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## **ON SEMISIMPLE RINGS THAT ARE CENTRALIZER NEAR-RINGS**

CARLTON JAMES MAXSON, MARTIN ROSS PETTET AND KIRBY C. SMITH

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Let  $G$  be a finite group with identity 0 and let  $\mathcal{A}$  be a group of automorphisms of  $G$ . The set  $C(\mathcal{A}; G) = \{f: G \rightarrow G \mid f(0) = 0, f(\gamma v) = \gamma f(v) \text{ for every } \gamma \in \mathcal{A}, v \in G\}$  is the centralizer near-ring determined by  $\mathcal{A}$  and  $G$ . In this paper we consider the following "representation" questions: (I) Which finite semisimple near-rings are of  $C(\mathcal{A}; G)$ -type? and (II) Which finite rings are of  $C(\mathcal{A}; G)$ -type?

1. Introduction. Let  $G$  be a finite group and let  $\Gamma$  denote a semigroup of endomorphisms of  $G$ . The set of functions  $C(\Gamma; G) = \{f: G \rightarrow G \mid f(0) = 0 \text{ and } f(\gamma v) = \gamma f(v) \text{ for every } \gamma \in \Gamma, v \in G\}$  forms a zero-symmetric near-ring under function addition and function composition. (Since all near-rings in this paper will be zero-symmetric this adjective will henceforth be omitted.) Such "centralizer near-rings" are indeed general, for it is shown in [7] that if  $N$  is any near-ring (with identity) then there exists a group  $G$  and a semigroup of endomorphisms  $\Gamma$  such that  $N \cong C(\Gamma; G)$ .

The structure of centralizer near-rings has been studied for various  $G$ 's and  $\Gamma$ 's, e.g. when  $\Gamma = \mathcal{A}$  is a group of automorphisms of a finite group  $G$  ([5]), or when  $\Gamma$  is a finite ring with 1 and  $G$  is a faithful, unital  $\Gamma$ -module ([6]). From a structure theorem due to Betsch [1] we have that a finite near-ring  $N$ , which is not a ring, is simple if and only if  $N \cong C(\mathcal{A}; G)$  where  $\mathcal{A}$  is a fixed point free group of automorphisms of a finite group  $G$ . (A group  $\mathcal{A}$  of automorphisms is fixed point free if the identity map in  $\mathcal{A}$  is the only element of  $\mathcal{A}$  that fixes a nonidentity element of  $G$ .)

Since every finite simple nonring is of " $C(\mathcal{A}; G)$ -type" it is natural to ask for which finite near-rings does there exist a finite group  $G$  and a group of automorphisms  $\mathcal{A}$  such that  $N \cong C(\mathcal{A}; G)$ , i.e. which finite near-rings are of  $C(\mathcal{A}; G)$ -type? In this paper we restrict our attention to the following more specific questions.

I. Which finite semisimple near-rings are of  $C(\mathcal{A}; G)$ -type?

II. Which finite rings are of  $C(\mathcal{A}; G)$ -type?

It will become clear in this paper that the "centralizer representation" problems I and II give rise to nontrivial group-theoretic, combinatoric problems.

In providing partial solutions to problems I and II we show that certain semisimple near-rings are not of  $C(\mathcal{A}; G)$ -type. Moreover

it is proven that the only possible rings of  $C(\mathcal{A}; G)$ -type are those that are direct sums of fields, but this is only a necessary condition. Information is obtained on which direct sums of fields are of  $C(\mathcal{A}; G)$ -type.

For definitions and basic results on near-rings the reader is referred to the book by Pilz [8]. A near-ring with 1 is simple if it has no nontrivial ideals. Since we are dealing exclusively with finite near-rings, we will regard a semi-simple near-ring as being one which is a direct sum of simple near-rings. For connections between our definition of semi-simplicity and near-ring radicals see [8], Chapters 4 and 5.

**2. Rings of  $C(\mathcal{A}; G)$ -type.** In this section we present results that characterize semisimple  $C(\mathcal{A}; G)$  near-rings. We also show that if a finite ring has a centralizer representation then this ring must be a direct sum of fields, a result that has been established independently by Zeller [10].

We begin by setting our notation and terminology.  $G$  will denote a finite group (normally written additively with identity 0) and  $\mathcal{A}$  a group of automorphisms of  $G$ . For  $v_0 \in G$ , let  $C_{\mathcal{A}}(v_0) = \{\alpha \in \mathcal{A} \mid \alpha v_0 = v_0\}$ , a subgroup of  $\mathcal{A}$ , and let  $N(C_{\mathcal{A}}(v_0))$  denote the normalizer of  $C_{\mathcal{A}}(v_0)$  in  $\mathcal{A}$ . Also let  $C_G(C_{\mathcal{A}}(v_0)) = \{v \in G \mid \alpha v = v \text{ for all } \alpha \in C_{\mathcal{A}}(v_0)\}$ , a subgroup of  $G$ . Finally for  $v \in G^* \equiv G - \{0\}$  let  $\theta(v) = \{\alpha v \mid \alpha \in \mathcal{A}\}$ , the orbit of  $G^*$  determined by  $v$  under  $\mathcal{A}$ .

