

# Pacific Journal of Mathematics

**A SPECTRAL CONTAINMENT THEOREM ANALOGOUS TO  
THE SEMIGROUP THEORY RESULT  $e^{t\sigma(A)} \subseteq \sigma(e^{tA})$**

D. M. TERLINDEN

## A SPECTRAL CONTAINMENT THEOREM ANALOGOUS TO THE SEMIGROUP THEORY RESULT $e^{t\sigma(A)} \subseteq \sigma(e^{tA})$

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**It is known that if  $A$  generates a  $(C_0)$  semigroup  $(e^{tA})$ , then  $e^{t\sigma(A)} \subseteq \sigma(e^{tA})$ , where  $\sigma$  denotes "spectrum." This result is generalized to the context of solution operators of certain  $n$ th order linear differential equations.**

1. **Introduction.** Let  $(T_t)$  be a  $(C_0)$  semigroup on a Banach space  $X$  with generator  $A$ . It is known [2, p. 457] that

$$(1.1) \quad e^{t\sigma(A)} \subseteq \sigma(T_t).$$

Since  $T_t f$  solves the differential equation

$$(1.2) \quad \begin{cases} x' = Ax \\ x(0) = f, \end{cases}$$

$T_t f$  can be formally written as  $e^{tA}$ , so that (1.1) can be written as

$$(1.3) \quad e^{t\sigma(A)} \subseteq \sigma(e^{tA}).$$

In this paper it will be shown that if one replaces the first derivative in (1.2) by an  $n$ th order linear differential expression  $L$  producing the equation

$$\begin{cases} Lx = Ax \\ x(0) = f \\ x^{(j)}(0) = 0 \quad \text{for } j = 2, \dots, n-1 \end{cases}$$

solved by  $S_t f$  for some linear operators  $S_t$  and replaces  $e^{tz}$ , the solution in  $C$  of

$$\begin{cases} x' = zx \\ x(0) = 1, \end{cases}$$

by  $\psi(t, z)$ , the solution of

$$\begin{cases} Lx = zx \\ x(0) = 1 \\ x^{(j)}(0) = 0 \quad \text{for } j = 2, \dots, n-1 \end{cases}$$

(so that formally  $S_t = \psi(t, A)$ ), then the analog

$$\begin{aligned} \psi(t, \sigma(A)) &\subseteq \sigma(S_t); \\ \psi(t, \sigma(A)) &\subseteq \sigma(\psi(t, A)) \end{aligned}$$

of (1.1) and (1.3) holds. The theorem will be stated precisely and be proved in §3.

2. Definitions and notation.

A. NOTATION. Let  $\phi$  be  $C$  or  $X$  valued in  $C^{(n)}$ . Let  $\hat{\phi} = (\phi, \phi', \dots, \phi^{(n-1)})$ , where  $n$ , the dimension of the vector, is the same as the order of differential expression appearing in context. If  $f$  is a constant, let  $\hat{f} = (f, 0, 0, \dots, 0)$ .

B. DEFINITION. Let  $X$  be a Banach space over  $C$ . Let  $L$  be a linear  $n$ th order differential expression of the form

$$L = \sum_{j=0}^n a_j(t) \frac{d^j}{dt^j},$$

where the  $a_j$  are continuous complex valued functions on  $[0, \infty)$  and  $a_n \equiv 1$ . Let  $A$  be a closed densely defined linear operator on  $X$ . The initial value problem

$$(P) \begin{cases} u \in C^{(n)}[0, \infty) \\ u(t) \in D_A \text{ for } t \in [0, \infty) \\ Lu = Au \\ \hat{u}(0) = \hat{f} \end{cases}$$

is *well posed* if there are bounded linear operators  $S_t \in \mathcal{B}(X)$  for  $0 \leq t < \infty$  and a vector subspace  $Y$  of  $X$  so that

- (1)  $Y \subseteq D_A$  and  $Y_0 = \{f: f \in Y \text{ and } Af \in Y\}$  is dense in  $D_A$  in the  $D_A$  norm. ( $\|x\|_{D_A} = \|x\| + \|Ax\|$ , the graph norm.)
- (2) If  $g \in Y$  and  $u(t) = S_t g$ , then  $u(t)$  solves (P) with  $f = g$ .
- (3) For each  $f \in X$ ,  $S_t f$  is continuous in  $t$ .
- (4) If  $u$  solves

$$\begin{cases} u \in C^n[0, \infty) \\ u(t) \in D_A \text{ for } t \in [0, \infty) \\ Lu = Au \text{ on } [0, \infty) \\ \hat{u}(0) = \hat{0}, \end{cases}$$

then  $u \equiv 0$  on  $[0, \infty)$ .

