpha = J(\beta)$, $n = K(\beta)$. Then there exists a compact space $K \supset X$ such that*

(i) $D(K) \leqq \beta + 1 = \alpha + n + 1$.

(ii) $K = \overline{R^{n+1}} \cup \bigcup_{i=1}^{\infty} H_i$, $H = \bigcup_{i=1}^{\infty} H_i$, $H \cap \overline{R^{n+1}} = \emptyset$.

(iii) $D(H_i) < \alpha$, $D(H) \leqq \alpha$.

(iv) *H_i is compact and the system $\{H_i\}$ is simple with respect to H.*

(v) *There exists a homeomorphism $i: \overline{R^{n+1}} \to R^{n+1}$ such that $i(X \cap \overline{R^{n+1}}) \subset A_n \subset R^{n+1}$.*

Proof. We shall prove our theorem by induction on β. If $\beta < \omega_0$ then we put $K = R^{n+1}$, $G = \emptyset$. Let $\beta \leqq \omega_0$. Since $D(X) \leqq \beta$, there exists a β-D-representation of X and by Lemma 8.8 there exists a separable space $Y' \supset X$ such that representation (14) is canonic and

$$X \cap R^{n+1} \subset A_n\,, \quad D(F_i) < \alpha\,. \tag{15}$$

Since the conjecture $C(\alpha)$ holds by inductive assumption, by Lemma 8.5 there exists a separable locally compact space G such that $\bigcup_{i=1}^{\infty} G_i = G$, G_i is compact, the system $\{G_i\}$ is simple with respect to G.

$$D(G_i) < \alpha\,, \quad D(G) \leqq \alpha \tag{16}$$

and there exists a homeomorphism $h: Y' \backslash R^{n+1} \to G$ such that $h(F_i) \subset G_i$. Let us put

$$G' = \omega \cup G$$

where ω is a compactification point and G' is a compactum. Then obviously $D(G') = D(G)$. We put:

$$K = R^{n+1} \times G',\ H_i = R^{n+1} \times G_i,\ H = R^{n+1} \times G,\ \overline{R^{n+1}} = R^{n+1} \times \omega\,. \tag{17}$$

Then from (16), (17) it follows that

$$D(K) = D(G') \oplus D(R^{n+1})^{15} = D(G) \oplus (n+1) \leqq \alpha + n + 1 = \beta + 1\,.$$

Since α is a limit number from (16), (17) it follows that

[15] Here we use the inequality $D(X \times Y) \leqq D(X) \oplus D(Y)$, where $\oplus$ is a natural sum of ordinal numbers, see [3], Henderson.

(18) $$D(H_i) \leqq D(G_i) \oplus D(R^{n+1}) = D(G_i) + n + 1 < \alpha .$$

Moreover the system H_i is simple with respect to H because the system $\{G_i\}$ simple with respect to G. Consequently, from (18) and the sum theorem for D-dimension (see [3]) it follows that $D(H) \leqq \alpha$. Thus conditions (i)–(iv) of our lemma are satisfied. We have only to prove the existence of homeomorphism $F: Y' \to K$ such that

(19) $$F(X) \cap \overline{R^{n+1}} \subset A_n \times \omega .$$

Then condition (v) will take place. Since R^{n+1} is AR space, there exists a retraction $r: Y' \to R^{n+1}$. We define a mapping $q: Y' \to G'$ by the equalities

$$q(y) = h(y) \quad \text{for} \quad y \in Y' \backslash R^{n+1} , \quad q(y) = \omega \in G' \quad \text{for} \quad y \in R^{n+1} .$$

Then obviously q is a homeomorphism on $Y' \backslash R^{n+1}$. We define a mapping $F: Y' \to K$ by the equalities:

$$\pi_1 \circ F(y) = q(y) , \quad \pi_2 \circ F(y) = r(y)$$

where $\pi_1: K = R^{n+1} \times G' \to G'$, $\pi_2: K \to R^{n+1}$ are projections. The condition (19) follows from (15). Let us prove that F is a homeomorphism. The mapping F is obviously injective and continuous. Moreover, since q is a homeomorphisms on $Y' \backslash R^{n+1}$ and r is a homeomorphism on $R^{n+1} \subset Y'$, the mapping F is a homeomorphism on $Y' \backslash R^{n+1}$ and on R^{n+1}. Besides that

$$F(Y' \backslash R^{n+1}) \subset H , \quad F(R^{n+1}) = \overline{R^{n+1}} , \quad F(Y' \backslash R^{n+1}) \cap F(R^{n+1}) = \varnothing .$$