The set  $\mathcal{S} = \{C_{\mathcal{A}}(v) \mid v \in G^*\}$  is partially ordered by inclusion, and we say  $C_{\mathcal{A}}(v)$  is maximal if it is maximal in  $\mathcal{S}$ . The following theorem appears in [5], but since it and its proof are basic to this paper we include it here for completeness.

**THEOREM 1.** *Let  $\mathcal{A}$  be a group of automorphisms of a finite group  $G$ . The following are equivalent.*

1.  $C(\mathcal{A}; G)$  is semi-simple.
2. Every element in  $\mathcal{S}$  is maximal.
3. The collection,  $\{C_G(C_{\mathcal{A}}(v)) \mid v \in G^*\}$ , of subgroups partitions  $G$ .

*Proof.* Suppose  $C(\mathcal{A}; G)$  is semisimple and there exist elements  $u, v \in G^*$  with  $C_{\mathcal{A}}(u)$  properly contained in  $C_{\mathcal{A}}(v)$ . Let

$$M = \{f \in C(\mathcal{A}; G) \mid C_{\mathcal{A}}(v) \subseteq C_{\mathcal{A}}(f(u)) \text{ and } f \text{ is zero off } \theta(u)\}.$$

Then  $M$  is a nonzero nilpotent  $C(\mathcal{A}; G)$ -subgroup and  $C(\mathcal{A}; G)$  is not semi-simple.

Suppose condition 2 holds, then if  $u \notin C_G(C_{\mathcal{A}}(v))$ ,  $C_G(C_{\mathcal{A}}(v)) \cap C_G(C_{\mathcal{A}}(u)) = \{0\}$ . So  $G$  is partitioned by the desired subgroups.

Assume now that condition 3 holds. For  $v \in G^*$  let  $T(v) = \cup \{\theta(w) \mid C_{\mathcal{A}}(w) = C_{\mathcal{A}}(v)\}$ , and let  $M(v) = \{f \in C(\mathcal{A}; G) \mid f \text{ is zero off } T(v)\}$ .  $M(v)$  is an ideal of  $C(\mathcal{A}; G)$ . We may select elements  $v_1, \dots, v_i \in G^*$  such that  $G = T(v_1) \cup \dots \cup T(v_i) \cup \{0\}$ , a disjoint union. We have  $C(\mathcal{A}; G) = M(v_1) \oplus \dots \oplus M(v_i)$ , a direct sum of ideals  $M(v_i)$ . It remains to show that each  $M(v_i)$  is simple. For each  $i$  let  $\bar{\mathcal{A}}_i = N_{\mathcal{A}}\{C_{\mathcal{A}}(v_i)\}/C_{\mathcal{A}}(v_i)$ . Then  $\bar{\mathcal{A}}_i$  can be regarded as a group of automorphisms on  $H_i = C_G(C_{\mathcal{A}}(v_i))$  by defining  $\bar{\beta}w = \beta w$  for all  $w \in H_i$ ,  $\bar{\beta} \in \bar{\mathcal{A}}_i$ . Moreover  $M(v_i) \cong C(\bar{\mathcal{A}}_i; H_i)$ , and since  $\bar{\mathcal{A}}_i$  acts fixed point free on  $H_i$ ,  $C(\bar{\mathcal{A}}_i; H_i)$  is a simple near-ring. So  $C(\mathcal{A}; G)$  is semi-simple.

When  $C(\mathcal{A}; G)$  is semi-simple the proof of Theorem 1 establishes that  $C(\mathcal{A}; G)$  is a direct sum of simple near-rings of  $C(\mathcal{A}; G)$ -type. We record this in the following corollary.

**COROLLARY 1.**  *$C(\mathcal{A}; G)$  is semi-simple if and only if there exist elements  $v_1, v_2, \dots, v_i$  in  $G^*$  with corresponding subgroups  $H_i \equiv C_G(C_{\mathcal{A}}(v_i))$  of  $G$  such that for every  $i$ ,  $\bar{\mathcal{A}}_i \equiv N(C_{\mathcal{A}}(v_i))/C_{\mathcal{A}}(v_i)$  acts fixed point free on  $H_i$  and*

$$C(\mathcal{A}; G) \cong C(\bar{\mathcal{A}}_1; H_1) \oplus \dots \oplus C(\bar{\mathcal{A}}_i; H_i).$$

**PROPOSITION 1.** *Assume  $C(\mathcal{A}; G)$  is simple. Then  $C(\mathcal{A}; G)$  is a ring if and only if it is a field. Moreover every field is a near-ring of  $C(\mathcal{A}; G)$ -type.*