This definition is formulated so as to correspond with that in [3]. As will be shown in 5.B, one can take the subspace  $Y$  to be all of  $D_A$ , in which case  $Y_0 = D_{A^2}$ . In this case (1) is automatically satisfied whenever  $\rho(A) \neq \emptyset$ , because then  $D_{A^2}$  is dense in  $X$ . Thus the reader may recast the definition in less complicated terms without affecting

the validity of the main theorem or its proof.

C. DEFINITION. The operators  $S_t$  described above such that  $S_t f$  solves (P) for  $f \in Y$  are called the *solution operators*.

D. DEFINITION. Let (P) be as in Definition B, the definition of "well posed." The corresponding *scalar equation* is

$$(p) \begin{cases} u \in C^n[0, \infty) \\ Lu = zu \\ \hat{u}(0) = \hat{1} . \end{cases}$$

( $u$  is complex valued.)

For each  $z \in C$ , this equation has a unique solution which will be denoted  $\psi(t, z)$ , so that formally  $S_t = \psi(t, A)$ .

### 3. Statement and proof of main theorem.

A. THEOREM. Let  $X$  be a Banach space,  $A$  be a closed but not necessarily bounded linear operator on  $X$ , and

$$(P) \begin{cases} u \in C^{(n)}[0, \infty) \\ u(t) \in D_A \text{ for } 0 \leq t < \infty \\ Lu(t) = Au(t) \text{ for } 0 \leq t < \infty \\ \hat{u}(0) = \hat{f} \end{cases}$$

be a well posed problem in  $X$  with solution operators  $S_t$  for  $0 \leq t < \infty$ . Let

$$(p) \begin{cases} u \in C^{(n)}[0, \infty) \\ Lu(t) = zu(t) \text{ for } 0 \leq t < \infty \\ \hat{u}(0) = \hat{1} \end{cases}$$

be the corresponding scalar equation with solution  $\psi(t, z)$ . Then

$$\psi(t, \sigma(A)) \subseteq \sigma(S_t) .$$

Note that denoting  $S_t$  by the suggestive notation  $\psi(t, A)$ , this statement become  $\psi(t, \sigma(A)) \subseteq \sigma(\psi(t, A))$ .

*Proof.* The reverse containment of the complements will be shown, i.e., that if  $\psi(t, \lambda) \in \rho(S_t)$ , then  $\lambda \in \rho(A)$ . Assume  $\lambda$  is such that  $\psi(t, \lambda) \in \rho(S_t)$ .

By the variation of parameters formula there is a scalar valued kernel  $K(t, s)$  which is  $C^{(n)}$  in  $t$  and  $C^{(1)}$  in  $s$  so that if  $\tilde{K}\phi$  is defined

by  $(\tilde{K}\phi)(t) = \int_0^t K(t, s)\phi(s)ds$  for any continuous scalar valued or  $X$  valued function  $\phi$ , then  $u = \tilde{K}\phi$  is the unique scalar or  $X$  valued solution of

$$\begin{cases} (\lambda - L)u = \phi \\ \hat{u}(0) = \hat{0}. \end{cases}$$

Consider the problem

$$(3.1) \quad \begin{cases} (\lambda - L)u = S_t f \\ \hat{u}(0) = \hat{0}. \end{cases}$$

For any fixed  $f \in X$ ,  $S_t f$  is continuous in  $t$ . Hence, given  $f \in X$ , (3.1) has a unique  $C^{(n)}$  solution  $\tilde{K}S_t f$ . Let  $N_t f = -\tilde{K}S_t f$ . Since  $S_s f$  is continuous in  $s$  for each fixed  $f$ , the  $\|S_s\|$  are uniformly bounded on compact intervals by the uniform boundedness principle. Since in addition  $K$  is continuous in  $s$ ,  $N_t$  is seen to be  $\in \mathcal{B}(X)$  for each fixed  $t$ .

Assume, as we may by 5.B, that  $Y = D_A$  so that  $S_t f$  solves  $Lu = Au$ ,  $\hat{u}(0) = 0$  for any  $f \in D_A$ .