Therefore, for proving that F is a homeomorphism, we have only to prove that for any sequence $\{F(x_n)\}$ such that

(20) $$F(x_n) \in H, \lim_{n \to \infty} F(x_n) = F(y) \in \overline{R^{n+1}}$$

we have

(21) $$\lim_{n \to \infty} x_n = y .$$

From condition (20) it follows that $\lim_{n \to \infty}(\pi_1 \circ F(x_n) = q(x_n) = h(x_n)) = \pi_1 \circ F(y) = \omega$. Since G_i and $G' = \pi_1(K)$ are compact, there exist two subsequences of natural numbers $\{n(k)\}$, $\{i(k)\}$ such that

(22) $$h(x_{n(k)}) \subset G_{i(k)} , \quad n(k+1) > n(k) + 2 , \quad i(k+1) > i(k) + 2 .$$

Since $h(F_i) \in G_i$ and systems $\{G_i\}$, $\{F_i\}$ are simple [from condition (22)], it follows that

(23) $$x_{n(k)} \subset F_{i(k)-1} \cup F_{i(k)} \cup F_{i(k)+1} .$$

Let $j(k)$ is such a number so that $x_{n(k)} \in F_{j(k)}$, then since $\{F_i\}$ is a simple system, from conditions (22), (23) it follows that $j(k+1) > j(k)$. Since (14) is a canonic representation, a sequence $\{x_{n(k)}\}$ has an accumulation point $y' \in R^{n+1}$ and consequently $y' = \lim\{z_j\}$ for some subsequence $\{z_j\}$ of the sequence $\{x_{n(k)}\}$. Moreover $F(y') = F(y)$ and since F is injective, we have $y = y'$. We have proved that for any sequence with condition (20) there exists a subsequence $\{z_j\}$ of a sequence $\{x_n\}$ such that $\lim z_j = y$. From this fact it follows (21). The lemma is proved. □

Theorem 8.1 obviously follows from Lemma 8.9. Let us prove Theorem 8.2.

DEFINITION 8.3. (See [20], Smirnov). For any ordinal number $\beta < \omega_1$ we shall define a compactum K_β. For $\beta < \omega_0 K_\beta$ is a β-dimensional cube. If β is a limit number we consider K_β to be a one-point compactification (with point p_β) of a discrete union of compact $K_\gamma\colon \gamma < \beta$. If $\beta = \alpha + n$, $\alpha = J(\beta)$, $n = K(\beta)$, we put $K_\beta = K_\alpha \times I^n$, where I^n is an n-dimensional cube.

In what follows we will consider α to be a limit number $< \omega_1$ and $n = 0, 1, 2, \cdots$. For any K_β

$$\operatorname{Ind} K_\beta = \beta . \tag{24}$$

(See [20], Smirnov.) Since for any compactum Z

$$D(Z) \geqq \operatorname{Ind} Z , \tag{25}$$

see [4], Henderson (Theorem 2), $D(K_\beta) \geqq \operatorname{Ind} K_\beta = \beta$. By transfinite induction it is easy to prove that $D(K_\beta) \leqq \beta$. Therefore

$$D(K_\beta) = \beta . \tag{26}$$

By definition we have

$$K_{\alpha+n+1} = \{p_\alpha\} \times I^{n+1} \cup \bigcup \{K_\gamma \times I^{n+1}\colon \gamma < \alpha\} . \tag{27}$$

Let S^n be a sphere which is a boundary of the cube I^{n+1}. We put:

$$X_{\alpha+n} = (\{p_\alpha\} \times S^n) \cup \bigcup \{K_\gamma \times I^{n+1}\colon \gamma < \alpha\} \subset K_{\alpha+n+1} . \tag{28}$$

Then $X_{\alpha+n} = Y \cup (\{p_\alpha\} \times S^n)$, where Y is a discrete union of compacta $K_\gamma \times I^{n+1}\colon \gamma < \alpha$. From (26) it follows that

$$D(Y) \leqq \sup\{D(K_\gamma \times I^{n+1})\colon \gamma < \alpha\} = \alpha .$$