*Proof.* Assume  $C(\mathcal{A}; G)$  is a ring and suppose  $\theta_1$  and  $\theta_2$  are distinct orbits in  $G^*$ . Since  $C(\mathcal{A}; G)$  is simple there exist elements  $v_i \in \theta_i$  such that  $C_{\mathcal{A}}(v_1) = C_{\mathcal{A}}(v_2)$ . Let  $e_{ij}: G \rightarrow G$ ,  $i, j = 1, 2$  be defined by

$$\begin{aligned} e_{ij}(\alpha v_k) &= \delta_{jk} \alpha v_i & \alpha \in \mathcal{A} \\ e_{ij}(x) &= 0 & x \notin \theta_1 \cup \theta_2. \end{aligned}$$

Then  $e_{ij} \in C(\mathcal{A}; G)$ . But  $e_{11}(e_{12} + e_{22}) \neq e_{11}e_{12} + e_{11}e_{22}$  and  $C(\mathcal{A}; G)$  is not a ring. So  $G^*$  is an orbit and  $C(\mathcal{A}; G)$  is a field.

If  $F$  is a finite field, let  $G = (F, +)$  and let  $\mathcal{A} = F^*$ , regarded as acting on  $G$  by left multiplication. Then  $F \cong C(\mathcal{A}; G)$ .

**THEOREM 2.**  *$C(\mathcal{A}; G)$  is a ring if and only if  $C(\mathcal{A}; G)$  is a direct sum of fields.*

*Proof.* Assume  $C(\mathcal{A}; G)$  is a ring. We show first that  $C(\mathcal{A}; G)$  is semisimple. Assume not; then there exist orbits  $\theta_1(v_1), \theta_2(v_2)$  of  $G^*$

such that  $C_{\mathcal{A}}(v_1) \not\cong C_{\mathcal{A}}(v_2)$ . If  $e_{ij}, i = 1, 2, j = 1, 2$  are defined as above then  $e_{11}, e_{22}, e_{21} \in C(\mathcal{A}; G)$ , and  $e_{22}(e_{21} + e_{11}) \neq e_{22}e_{21} + e_{22}e_{11}$ .

So  $C(\mathcal{A}; G)$  is semi-simple and  $C(\mathcal{A}; G) \cong C(\mathcal{A}_i; H_i) \oplus \dots \oplus C(\mathcal{A}_i; H_i)$  as in the corollary to Theorem 1. This means each  $C(\mathcal{A}_i; H_i)$  is a ring, and by Proposition 1 must be a field.

As a result of the arguments above we have the following structural result.

**COROLLARY 2.** *If  $N$  is a finite semi-simple near-ring with  $N = S_1 \oplus \dots \oplus S_i$  where each  $S_i$  is simple, and if for some  $j, S_j$  is a ring which is not a field, then  $N$  is not of  $C(\mathcal{A}; G)$ -type.*

**3. Centralizer representations of direct sums of fields.** From Theorem 2 the only time  $C(\mathcal{A}; G)$  is a ring is when it is a direct sum of fields. Thus, it is natural to investigate the problem of when a direct sum of fields has a centralizer representation. We shall show that *not* all direct sums of fields are near-rings of  $C(\mathcal{A}; G)$ -type. For notation, let  $GF(q)$  denote the finite field with  $q$  elements where  $q = p^t$  for some prime  $p$ . If  $C(\mathcal{A}; G)$  is direct sum of fields then from Corollary 1 we have

$$C(\mathcal{A}; G) \cong C(\mathcal{A}_i; H_i) \oplus \dots \oplus C(\mathcal{A}_i; H_i)$$

where each  $C(\mathcal{A}_i; H_i)$  is a finite field. From Theorem 1 and its proof, and from Corollary 1, we have the following necessary and sufficient conditions for  $GF(q_1) \oplus \dots \oplus GF(q_t), q_i = p_i^{n_i}$  to be a near-ring of  $C(\mathcal{A}; G)$ -type:

- (i) There exists a finite group  $G$  and a group of automorphisms  $\mathcal{A}$  such that any one of the conditions of Theorem 1 is satisfied.
- (ii)  $G^*$  has exactly  $t$  orbits under  $\mathcal{A}$ .
- (iii) Every nonzero element in  $G$  has prime order.
- (iv) If  $v, v' \in G^*$  belong to different orbits then  $C_{\mathcal{A}}(v)$  and  $C_{\mathcal{A}}(v')$  are not conjugate subgroups of  $\mathcal{A}$ .
- (v) There exist elements  $v_1, \dots, v_t \in G^*$ , no two in the same orbit, such that for each  $i, N(C_{\mathcal{A}}(v_i))/C_{\mathcal{A}}(v_i) \cong GF(q_i)^*$ .

The following group theoretic result indicates that property (iii) places a rather strong restriction on the structure of the group  $G$ . The theorem is certainly known but we are not aware of any explicit reference in the literature so, for the reader's convenience, we have included a proof that is, for the most part, elementary.

**THEOREM 3.** *Let  $G$  be a finite group such that every non-identity element of  $G$  has prime order. Then one of the following holds:*

- (a)  $G$  is a  $p$ -group of exponent  $p$  for some prime  $p$ ,
- (b)  $G$  is a Frobenius group with kernel of order  $p^a$  and com-

plement of order  $q$ , where  $p$  and  $q$  are distinct primes,

(c)  $G$  is isomorphic to  $A_5$ , the alternating group on five elements.