If  $f \in D_A$ , then  $LS_t f = AS_t f$ , and since  $LS_t f$  is continuous in  $t$ , so is  $AS_t f$ . Thus,  $S_t f \in D_A$  and  $S_t f$  is continuous in  $t$  with respect to the  $D_A$  norm. Hence, since  $N_t f = -\tilde{K}S_t f = -\int_0^t K(t, s)S_s f ds$ ,  $N_t f \in D_A$ .

Furthermore

$$(3.2) \quad \begin{aligned} -(\lambda - A)N_t f &= \int_0^t K(t, s)(\lambda - A)S_s f ds \\ &= \int_0^t K(t, s)(\lambda - L)S_s f ds \\ &= \tilde{K}(\lambda - L)S_t f, \end{aligned}$$

since  $A$  is closed and  $S_t f$  solves

$$\begin{cases} Lu = Au \\ \hat{u}(0) = \hat{f}. \end{cases}$$

Since  $-(\lambda - A)N_t f = \tilde{K}(\lambda - L)S_t f$ , by the variation of parameters formula  $-(\lambda - A)N_t f$  must solve

$$\begin{cases} (\lambda - L)u(\lambda - L)S_t f \\ \hat{u}(0) = \hat{0}. \end{cases}$$

Clearly  $S_t f$  solves

$$\begin{cases} (\lambda - L)u = (\lambda - L)S_t f \\ \hat{u}(0) = \hat{f}. \end{cases}$$

Hence  $S_t f + (\lambda - A)N_t f$  solves

$$(3.3) \quad \begin{cases} (\lambda - L)u = 0 \\ \hat{u}(0) = \hat{f} . \end{cases}$$

But (3.3) is obviously solved by  $\psi(t, \lambda)f$ . By uniqueness in the initial value problem  $S_t f + (\lambda - A)N_t f = \psi(t, \lambda)f$ . Hence

$$(3.4) \quad (\lambda - A)N_t f = (\psi(t, \lambda) - S_t)f \quad \text{for all } f \in D_A .$$

Since  $A$  and  $S_s$  commute by Lemma 5.A,  $(\lambda - A)N_t f = (\lambda - A) \int_0^t -K(t, s)S_s f ds = \int_0^t -K(t, s)S_s(\lambda - A)f ds = N_t(\lambda - A)f$ . Hence from (3.4)

$$(3.5) \quad N_t(\lambda - A)f = (\psi(t, \lambda) - S_t)f \quad \text{for all } f \in D_A .$$

By using the density of  $D_A$  in  $X$  and closure and continuity properties, one concludes from (3.4) that

$$(3.6) \quad (\lambda - A)N_t f = (\psi(t, \lambda) - S_t)f, \quad f \in X .$$

Since  $\psi(t, \lambda) \in \rho(S_t)$ ,  $(\psi(t, \lambda) - S_t)$  is a bijection. Thus, (3.6) shows that  $(\lambda - A)$  is onto, and (3.5) shows that  $(\lambda - A)$  is one to one. Thus  $(\lambda - A): D_A \rightarrow X$  is a bijection, and from the closed graph theorem,  $(\lambda - A)^{-1}$  exists in  $\mathcal{B}(X)$ , i.e.,  $\lambda \in \rho(A)$ . This completes the proof.

*Note.* For  $\lambda$  such that  $\psi(t, \lambda) \in \rho(S_t)$ ,  $(\lambda - A)^{-1} = N_t(\psi(t, \lambda) - S_t)^{-1}$ . This fact follows from (3.6) and from the fact that  $(\lambda - A)^{-1}$  exists as a bounded linear operator.

4. **Remarks.** There are a variety of well posed second order problems to which the theorem applies. Any problem solved by a cosine function [1, Ch. 2, §8] is well posed as are all problems considered by Stafney in [3].

Containment may be strict in the semigroup  $e^{t\sigma(A)} \subseteq \sigma(e^{tA})$  case. A genuine second order example of strict containment in the main theorem also exists where  $L = d^2/dt^2$ . Hille and Phillips mention [2, §26.16] a  $(C_0)$  group  $(T_t)$  whose generator  $A$  has empty spectrum. Then

$$\begin{cases} x'' = A^2 x \\ x(0) = f \\ x'(0) = 0 \end{cases}$$

is well posed and solved by the cosine function  $S_t = 1/2(T_t + T_{-t})$ . Yet, spectral equality fails in  $\cosh(t\sqrt{\sigma(A^2)}) \subseteq \sigma(S_t)$ , because

$\sigma(A^2) = \emptyset$ , but  $\sigma(S_i) \neq \emptyset$ .