Since Y is open in $X_{\alpha+n}$ and $\{p_\alpha\} \times S^n$ is closed we have the inequality (see [3], Henderson, Theorem 4)

$$D(X_{\alpha+n}) \leqq D(Y) + D(p_\alpha \times S^n) = \alpha + n .$$

Let A be an n-dimensional face of the cube I^{n+1}, then $K_\alpha \times A \subset X_{\alpha+n} \subset K_{\alpha+n+1}$ and $K_\alpha \times A$ is homeomorphic to $K_{\alpha+n}$. Consequently

$$D(X_{\alpha+n}) \geqq D(K_\alpha \times A) = D(K_{\alpha+n}) = \alpha + n .$$

Thus $D(X_{\alpha+n}) = \alpha + n$. Let $\{A_i, B_i\} i = 1, \cdots, n+1$ be a system of all pairs of opposite faces in cube I^{n+1}, then clearly $\{K_\alpha \times A_i, K_\alpha \times B_i\}$ is a system of compact disjoint pairs in $X_{\alpha+n}$. Let $C \supset X_{\alpha+n}$ by any compact space, then we have to prove that

$$D(C) \geqq \alpha + n + 1 = \beta + 1 . \tag{29}$$

We put $F_i = K_\alpha \times A_i$, $G_i = K_\alpha \times B_i$. Let D_i be a partition between F_i and G_i in C.

Since clearly

$$F_i \cap K_\gamma \times I^{n+1} = K_\gamma \times A_i , \quad G_i \cap K_\gamma \times I^{n+1} = K_\gamma \times B_i (\gamma < \alpha)$$

the set $C_i^\gamma = D_i \cap K_\gamma \times I^{n+1}$ is a partition between $K_\gamma \times A_i$ and $K_\gamma \times B_i$ in $K_\gamma \times I^{n+1} = K_{\gamma+n+1}$ for all $\gamma < \alpha$. Consequently, by virtue of [13], Luxemburg, Lemma 4.8 we have for all $\gamma < \alpha$

$$\operatorname{Ind}\left[\bigcap_{i=1}^{n+1} C_i = \left[\bigcap_{i=1}^{n+1} D_i \cap K_\gamma \times I^{n+1}\right]\right] \geqq \gamma .$$

Therefore $\operatorname{Ind}(\bigcap_{i=1}^{n+1} D_i) \geqq \sup\{\gamma : \gamma < \alpha\} = \alpha$ and by virtue of [13], Luxemburg, Lemma 4.7

$$\operatorname{Ind} C \geqq \alpha + n + 1 = \beta + 1 . \tag{30}$$

Inequality (29) now follows from (25), (30). The theorem is proved. □

9. On completion of metric spaces. It is known, see [15], Nagata, that every finite dimensional subspace $X \subset Y$ is contained in G_δ-set $G \subset Y$ with the same dimension $\operatorname{Ind} G = \operatorname{Ind} X$. Here we extend this result to infinite dimensional case.

THEOREM 9.1. *For any subspace X of a space Y, such that the dimension* $\operatorname{Ind} X$ *exists, there is a G_δ-set G, such that* $\operatorname{Ind} G = \operatorname{Ind} X$, *$X \subset G \subset Y$, $P(X) = P(G)$.*

We note that by Lemma 2.8 we have also $D(X) = D(G)$.

COROLLARY 9.1. *For any space X with dimension* $\operatorname{Ind} X$ *there exists a completion $G = X$*[16] *such that* $\operatorname{Ind} X = \operatorname{Ind} G$, *weight $X =$*

[16] We call a space $G \supset X$ a completion of a space X if G is an absolute G_δ-set. It is equivelant to the fact that we can introduce in G a complete metric (see [8]).

weight G.

Proof. Let $Y \supset X$ be any complete space and G_0 be a G_δ-set in Y such that $X \subset G_0 \subset Y$, $\operatorname{Ind} G_0 = \operatorname{Ind} X$. Such G_0 exists by Theorem 9.1. We put $G = G_0 \cap \bar{X}$. Then G is G_δ in Y, and consequently is an absolute G_δ-set. Also, $\operatorname{Ind} G \leqq \operatorname{Ind} G_0 = \operatorname{Ind} X$. Inequality $\operatorname{Ind} G \geqq \operatorname{Ind} X$ follows from Corollary 2.3 and (SM1) §2. □

For proving Theorem 9.1 we need some preliminary results.