*Proof Case 1.* Assume  $G$  is solvable and not a  $p$ -group. Then every minimal normal subgroup of  $G$  is abelian ([4], page 23), so the Fitting subgroup  $F(G)$  is nontrivial. The nilpotent group  $F(G)$  must be a  $p$ -group for some prime  $p$ , for otherwise if  $x$  and  $y$  in  $F(G)$  have distinct prime orders,  $xy = yx$  has composite order. Let  $\bar{G} = G/F(G)$ , and let  $V = F(G)/\Phi(F(G))$ , the Frattini factor group of  $F(G)$ .  $V$  is a vector space over  $GF(p)$  ([4], page 174, Theorem 1.3) and  $\bar{G}$  acts faithfully by conjugation as a group of linear transformations on  $V$  ([4], page 229, Theorem 3.4).

Let  $\bar{N} = N/F(G)$  be a minimal normal subgroup of  $\bar{G}$ , so  $\bar{N}$  is an elementary abelian  $q$ -group for some prime  $q \neq p$ . Since all elements of  $G$  have prime order,  $\bar{N}$  acts fixed point freely on  $V$ . By Theorem 3.3, page 69 of [4] we have  $|\bar{N}| = q$ . It suffices now to prove  $\bar{G} = \bar{N}$ .

Suppose  $\bar{G} \neq \bar{N}$  and let  $\bar{M}/\bar{N}$  be a subgroup of prime order  $r$  in  $\bar{G}/\bar{N}$ . Now  $r \neq q$  for if so, then  $\bar{M}$  would be elementary abelian of order  $q^2$ , which is not allowed by Theorem 3.3 of [4].  $\bar{M}$  must be a Frobenius group, so let  $\bar{M} = \bar{N}\langle x \rangle$ , where  $x$  has order  $r$ .

Regarding  $\bar{M}$  as a set of linear transformations on  $V$ , we see that  $\sum_{n \in \bar{N}} n$  maps  $V$  into  $C_V(\bar{N}) = 1$ , so  $\sum n = 0$ . Similarly,  $\sum_{m \in \bar{M}} m = 0$ . Since  $\bar{M}^*$  is partitioned by  $\bar{N}^*$  and the  $q$  conjugates of  $\langle x \rangle^*$  then

$$\begin{aligned} 0 &= \sum_{m \in \bar{M}} m = \sum_{n \in \bar{N}} n + \sum_g (x + x^2 + \dots + x^{r-1})^g \\ &= 0 + \sum_g \left[ \sum_{i=0}^{r-1} x^i \right]^g - q^r. \end{aligned}$$

Therefore  $\sum_{i=0}^{r-1} x^i \neq 0$ .

Let  $v \in V^*$  such that  $v^y \neq 1$  where  $y = \sum_{i=0}^{r-1} x^i$ . If  $r = p$  then  $v^y = v^{v^x} \dots v^{x^{p-1}} = v(x^{-1}vx)(x^{-2}vx^2) \dots (x^{-(p-1)}vx^{p-1}) = (vx^{-1})^p \neq 1$ . So  $vx^{-1}$  has order at least  $p^2$  in the  $p$ -group  $\langle x \rangle V$ , impossible. On the other hand, if  $r \neq p$ , the fact that  $x$  does not satisfy the polynomial  $1 + \alpha + \dots + \alpha^{r-1} = (\alpha^r - 1)/(\alpha - 1)$ , but does satisfy  $\alpha^r - 1$  means that 1 is an eigenvalue for  $x$  on  $V$ . Then  $x^{-1}wx = w^x = w$  for some  $w \in V^*$ , so  $wx$  has order  $pr$ , also impossible. Hence  $\bar{G} = \bar{N}$ .

*Case 2.* Assume  $G$  is not solvable. Then  $G$  has even order by the Feit-Thompson theorem. Let  $S$  be a Sylow 2-subgroup of  $G$ . Every element of  $S^*$  has order 2 so  $S$  is abelian. This means for every  $x \in S^*$  we have  $S \subseteq C(x)$  where  $C(x)$  is the centralizer of  $x$ . On the other hand  $C(x)$  is a 2-group if  $x \in S^*$ , otherwise  $G$  has elements of composite order. Hence  $C(x) = S$  for every  $x \in X^*$ .

If  $|S| = 2$  then  $G$  has a normal 2-complement (see e.g. [4], Theorem 7.6.1, page 257) which implies  $G$  is solvable. Hence we may assume  $|S| > 2$ . By a result of Brauer-Suzuki-Wall ([2], or for a more elementary reference see [3]), either  $S$  is a normal subgroup of  $G$  or else  $G$  isomorphic to  $SL(2, 2^n)$  where  $|S| = 2^n$ . In the former situation,  $G/S$  has odd order so it is solvable. Then  $G$  is solvable, contradiction. Thus  $G$  is isomorphic to  $SL(2, 2^n)$  for some  $n \geq 2$ . Since  $SL(2, 2^n)$  contains cyclic subgroups of order  $2^n - 1$  and  $2^n + 1$  ([4], Theorem 8.3 page 42) then  $2^n - 1$  and  $2^n + 1$  must be primes. But  $2^n - 1$  prime implies  $n$  is prime, and  $2^n + 1$  prime implies  $n$  is a power of 2. Hence  $n = 2$  and  $G$  is isomorphic to  $SL(2, 4) \cong A_5$ .