It appears that the initial conditions on  $(p_n)$  and  $(P_n)$  in the statement of the theorem can be altered to other conditions differing in only one coordinate from the zero initial conditions. We think our proof could be altered to reflect the change.

### 5. Commutativity results.

A. LEMMA. *Let*

$$\begin{cases} Lx = Ax \\ \hat{x}(0) = \hat{f} \end{cases}$$

be a well posed problem in a Banach space  $X$ . For all  $t > 0$ , if  $f \in D_A$ , then  $S_t f \in D_A$ , and  $AS_t f = S_t Af$ .

*Proof.* This lemma can be proved exactly as the special case in [3] is proved.

In case  $\rho(A) \neq \emptyset$ , a more direct proof can be constructed by noting that  $(\lambda - A)^{-1}S_t(\lambda - A)f$  and  $S_t f$  both solve the same initial value problem and so must be equal.

B. COROLLARY. *In the Definition 2.B of "well posed,"  $Y$  can be chosen to be all  $D_A$ .*

*Proof.* Suppose that  $Y$  and  $Y_0$  satisfy the conditions of 2.B. The only condition on  $Y$  that is not obviously met by  $D_A$  is that  $u = S_t f$  satisfy

$$(5.1) \quad \begin{cases} u \in C^{(n)}[0, \infty) \\ Lu = Au \\ \hat{u}(0) = \hat{f} \end{cases}$$

for each  $f \in D_A$ .

Let  $Y$  and  $Y_0$  be as originally given. Choose  $f \in D_A$ . Choose a sequence  $(f_n)$  from  $Y$  so that  $f_n \rightarrow f$  in the  $D_A$  norm, that is  $f_n \rightarrow f$  and  $Af_n \rightarrow Af$  in the norm of  $X$ .

Then since the norms of  $S_t$  are bounded on finite  $t$  intervals,  $S_t Af_n \rightarrow S_t Af$  uniformly on finite  $t$  intervals. Since  $S_t$  and  $A$  commute

$$(5.2) \quad AS_t f_n \rightarrow AS_t f \quad \text{uniformly on finite } t \text{ intervals.}$$

For any  $\psi \in C([0, \infty), X)$  define  $\tilde{K}\psi$  to the unique solution of

$$\begin{cases} Lu = \psi \\ \hat{u}(0) = \hat{0} \end{cases}$$

In particular this solution  $u$  is in  $C^{(n)}$ . By the variation of parameters formula  $\tilde{K}\psi$  is given by

$$(\tilde{K}\psi)(t) = \int_0^t K(t, s)\psi(s)ds$$

for some continuous kernel  $K$ .

Let  $\phi$  be the unique (scalar) solution of

$$\begin{cases} Lu = 0 \\ \hat{u}(0) = \hat{1}. \end{cases}$$

Then  $\phi g$  is the unique  $X$  valued solution of

$$\begin{cases} Lu = 0 \\ \hat{u}(0) = \hat{g}. \end{cases}$$

Both  $S_t f_n$  and  $\tilde{K}AS_t f_n + \phi f_n$  solve

$$\begin{cases} Lu = Au \\ \hat{u}(0) = \hat{f}_n, \end{cases}$$

and so by uniqueness for the well posed problem,

$$(5.3) \quad S_t f_n = \tilde{K}AS_t f_n + \phi f_n.$$

Since  $\tilde{K}$  is defined by an integral with continuous kernel and since by (5.1)  $AS_t f_n \rightarrow AS_t f$  uniformly on finite  $t$  intervals,

$$\tilde{K}AS_t f_n \longrightarrow \tilde{K}AS_t f.$$

Clearly  $\phi f_n \rightarrow \phi f$ . Hence, from (5.3), taking limits of both sides,

$$(5.4) \quad S_t f = \tilde{K}AS_t f + \phi f.$$

$\tilde{K}AS_t f$  solves  $Lu = AS_t f$ ,  $AS_t f$  is continuous, and  $\phi f$  solves  $Lu = 0$ . Hence, both  $\tilde{K}AS_t f$  and  $\phi f$  are in  $C^{(n)}$ , and their sum  $S_t f$  is also in  $C^{(n)}$ . This establishes the first condition in (5.1). Applying  $L$  to both sides of (5.4),

$$(5.5) \quad LS_t f = AS_t f.$$

Since  $(\tilde{K}AS_t f)_{t=0} = \hat{0}$  and  $(\phi(t)f)_{t=0} = \hat{\phi}(0)f = \hat{1}f = \hat{f}$ ,  $(S_t f)_{t=0} = \hat{f}$ . Thus  $u(t) = S_t f$  satisfies the other conditions of (5.1).