(S2) [19], Sklyarenko. If in a space X there exists a convergent system (see Definition 2.2) of weakly infinite dimensional open sets with weakly infinite dimensional limit then X is weakly infinite dimensional.

LEMMA 9.1. *Let $K(K \subset X)$ be a limit of a system of open sets $\{\Gamma_n\}$ in a space X, then if $K \subset Y \subset X$ the system $\{\Gamma_n \cap Y\}$ is convergent in Y and K is a limit of this system in Y.*

The lemma is evident.

LEMMA 9.2. *Let a space X have the dimension* $\operatorname{Ind} X$ and $Y \subset X$ *is a subspace such that $Y \supset P(X)$. Then the dimension* $\operatorname{Ind} Y$ *exists and* $\operatorname{Ind} Y \leqq \operatorname{Ind} X$ *and Y is weakly infinite dimensional.*

Proof. Let U_n be the set defined by equality (1) §2. Since $\operatorname{Ind} X$ exists, the space X is weakly infinite dimensional by (SM1) §2, consequently by Theorem (S1) §2 the set $X \backslash \bigcup_{n=1}^{\infty} U_n = P(X)$ is a limit of the system $\{U_n\}$. By Lemma 9.1 $P(X)$ is a limit of the system $\{U_n \cap Y\}$. Since $\dim U_n \cap Y \leqq \dim U_n \leqq n$ sets $U_n \cap Y$ are weakly infinite dimensional. Moreover $P(X)$ is closed in X and, consequently, is also weakly infinite dimensional. Therefore, by virtue of (S2) Y is weakly infinite dimensional. Our lemma now follows from Corollary 2.3. □

Proof of Theorem 9.1. We shall prove this theorem by induction on $\operatorname{Ind} X$. If $\operatorname{Ind} X$ is a finite number then this theorem is known (see [15]), and in this case $P(X) = \varnothing$. Let $\operatorname{Ind} X \geqq \omega_0$. Then $P(X)$ is a nonempty compactum (Corollary 2.1). We put

$$V_n = Y \backslash \overline{O_{1/n} P(X)}\,. \tag{1}$$

Then by Theorem SM1 §2 and by Corollary 2.1 the set $V_n \cap X$ is finite dimensional. Then by inductive assumption there exists G_δ-sets G_n such that

(2) $$V_n \supset G_n \supset V_n \cap X\,, \quad \operatorname{Ind} G_n < \omega_0\,.$$

We put

(3) $$F = \bigcup_{n=1}^{\infty} (V_n \backslash G_n)\,, \quad D_n = (Y \backslash F) \cap V_n\,.$$

Then F is F_σ set in Y, D_n is open in $Y \backslash F$ and

(4) $$X \subset Y \backslash F\,, \quad D_n \subset G_n\,.$$

Consequently, by virtue of (2), D_n is finite dimensional. By virtue of (3) we have

(5) $$\phi = P(X) \cup \bigcup_{n=1}^{\infty} D_n = Y \backslash F\,.$$

Since $P(X)$ is compact, by virtue of (1), (3) the system $\{D_n\}$ is convergent in ϕ and $P(X)$ is a limit of this system. Since $P(X)$ is closed in X, it is weakly infinite dimensional. Since D_n are finite dimensional and consequently weakly infinite dimensional, ϕ is also weakly infinite dimensional by

THEOREM S2. *Since F is F_σ-set, $\phi = Y \backslash F$ is a G_δ-set in Y. Moreover, since $X \subset \phi$ and D_n are fininite dimensional, we have*

(6) $$P(\phi) = P(X)\,.$$

Let $\operatorname{Ind} X = \alpha \geqq \omega_0 \gamma$ then there exists a countable system $\{U_k\}$ of open in Y sets such that a system $\{U_k\}$ forms a large base in $P(X)$ and

(7) $$\operatorname{Ind} Fr\, U_k \cap X \leqq \beta_k < \alpha\,.$$

By inductive assumption for any k there exists such G_δ-set W_k in Y so that

(8) $$W_k \supset Fr\, U_k \cap X\,, \quad \operatorname{Ind} W_k \leqq \beta_k\,, \quad P(W_k) = P(Fr\, U_k \cap X)\,.$$