REMARK. By invoking a deep result of Suzuki on partitioned groups [9], the following stronger result can be proved: If the near-ring  $C(\mathcal{A}; G)$  is semi-simple and  $F(G) = 1$ , then  $G \cong SL(2, 2^n)$  for some  $n$ .

COROLLARY 3. *Assume  $C(\mathcal{A}; G)$  is a direct sum of fields  $F_i$ ,  $i = 1, \dots, n$ . Let  $S = \{p_i \mid p_i \text{ is the characteristic of } F_i\}$ . Then*

(i)  $|S| \leq 3$ ,

(ii) *if  $|S| = 3$  then  $C(\mathcal{A}; G) \cong GF(2) \oplus GF(3) \oplus GF(5)$  where  $G \cong A_5$  and  $\mathcal{A} = \text{Aut}(G)$ ,*

(iii) *if  $|S| = 2$ , then for some  $q \in S$ , all components  $F_i$  of  $C(\mathcal{A}; G)$  with characteristic  $q$  are isomorphic to  $GF(q)$ .*

*Proof.* Part (i) is immediate from Theorem 3. For part (ii) we have  $G \cong A_5$  due to Theorem 3 and the remarks preceding it. If  $\mathcal{A} = \text{Aut}(A_5)$  then  $\Phi \in \mathcal{A}$  has the form  $\Phi(x) = yxy^{-1}$  where  $y$  is a fixed element in  $S_5$ . Hence  $A_5$  has three nontrivial orbits, one for each type of cycle structure. We have

$$C_G(C_{\mathcal{A}}(123)) = \langle (123) \rangle \cong Z_3$$

$$C_G(C_{\mathcal{A}}(12)(34)) = \langle (12)(34) \rangle \cong Z_2$$

$$C_G(C_{\mathcal{A}}(12345)) = \langle (12345) \rangle \cong Z_5$$

Computations show that

$$N(C_{\mathcal{A}}(123))/C_{\mathcal{A}}(123) \cong Z_2, N(C_{\mathcal{A}}(12)(34))/C_{\mathcal{A}}(12)(34) \cong \{I\}$$

and  $N(C_{\mathcal{A}}(12345))/C_{\mathcal{A}}(12345) \cong Z_4$ . Hence  $C(\mathcal{A}; G) \cong GF(2) \oplus GF(3) \oplus GF(5)$ .

It remains to show that no other group  $\mathcal{A}$  of automorphisms of  $G = A_5$  gives rise to a near-ring which is a direct sum of fields. We may assume  $\mathcal{A} \subseteq S_5$  where  $\mathcal{A}$  acts on  $A_5$  by conjugation. If  $x$  is a 5-cycle then  $x \in A_5$  and  $C_{\mathcal{A}}(x)$  is a subgroup of  $\langle x \rangle$ . Since

$C(\mathcal{A}; A_5)$  is semisimple we must have  $C_{\mathcal{A}}(x) = \langle x \rangle$ . Thus  $\mathcal{A}$  contains all 5-cycles in  $S_5$ . Since the set of 5-cycles generates a normal subgroup of  $A_5$ , and  $A_5$  is simple, we have  $A_5 \subseteq \mathcal{A}$ . Thus  $\mathcal{A} = A_5$ . The near ring  $C(A_5; A_5)$  is semi-simple but is not a direct sum of fields. So we have  $\mathcal{A} = S_5$ .

Part (iii) follows from the fact that in part b) of Theorem 3, a Sylow  $q$ -subgroup of  $G$  has order  $q$ .

The preceding theorem places a restriction on which direct sums of fields can be realized as a centralizer near-ring. The following two theorems give more information about when a direct sum of two fields with different characteristics is a centralizer near-ring.

**THEOREM 4.** *Let  $G$  be a finite group and  $\mathcal{A}$  a subgroup of  $\text{Aut } G$  such that  $\mathcal{A}$  has exactly two orbits in  $G^*$ . If  $G$  does not have prime power order, then for distinct primes  $p$  and  $q$*

- (i)  $G$  is a Frobenius group  $[V]Q$ , with  $V$  an elementary abelian normal subgroup of order  $p^n$  and  $Q$  a cyclic group of order  $q$ , and
- (ii)  $p$  is a generator of  $GF(q)^*$ .