C. LEMMA. *Let  $X$  be a Banach space and let*

$$\begin{cases} Lu = Au \\ \hat{u}(0) = \hat{f} \end{cases}$$



be a well posed problem in  $X$  with solution operators  $S_t$ . Then the solution operators commute with each other (that is,  $S_s S_t = S_t S_s$ ,  $s, t \geq 0$ ).

*Proof.* Consider  $s \geq 0$  as fixed and  $t$  as variable, so that  $L$  depends on  $t$  and takes derivatives in  $t$ . By Corollary B, take  $Y$  in the Definition 2.B of "well posed" to be all of  $D_A$ . For any  $f \in D_A$ ,  $S_s f \in D_A$  by Lemma A (commutativity result). Then by the definition of  $S_t$ ,

$$(5.6) \quad \begin{cases} LS_t S_s f = AS_t S_s f \\ (S_t S_s f)_{t=0} = (S_s f)^\wedge . \end{cases}$$

But using the fact that  $S_s$  is linear and continuous,  $S_s$  commutes with differentiation.  $S_s$  also commutes with scalar multiplication. Hence  $S_s$  commutes with  $L$ . Thus

$$LS_s S_t f = S_s LS_t f = S_s AS_t f = AS_s S_t f ,$$

and

$$(S_s S_t f)_{t=0} = (S_s f)^\wedge .$$

Hence,

$$(5.7) \quad \begin{cases} LS_s S_t f = AS_s S_t f \\ (S_s S_t f)_{t=0} = (S_s f)^\wedge . \end{cases}$$

Now (5.6) and (5.7) show that both  $S_t S_s f$  and  $S_s S_t f$  solve

$$\begin{cases} Lu = Au \\ \hat{u}(0) = (S_s f)^\wedge , \end{cases}$$

and so since  $Lu = Au$  is well posed, the two solutions must be the same. Thus

$$S_s S_t f = S_t S_s f$$

for all  $f \in D_A$ . Since  $D_A$  is dense in  $X$  and  $S_t$  and  $S_s$  are continuous,  $S_s S_f = S_t S_s$  on  $X$ .

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Received December 1, 1980. This paper is based on Chapter 1 of the author's 1980 Ph. D. dissertation at University of California, Riverside, under J. D. Stafney.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966, Regular subscription rate: \$114.00 a year (6 Vol., 12 issues). Special rate: \$57.00 a year to individual members of supporting institution.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).  
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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# Pacific Journal of Mathematics

Vol. 101, No. 2      December, 1982

<b>Jean Bourgain</b> , A Hausdorff-Young inequality for $B$ -convex Banach spaces .....	255
<b>J. L. Brenner and Lorraine L. Foster</b> , Exponential Diophantine equations .....	263
<b>Henry H. Glover and William Duncan Homer, II</b> , Fixed points on flag manifolds .....	303
<b>Lothar Hahn</b> , A note on stochastic methods in connection with approximation theorems for positive linear operators .....	307
<b>James P. Henderson</b> , Approximating cellular maps between low-dimensional polyhedra .....	321
<b>V. K. Jain</b> , Certain transformations of basic hypergeometric series and their applications .....	333
<b>Charles David Keys</b> , On the decomposition of reducible principal series representations of $p$ -adic Chevalley groups .....	351
<b>M. S. Klamkin and A. Meir</b> , Ptolemy's inequality, chordal metric, multiplicative metric .....	389
<b>Robert F. Lax</b> , Independence of normal Weierstrass points under deformation .....	393
<b>Leonid A. Luxemburg</b> , On compactifications of metric spaces with transfinite dimensions .....	399
<b>Carlton James Maxson, Martin Ross Pettet and Kirby C. Smith</b> , On semisimple rings that are centralizer near-rings .....	451
<b>Teodor C. Przymusiński</b> , Extending functions from products with a metric factor and absolutes .....	463
<b>Giorgio Talenti</b> , A note on the Gauss curvature of harmonic and minimal surfaces .....	477
<b>D. M. Terlinden</b> , A spectral containment theorem analogous to the semigroup theory result $e^{t\sigma(A)} \subseteq \sigma(e^{tA})$ .....	493