We put $H_K = Fr\, U_K \backslash W_K$, then H_K is F_σ-set in Y and $H_K \cap X = \varnothing$. Consequently the set $G = \phi \backslash \bigcup_{k=1}^{\infty} H_K$ is a G_δ-set and

(9) $$X \subset G\,, \quad Fr\, U_k \cap X \subset Fr\, U_k \cap G \subseteqq W_k\,.$$

From conditions (8), (9) it follows that $P(Fr\, U_k \cap X) = P(Fr\, U_k \cap G) = P(W_k)$ and by Lemma 9.2 we have

(10) $$\operatorname{Ind} Fr\, U_k \cap G \leqq \operatorname{Ind} W_k \leqq \beta_k < \alpha\,.$$

Since $X \subset G \subset \phi$, by virtue of (6) we have

$$P(G) = P(X) = P(\phi)\,.$$

Consequently by Lemma 9.2 G is weakly infinite dimensional, because ϕ is weakly infinite dimensional.

Since $\{V_k\}$ is a large base in $P(G)$, for any closed subsets F_1, F_2 in $P(G)$ there exists a set U_k such that $\mathrm{Fr}\, U_k \cap G$ is a partition in G between F_1 and F_2. Therefore from condition (10) and Lemma 2.5 it follows that

$$\operatorname{Ind} G \leqq \alpha .$$

The inequality $\operatorname{Ind} G \geqq \operatorname{Ind} X = \alpha$ follows from Corollary 2.3. □

THEOREM 9.2. *Let X be a separable subspace of a space Y, and the dimension* $\operatorname{ind} X$ *exists. Then there exists a G_δ-set $G \subset Y$ such that* $\operatorname{ind} G = \operatorname{ind} X$, $X \subset G$.

Problem. Is this theorem true for a nonseparable space? The answer is still unknown even for finite dimensional spaces.

COROLLARY 9.2. *For any separable space X with dimension* $\operatorname{ind} X$ *there exists a separable completion $G \supset X$ with the same dimension* $\operatorname{ind} G = \operatorname{ind} X$.

The proof is similar to the one of Corollary 9.1.

Proof of Theorem 9.2. We shall prove this theorem by induction on $\operatorname{ind} X$. For $\operatorname{ind} X = -1$ the theorem is evident. Let $\operatorname{ind} X = \alpha > -1$. Since X is separable for any $n = 1, 2, \cdots$, there exists a countable system of open in Y sets $\{V_{ni}\}$ such that

$$\operatorname{ind} (Fr\, V_{ni}) \cap X \leqq \beta_{ni} < \alpha$$

and

$$X \subset H_n = \bigcup_{i=1}^{\infty} V_{ni} , \quad \operatorname{diam} V_{ni} < \frac{1}{n} . \tag{11}$$

By inductive assumption for every pair (i, n) there exists a set G_{in} of a type G_δ in Y such that

$$Fr\, V_{ni} \cap X \subset G_{nk} , \quad \operatorname{ind} G_{ni} = \operatorname{ind}(Fr\, V_{ni} \cap X) = \beta_{ni} < \alpha . \tag{12}$$

We put

$$A_n = H_n / \bigcup_{i=1}^{\infty} (Fr\, V_{ni} \backslash G_{ni}) , \quad G = \bigcap_{n=1}^{\infty} A_n . \tag{13}$$

Then $A_n \supset X$ and A_n is a G_δ-set in Y. Consequently $G \supset X$ and G is also a G_δ-set in Y. Let us prove the inequality

$$\text{ind}\, G \leqq \alpha \,. \tag{14}$$

Let $x \in G$, $\varepsilon > 0$ and $1/n < \varepsilon$ for some n. Then by virtue of (11) there exists an open set V_{ni} such that $x \in V_{ni}$. From (12) and (13) it follows that

$$Fr\, V_{ni} \cap G \subset Fr\, V_{ni} \cup A_n \subset G_{ni}$$

and consequently $\text{ind}(Fr\, V_{ni} \cap G) \leqq \text{ind}\, G_{ni} < \alpha$. Hence inequality (14) is true. Since $X \subset G$ we also have $\text{ind}\, G \geqq \text{ind}\, X$. □

THEOREM 9.3. *Let $X \subset Y$ then there exists a G_δ-set G in Y such that $D(X) = D(G)$, $X \subset G$.*

COROLLARY 9.3. *For any space X there exists a completion $G \supset X$ such that $D(G) = D(X)$, weight G = weight X.*

The proof is similar to the one of Corollary 9.1.