*Proof.* Since  $G$  is not a  $p$ -group there exist distinct primes  $p$  and  $q$  such that the two orbits consist of the elements of order  $p$  and the elements of order  $q$  respectively. By Theorem 3,  $G$  is a Frobenius group with a  $p$ -group  $V$  as kernel and with a complement  $Q$  of order  $q$ . Since  $V$  is characteristic in  $G$ , the center of  $V$  is  $\mathcal{A}$ -invariant so the transitivity of  $\mathcal{A}$  on elements of order  $p$  implies that  $V$  is abelian. This proves (i).

If  $\alpha \in \mathcal{A}$ ,  $Q^\alpha$  is a Sylow  $q$ -subgroup of  $G$  so  $Q^\alpha = g^{-1}Qg$  for some  $g \in G$ . Since  $G = VQ = QV$ ,  $g$  can be selected to be in  $V$  so  $Q^\alpha = v^{-1}Qv = Q^{i_v}$  where  $i_v$  is the inner automorphism of  $G$  induced by  $v$ . So  $\alpha i_v^{-1} \in N_{\text{Aut } G}(Q) \equiv N$  and  $\alpha \in N i_v$ . We now have  $\mathcal{A} \subseteq N I_v$  where  $I_v$  is the group of inner automorphisms of  $G$  induced by elements of  $V$ . Since  $V$  is a characteristic subgroup of  $G$  then  $I_v$  is normal in  $\text{Aut } G$  so  $N I_v = I_v N$ .

Since  $\mathcal{A}$  acts transitively on  $V^*$  so does  $N$ . We claim  $N$  is also transitive on  $Q^*$ . For if  $x, y \in Q^*$  then  $x^\alpha = y$  for some  $\alpha \in \mathcal{A}$ . Writing  $\alpha = i_v n$  where  $v \in V$ ,  $n \in N$ , we have  $x^{i_v n} = y$ , so  $x^{i_v} = y^{n^{-1}} \in Q^{n^{-1}} = Q$ . Hence  $x^{-1}v^{-1}xv = x^{-1}x^{i_v} \in Q$ . On the other hand, since  $V$  is normal in  $G$ ,  $x^{-1}v^{-1}xv \in V$ , so  $x^{-1}v^{-1}xv \in Q \cap V = \{1\}$ . Therefore  $x^{i_v} = x$  and  $x^n = x^{i_v n} = y$ .

$Q$  acts faithfully on  $V$  so we may let  $Q = \langle T \rangle$  where  $T$  is a linear transformation on  $V$  regarded as a vector space over  $GF(p)$ . Suppose  $W$  is an irreducible  $Q$ -submodule of  $V$ . Since  $Q$  is invariant under  $N$ ,  $W^n$  is an irreducible  $Q$ -submodule for every  $n \in N$ . The



transitivity of  $N$  on  $V^*$  implies that every element of  $V^*$  belongs to some irreducible  $Q$ -submodule  $V$  and hence for every  $v \in V^*$  there exists an irreducible polynomial (over  $GF(p)$ ),  $f_v(x)$ , such that  $f_v(T)v = 0$ . If  $v, w \in V^*$  then  $f_v(T)f_w(T)(v + w) = 0$  so  $f_{v+w}(x)$  divides  $f_v(x)f_w(x)$ . Hence we may assume  $f_{v+w}(x) = f_v(x)$ , implying  $f_v(T)w = 0$  so  $f_v(x) = f_w(x)$ . Hence  $f_v(x) = f_w(x)$  for all  $v, w \in V^*$  and the minimal polynomial  $f(x)$  of  $T$  on  $V$  is irreducible.

Since  $T^q = I$ ,  $f(x)$  divides  $x^q - 1 = (x - 1)c(x)$  where  $c(x) = x^{q-1} + \dots + x + 1$ . Since  $T$  fixes no element of  $V^*$ ,  $f(x)$  divides  $c(x)$ . On the other hand if  $\alpha$  is an eigenvalue of  $T$  in some extension field of  $GF(p)$  then the transitivity of  $N$  on  $Q^*$  implies  $T$  is similar in  $GL(V)$  to  $T^k$  for every  $k$  with  $1 \leq k \leq q - 1$ , so  $\alpha^k$  is an eigenvalue for  $T$  for every such  $k$ . Hence, all  $q$ th roots of 1 (except 1) are eigenvalues for  $T$  and thus roots of  $f(x)$ . It follows that  $f(x) = x^{q-1} + \dots + x + 1 = c(x)$  and  $c(x)$  is irreducible over  $GF(p)$ . Therefore any extension of  $GF(p)$  containing a  $q$ th root of 1 has degree at least  $q - 1$ . Since  $GF(p^k)$  contains a  $q$ th root of 1 precisely when  $q$  divides  $|GF(p^k)^*| = p^k - 1$ , this means that  $p^{q-1}$  is the smallest power of  $p$  which is congruent to 1 modulo  $q$ . In other words,  $p$  generates  $GF(q)^*$ .

As an application of this group theoretic property we obtain the following centralizer representation result, the "if" part being established by Theorem 5 below.

**COROLLARY 4.** *Let  $p$  and  $q$  be distinct primes. There is a group  $G$  and a subgroup  $\mathcal{A}$  of  $\text{Aut } G$  such that  $C(\mathcal{A}; G) \cong GF(p) \oplus GF(q)$  if and only if either  $p$  generates  $GF(q)^*$  or  $q$  generates  $GF(p)^*$ .*

Corollary 4 partially generalizes to the case in which  $p^n$  generates  $GF(q)^*$ . This is given in the next theorem.

**THEOREM 5.** *Suppose  $p$  and  $q$  are distinct prime such that  $p^n$  is a generator of  $GF(q)^*$ . Then there exists a group  $G$  and a subgroup  $\mathcal{A}$  of  $\text{Aut } G$  such that  $C(\mathcal{A}; G) \cong GF(p^n) \oplus GF(q)$ .*

*Proof.* Let  $m$  be any integer divisible by  $n(q - 1)$  and let  $V = GF(p^m)$  considered as a vector space over  $GF(p)$ . Since  $n$  divides  $m$  we have  $GF(p^n) \subseteq GF(p^m)$  and the Galois group  $B = \text{Gal}(GF(p^m)/GF(p^n))$  is cyclic, generated by the automorphism  $\theta: \alpha \rightarrow \alpha^{p^n}, \alpha \in GF(p^m)$ .

For every  $\alpha \in GF(p^m)^*$  and  $\sigma \in B$  define the  $GF(p^n)$ -linear transformation  $T_{\sigma, \alpha}$  of  $V$  by  $vT_{\sigma, \alpha} = \alpha v^\sigma$ . Let  $T = \{T_{\sigma, \alpha} | \alpha \in GF(p^m)^*, \sigma \in B\}$  and  $M = \{T_{1, \alpha} | \alpha \in GF(p^m)^*\}$ . The set  $T$  forms a group where  $T_{\sigma, \alpha}T_{\tau, \beta} = T_{\sigma\tau, \alpha\tau_\beta}$ , and  $M \trianglelefteq T$  with  $M \cong GF(p^m)^*$  which is cyclic. Also, let  $H = \{T_{\sigma, 1} | \sigma \in B\}$ , a subgroup of  $T$  isomorphic to  $B$ . We have  $M \cap H = \{1\}$  and  $T = MH$ .

Since  $q - 1$  divides  $m$  then  $q$  divides  $p^m - 1$ . But  $M$  is cyclic of order  $p^m - 1$  so  $M$  contains a characteristic subgroup  $Q$  of order  $q$ . Also  $Q$  is normal in  $T$ . Let  $G$  be the semidirect product  $[V]Q$ , so  $G$  is a Frobenius group and is a normal subgroup of the semidirect product  $A = [V]T$ . We have  $C_A(G) \subseteq C_A(V) = \{1\}$ , so  $A$  acts faithfully on  $G$  by conjugation as a group of automorphisms.

Since  $\theta: \alpha \rightarrow \alpha^{p^n}$  generates  $B$ , the fact that  $p^n$  is a generator of  $GF(q)^*$  implies that the powers  $1, p^n, p^{2n}, \dots$  of  $p^n$  are congruent modulo  $q$  to the integers  $1, 2, 3, \dots, q - 1$  (in some order) and hence, that  $H$  is transitive on  $Q^*$ . Since  $G \subseteq A$  and since all Sylow  $q$ -subgroups of  $G$  are conjugate in  $G$ , it follows  $A$  is transitive on elements of order  $q$ .  $A$  is also transitive on elements of order  $p$  in  $G$  (i.e., on  $V^*$ ), since  $M$  is.  $G$  is a Frobenius group so all its elements have order  $p$  or  $q$  (otherwise some nontrivial element of order  $q$  would centralize an element of order  $p$ ). Thus,  $A$  has precisely two orbits in  $G$ , of sizes  $|V^*| = p^m - 1$  and  $|G| - |V| = p^m q - p^m = p^m(q - 1)$ .

If  $v_0 \in V^*$  and  $x_0 \in Q^*$ , then  $V \subseteq C_A(v_0)$ ,  $C_V(x_0) = \{0\}$ ,  $Q \subseteq C_A(x_0)$  and  $C_Q(v_0) = \{1\}$ . Hence, stabilizers in  $A$  of elements of  $G$  are incomparable and  $C(A; G)$  is semi-simple by Theorem 1. Also, if  $H_1 = \{x \in G \mid C_A(x) = C_A(x_0)\} = C_G(C_A(x_0))$  and  $H_2 = C_G(C_A(v_0))$ , then  $C(A; G) \cong C(A_1; H_1) \oplus C(A_2; H_2)$  where  $A_1 = N_A(C_A(x_0))/C_A(x_0)$  and  $A_2 = N_A(C_A(v_0))/C_A(v_0)$ .

Since  $x_0 \in H_1$  and the Sylow  $q$ -subgroups of  $G$  have order  $q$ ,  $H_1 = Q$ . Since  $A$  is transitive on  $Q^*$ , so also is  $A_1$ . Since  $\text{Aut } Q$  is abelian,  $A_1$  is abelian and  $C(A_1; H_1) \cong GF(q)$ .