LEMMA 9.3. Let $\mathscr{A} = \cup \{A_\mu : \mu \in \mathscr{M}\}$ be a locally finite system of F_σ-sets in a space X. Then the set $A = \cup \{A_\mu : \mu \in \mathscr{M}\}$ is also a F_σ-set in X.

The lemma follows from the fact that a union of locally finite system of closed sets is also a closed set.

Proof of Theorem 9.3. We will prove this theorem by induction on $D(X)$. If $D(X) < \omega_0$, then $D(X) = \text{Ind}\, X$ and our assertion is true. Let $D(X) = \beta \geqq \omega_0$ and the equality

$$X = \cup \{A_\delta : \delta \leqq \alpha = J(\beta)\}$$

be a β-D-representation of X. We put

$$Z = Y \backslash \bar{A}_\alpha$$

where $\bar{A}_\alpha$ is a closure of A_α in Y. Let $U_\delta = X \backslash \cup \{A_{\delta'} : \delta' > \delta\}$ where $\delta < \alpha$. Then from condition (e) of Definition 0.3 we have

$$X \backslash A_\alpha = \cup \{U_\delta : \delta < \alpha\} \tag{15}$$

and by Lemma 8.3 in [13], Luxemburg, U_δ is open in X and

$$D(U_\delta) < \alpha = J(\beta) \leqq \beta \,. \tag{16}$$

Let V_δ be an open in Y set such that $V_\delta \subset Z$ and

$$V_\delta \cap X = U_\delta \,. \tag{17}$$

We put $V = \cup \{V_\delta : \delta < \alpha\}$ then $V \subset Z$ and V is an open set in Y.

Since V is paracompact, we can find a closed in V locally finite refinement $\mathscr{F} = \{F_\mu: \mu \in \mathscr{M}\}$ of the covering $\{V_\delta: \delta < \alpha\}$. Consequently from (16), (17) it follows that

$$D(F_\mu \cap X) < \alpha \quad (\mu \in \mathscr{M}) .$$

By inductive assumption for every $\mu \in \mathscr{M}$ there exists a G_δ-set G_μ in F_μ such that

$$X \cap F_\mu \subset G_\mu \subset F_\mu \quad D(G_\mu) = D(F_\mu \cap X) < \alpha .$$

Since the system $\mathscr{F}$ is locally finite in V, the system $\mathscr{L} = \{L_\mu = F_\mu \backslash G_\mu: \mu \in \mathscr{M}\}$ is also locally finite in V. Moreover, since G_μ is a G_δ-set in V the set L_μ is F_σ-set in V and by Lemma 9.3 the set $L = \bigcup \{L_\mu: \mu \in \mathscr{M}\}$ is F_σ in V. We put $G_1\ V \backslash L$, then clearly, G_1 is a G_δ-set in V (and consequently in Y) and

$$Z \supset G_1 \supset X \backslash A_\alpha .$$

Since $F_\mu \cap G_1 = F_\mu \backslash L$, sets $F_\mu \backslash L$ are closed in G_1. Moreover, the system $\{F_\mu \backslash L, \mu \in \mathscr{M}\}$ is locally finite in V (and consequently in G_1) and $F_\mu \backslash L \subset G_\mu$. Therefore

$$D(F_\mu \backslash L) \leqq D(G_\mu) < \alpha .$$

By the sum theorem for locally finite union of closed sets (see [3], Henderson) we get

$$(18) \quad D(G_1) = D(\bigcup \{F_\mu \backslash L: \mu \in \mathscr{M}\}) \leqq \sup\{D(F_\mu \backslash L): \mu \in \mathscr{M}\} \leqq \alpha = J(\beta) .$$

Let us consider the set $A_\alpha \subset Y \backslash Z \subset Y \backslash G_1$. Since $\operatorname{Ind} A_\alpha \leqq K(\beta) < \omega_0$ (see condition (d) of Definition 0.3) there exists a G_δ-set G_2 in Y such that

$$A_\alpha \subset G_2 , \quad D(A_\alpha) = \operatorname{Ind} A_\alpha = D(G_2) \leqq K(\beta) .$$