It remains to show that  $C(A_2; H_2) \cong GF(p^n)$ . First we claim  $H_2$  is an  $n$ -dimensional subspace of  $V$ . For this we may assume  $v_0 \in GF(p^n) \subseteq GF(p^m) = V$  (since  $A$  is transitive on  $V^*$ ), so  $H \subseteq C_A(v_0)$ , and  $H_2 = C_G(C_A(v_0)) \subseteq C_G(H) = GF(p^n)$ . On the other hand, the stabilizer in  $A$  of any element of  $GF(p^n)^*$  is  $VH$  since no element of  $M^*$  fixes an element of  $V^*$ . So  $GF(p^n) \subseteq H_2$ . Hence  $H_2 = GF(p^n)$  if  $v_0 \in GF(p^n)$  proving the claim.

Now  $A_2$  is transitive on  $H_2$  since  $A$  is, so  $C(A_2; H_2)$  is a near-field of order  $p^n$ . But if  $v_0 \in GF(p^n)$  we have  $C_A(v_0) = VH$  so  $A_2 = N_A(VH)/VH = VHN_M(VH)/VH \cong N_M(VH)$  using the facts that  $A = VMH$  and  $VH \cap M = \{1\}$ . Since  $M$  is abelian,  $A_2$  is abelian and  $C(A_2; H_2) \cong GF(p^n)$ .

Note that, by Corollary 3, (iii), a proof of the converse of Theorem 5 would completely classify those near-rings of  $C(\mathcal{A}; G)$ -type which are a direct sum of two fields of different characteristic.

In our final representation theorem we show that a direct sum of a tower of finite fields can be obtained as a centralizer near-ring.

**THEOREM 6.** *Let  $F_1 \subseteq F_2 \subseteq \dots \subseteq F_t$  be fields. Then there exists a vector space  $V$  over  $F_1$  and a group  $\mathcal{A}$  of linear transformations on  $V$  such that  $C(\mathcal{A}; V) \cong F_1 \oplus F_2 \oplus \dots \oplus F_t$ .*

*Proof.* Let  $F_i = GF(p^{n_i}), i = 1, 2, \dots, t$ . Then  $n_i$  divides  $n_{i+1}$ . We construct the vector space  $V$  as follows. Let  $W_t$  be a (finite dimensional) vector space over  $F_t$ , let  $W_{t-1}$  be any vector space over  $F_{t-1}$  that contains  $W_t$  as a proper subspace, let  $W_{t-2}$  be any vector space over  $F_{t-2}$  that contains  $W_{t-1}$  as a proper subspace, etc. Hence  $W_t \subset W_{t-1} \subset \dots \subset W_2 \subset W_1 \equiv V$ , where each containment is proper and  $W_i$  is a vector space over  $F_i$ . Let  $\mathcal{A}$  be the set of invertible  $F_1$ -linear transformations on  $V$  defined as follows:  $A \in \mathcal{A}$  if and only if for each  $i, W_i$  is  $A$ -invariant and  $A$  restricted to  $W_i$  is  $F_i$ -linear.

We claim that  $C(\mathcal{A}; V) \cong F_1 \oplus \dots \oplus F_t$ . It is clear that  $V^*$  has  $t$  orbits under  $\mathcal{A}$ , namely  $W_t^*, W_{t-1} - W_t, \dots, W_1 - W_2$ . If  $v_i \in W_i - W_{i+1}$  then  $C_V(C_{\mathcal{A}}(v_i)) = F_i v_i$ . Let  $\mathcal{A}_i = N_{\mathcal{A}}(C_{\mathcal{A}}(v_i))$ . If  $S \in \mathcal{A}_i$  and  $A \in C_{\mathcal{A}}(v_i)$  then  $S^{-1}ASv_i = v_i$ , that is  $ASv_i = Sv_i$ . Hence  $Sv_i \in C_V(C_{\mathcal{A}}(v_i))$  meaning  $Sv_i = \alpha v_i$  for some  $\alpha \in F_i^*$ . This implies  $\mathcal{A}_i \cong \mathcal{A}_i/C_{\mathcal{A}}(v_i)$  is isomorphic to  $F_i^*$ . This implies

$$\begin{aligned} C(\mathcal{A}; V) &\cong C(F_t^*; F_t v_t) \oplus \dots \oplus C(F_1^*; F_1 v_1) \\ &\cong F_t \oplus \dots \oplus F_1. \end{aligned}$$

We conclude this section (and the paper) with a couple of open problems relative to representing  $C(\mathcal{A}; G)$  as the direct sum of two fields. The first question concerns the converse of Theorem 5 while the second question deals with the theorem above.

*Problem 1.* If  $C(\mathcal{A}, G) \cong GF(p^n) \oplus GF(q)$ , is  $p^n$  a generator of  $GF(q)^*$ ?

*Problem 2.* If  $C(\mathcal{A}, G) \cong GF(p^a) \oplus GF(p^b)$  and  $a < b$ , does  $a$  divide  $b$ ?

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