We put $G_3 = G_2 \cap (Y \backslash Z)$, then clearly G_3 is closed in $G = G_3 \cup G_1$ and G_3 is a G_δ-set in Y, because $Y \backslash Z$ is a closed set. Moreover, we have obviously that

$$(19) \qquad X \subset G , \quad D(G_3) \leqq D(G_2) \leqq K(\beta)$$

and G is a G_δ-set in Y as a union of G_δ-sets G_1 and G_3. Since G_3 is closed in G, the set $G_1 = G \backslash G_3$ is open in G and consequently (see [3]) from (18), (19) it follows that:

$$D(G) \leqq D(G_1) + D(G_3) \leqq J(\beta) + K(\beta) = \beta = D(X) .$$

The inequality $D(X) \leqq D(G)$ follows from the inclusion $X \subset G$. □

10. On the necessary and sufficient conditions of the existence

of transfinite dimensions. As it was mentioned in the introduction, dimensions $\operatorname{ind} X$ and $\operatorname{Ind} X$ do not exist for every space X. In this section we shall consider that D-dimension of a space X exists if $D(X)$ is an ordinal number. The following theorem gives a criterion of the existence of transfinite dimensions in terms of compactifications.

THEOREM 10.1. *Let X be a separable space, then*

(a) $\operatorname{Ind} X$ *exists* $\Leftrightarrow$ *there is a countable dimensional*[17] *compactification* $cX \supset X$ *such that* $P(cX) = P(X)$.

(b) $\operatorname{ind} X$ *exist* $\Leftrightarrow$ *there is a countable dimensional compactification* $cX \supset X$.

(c) $D(X)$ *exists* $\Leftrightarrow$ *there is a weakly countable dimensional compactification* $cX \supset X$.

To prove this we need the following assertions:

(1) If $\operatorname{Ind} X$ exists then X is countable dimensional, see [21], Smirnov.

(2) If X is a countable dimensional compactum then $\operatorname{Ind} X$ exists, see [22], Smirnov.

(3) If X is a complete countable dimensional space then $\operatorname{ind} X$ exists, see [5], Hurewicz and Wallman.

(4) If $\operatorname{ind} X$ exists, X is a separable space, then X is countable dimensional, see [5], Hurewicz and Wallman.

(5) If X is a complete separable space, then $D(X)$ exists $\Leftrightarrow$ is a weakly countable dimensional[18], see [4], Henderson.

Proof of Theorem 10.1. (a) Let $\operatorname{Ind} X$ exist, then the existence of such a compactification cX follows from Corollary 1.1 and (1). If there exists a countable dimensional compactification $cX \supset X$ then by virtue of (2) $\operatorname{Ind} X$ exists and our assertion follows from Lemma 8.2.

(b) Let $\operatorname{ind} X$ exist, then by Corollary 9.2 there exists a separable absolute G_δ-set $pX \supset X$ such that $\operatorname{ind} pX = \operatorname{ind} X$. Since pX is an absolute G_δ-set there exists a compactification $cX \supset pX$ such that the set $cX \backslash pX$ is countable dimensional see [9], Lelek. Since $\operatorname{ind} pX$ exists, by virtue of (4), the space pX is countable dimensional, consequently the compactum cX is also countable dimensional. If a space X has a countable dimensional compactification $cX \supset X$, then by virtue of (3) $\operatorname{ind} cX$ exists. Consequently, $\operatorname{ind} X$ also exists (see (1), introduction).

Property (c) follows from (5), Theorem 8.1 and (1) introduction. □

17 A space X is countable dimensional if X is a union of countable number of zero-dimensional sets.

18 It is easy to prove this theorem also for nonseparable spaces.

COROLLARY 10.1. *For any separable space X with dimension* $\operatorname{ind} X$ *there exists a compactification cX such that the dimension* $\operatorname{ind} cX$ *also exists. This corollary follows from Theorem* 10.1(b) *and* (3).

Theorem 7.1 shows that we can not require that the equality $\operatorname{ind} cX = \operatorname{ind} X$ holds for some compactification $cX \supset X$. However I think that the technique of the proof of Theorem 1.1 will permit proving the following.

Conjecture. If X is a separable space and $\operatorname{ind} X = \alpha + p$, where $\alpha = J(\alpha + p)$, $p = K(\alpha + p) = 0, 1, 2, \cdots$, then there exists a compactification $cX \supset X$ such that $\operatorname{ind} cX \leqq \alpha + 2p + 1$.

We also note that using Theorems 9.1 and 9.3 we can easily obtain criteria of the existence of dimensions $\operatorname{Ind} X$ and $D(X)$ in terms of completions for nonseparable spaces.

REFERENCES

1. H. G. Bothe, *Eine Einbettung m-dimensionaler Mengen in einen (m+l)-dimensionalen absoluten Retrakt*, Fund. Math., **52** (1963), 209-224.

2. S. Hanai, and K. Morita, *Closed mappings and metric spaces*, Proc. Japan Acad., **32** (1956), 10-14.

3. D. W. Henderson, *D-dimension, I. A new transfinite dimension,* Pacific J. Math., **26** (1968), 91-107.

4. ———, *D-dimension, II. Separable spaces and compactifications,* Pacific J. Math., **26** (1968), 109-113.

5. W. Hurewicz and H. Wallman, *Dimension Theory,* Princeton Math., Ser. vol. 4, Princeton Univ. Press, Princeton, N. J., 1941.

6. M. Katetov, *On the dimension of non-separable spaces I,* (Russian), Czechoslovak. Mat. J., **2** (77) (1952), 333-368 (1953). (Russian, English summary).

7. I. M. Kozlovsky, *Two theorems on metric spaces*, (Russian), Dokl. Akad. Nauk SSSR., **204** (1972), 784-787 = Soviet Math. Dokl., **13** (1972), 743-747.

8. K. Kuratowski, *Topology*, Vol. I. New edition. Academic Press, New York-London, Panstwowe Wydawnictwo Naukowe, Warsaw, 1966, xx + 560.

9. A. Lelek, *On the dimensionality of remainders in compactifications*, (Russian), Dokl. Acad. Nauk SSSR., **160** (1965), 534-537.

10. B. T. Levsenko, *Spaces of transfinite dimension*, (Russian), Mat. Sb. (N. S.), **67** (109) (1965), 255-266.

11. L. A. Luxemburg, *On transfinite D-dimension*, Theses of the 6th All Union Topological Conference, Tbilisi, 1972 (Russian).

12. ———, *On compact spaces with noncoinciding transfinite dimensions*, Dokl. Akad. Nauk SSSR., TOM 212 (1973) No. 6 (Russian) = Soviet Math. Dokl., **4** (1973) No. 5, 1593-1597.

13. ———, *On compact metric spaces with noncoinciding transfinite dimensions*, Pacific J. Math., **93** (1981), 339-386.

14. ———, *On transfinite inductive dimensions*, Dokl. Akad. Nauk SSSR., **209** (1973) No.6, 295-298 (Russian) = Soviet Math. Dokl., **14** (1973), 388-393.

15. J. I. Nagata, *Modern Dimension Theory*, Bibliotheca Mathematica, vol. VI. Interscience Publishers, [John Wiley and Sons, Inc.] New York, 1965.

16. B. A. Pasynkov, *Partial topological products*, (Russian), Trudy Moskov. Mat. Obsc., **13** (1965), 136-245.
17. B. A. Pasynkov, *The dimensionality of normal spaces*, (Russian), Dokl. Akad. Nauk SSSR., **201** (1971), 1049-1052 = Soviet Math. Dokl., **12** (1971), 1784-1787.
18. P. Roy, *Failure of equivalence of dimension concepts for metric spaces*, Bull. Amer. Math. Soc., **68** (1962), 609-613.
19. E. G. Sklyarenko, *Dimensionality properties of infinite dimensional spaces*, (Russian), Izv. Akad. Nauk SSSR. Ser. Mat., **23** (1959), 1203-1206. Amer. Math. Transl., (2) **21** (1962), 25-50.
20. Ju. M. Smirnov, *Universal spaces for certain classes of infinite-dimensional spaces*, (Russian), Izv. Akad. Nauk SSSR. Ser. Mat., **23** (1959), 185-196; English transl., Amer. Math. Soc. Transl., (2) **21** (1962), 21-33.
21. Yu. M. Smirnov, *Some remarks on transfinite dimension*, (Russian), Dokl. Akad. Nauk SSSR., **141** (1961), 814-817.
22. ———, *On transfinite dimension*, Matem. Sbornik., **58** No. 4 (1962), 415-422 (Russian).
23. A. Zarelua, *The Equality of dimensions and bicompact extensions*, (Russian), Dokl. Akad. Nauk SSSR., **144** (1962), 713-716.

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