ON THE IMAGE OF THE GENERALIZED GAUSS MAP OF A COMPLETE MINIMAL SURFACE IN $\mathbb{R}^4$

Chi Cheng Chen
ON THE IMAGE OF THE GENERALIZED GAUSS MAP
OF A COMPLETE MINIMAL SURFACE IN $R^4$

CHI CHENG CHEN

The generalized Gauss map of an immersed oriented surface $M$ in $R^4$ is the map which associates to each point of $M$ its oriented tangent plane in $G_{2,4}$, the Grassmannian of oriented planes in $R^4$. The Grassmannian $G_{2,4}$ is naturally identified with $Q_2$, the complex hyperquadric

$$\left\{ (z_1, z_2, z_3, z_4) \mid \sum_{k=1}^4 z_k^2 = 0 \right\} \text{ in } P^3(C).$$

The normalized Fubini-Study metric on $P^3(C)$ with holomorphic curvature 2 induces an invariant metric on $Q_2 \cong G_{2,4}$, which corresponds exactly to the metric on the canonical representation of $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$ in $R^6$ as $\{ X \in R^6 \mid x_1^2 + x_2^2 + x_3^2 = (1/2), x_4^2 + x_5^2 + x_6^2 = (1/2) \}$. The product representation above allows us to associate with any map $g$ in $Q_2$ two canonical projections $g_1, g_2$. In the case where $g$ is complex analytic map defined on some Riemann surface $S_0$, the projections $g_1, g_2$ are complex analytic also. Detailed treatment can be found in the recent work of Hoffman and Osserman.

The study of the image of the Gauss map of a complete minimal surface in $R^3$ was motivated in one way to generalize a classical theorem of S. Bernstein [1], and was initiated by Osserman [7, 8, 9]. And the value distribution of the generalized Gauss map of a complete minimal surface in $R^4$, due to the product representation of $Q_2$, can therefore be studied in a similar manner. In fact, results treating the case in $R^3$ have been extended to that in $R^4$ by Chern [3], Osserman [9], Hoffman and Osserman [5]. Very recently, Xavier [10] has made a remarkable improvement in the study of the case in $R^3$. Therefore it’s quite natural to extend it to the case in $R^4$, which will be shown in the following theorem.

**THEOREM 1.** Let $S$ be a complete minimal surface in $R^4$ with $g$ its generalized Gauss map and $g_1, g_2$, the corresponding projections. Then $S$ must be a plane if

(i) both $g_1$ and $g_2$ omit more than 6 points, or

(ii) one projection is constant and the other omits more than 4 points.

**Proof.** Let $S$ be given by

(1) $X: S_0 \longrightarrow R^4$
where $S_0$ is a Riemann surface. Its generalized Gauss map can be expressed by

$$g=[\phi_1(\zeta), \phi_2(\zeta), \phi_3(\zeta), \phi_4(\zeta)]$$

where

$$\phi_k(\zeta) = 2\frac{\partial x_k}{\partial \zeta}$$

with $\zeta$ a local complex parameter. And the projection $g_1, g_2$ are expressed by

$$g_1 = \frac{\phi_3 + i\phi_4}{\phi_1 - i\phi_2}, \quad g_2 = \frac{\phi_3 - i\phi_4}{\phi_1 - i\phi_2}.$$  

The induced metric is given by

$$ds^2 = \frac{1}{4} f(|g_1|^2)(1 + |g_1|^2)(1 + |g_2|^2)|d\zeta|^2$$

where $f(\zeta) = \phi_1 - i\phi_2$. For detailed explanation, see Osserman [9].

Without loss of generality, we may assume $S_0$ to be simply connected. Combining our hypothesis in (i), (ii) with the Koebe uniformization theorem and the Picard's theorem, we may assume further that $S_0$ is the unit disk $D = \{\zeta \in \mathbb{C} | |\zeta| < 1\}$.

A crucial lemma used by Xavier [10] can be adapted easily in our case as:

**Lemma.** Let $g_1: D \rightarrow C - \{0, a\} (a \neq 0)$, $g_2 = D \rightarrow C - \{0, b\} (b \neq 0)$ be holomorphic functions. Then

$$\int \prod_{j=1}^2 \left( \frac{|g_j'|}{|g_j|^\alpha + |g_j'|^{1-\alpha}} \right)^p |d\zeta| < \infty$$

for any $\alpha = 1 - 1/k, k \in \mathbb{Z}^+$ and $0 \leq p < 1/2$, where $\zeta = \xi + i\eta$.

Now we proceed our proof. Suppose $S$ is not a plane. Under the hypothesis in (i) or (ii), we may assume that both $g_1$ and $g_2$ are holomorphic.

For the case (i), suppose $g_1$ omits $a_1, \ldots, a_6$ in $C$ and $g_2$ omits $b_1, \ldots, b_6$ in $C$. Consider the function

$$h = g_1'g_2'^{-1/p} \prod_{i=1}^6 (g_1 - a_i)^{-\alpha} \prod_{j=1}^6 (g_2 - b_j)^{-\alpha},$$

where $\alpha = 1 - 1/k$ with $10/11 \leq \alpha < 1$ and $p = 5/12\alpha$.

For the case (ii), suppose $g_1$ constant and $g_2$ omits $b_1, \ldots, b_6$ in $C$. And consider the function
where \( \alpha = 1 - 1/k \) with \( 10/11 \leq \alpha < 1 \) and \( p = 3/4\alpha \).

In both cases, using the same arguments in [10], we can see that from one hand, essentially due to a theorem of Yau [11, Th. 1].

\[
|h| \in L^p(S_\circ)
\]

and from the other hand, by direct calculation, we get

\[
|h| \in L^p(S_\circ)
\]

which is impossible. \( \square \)

Next we shall extend a theorem of Gackstatter [6] on complete abelian minimal surfaces in \( R^4 \) to those in \( R^4 \).

**Theorem 2.** Let \( S \) be a complete abelian minimal surface in \( R^4 \), and \( g \) its generalized Gauss map. Then \( S \) must be plane if either

1. one projection, say \( g_1 \), omits more than 4 points and the other projection \( g_2 \) omits more than 3 points, or
2. \( g_1 \) is constant and \( g_2 \) omits more than 3 points.

**Proof.** By a complete abelian minimal surface \( S \) in \( R^4 \). We mean that \( S \) can be constructed out of a meromorphic differential \( f d\zeta \) and two meromorphic functions \( g_1, g_2 \) on a compact Riemann surface \( \tilde{M} \) with the metric

\[
d\tilde{s}^2 = \frac{|f|^2}{4}(1 + |g_1|^2)(1 + |g_2|^2)|d\zeta|^2
\]

which never vanishes. And the construction is made in the sense of L. Bers [2] such that the immersion is given by the formula

\[
x = \text{Re} \int \frac{f}{2}(1 + g_1g_2, i(1 - g_1g_2), g_1 - g_2, -i(g_1 + g_2))d\zeta
\]

on a covering space \( \tilde{M} \) over \( \overline{ \tilde{M} - \{p | d\tilde{s}^2(p) = \infty \} } \) as long as (10) is well-defined. The boundary points to the metric \( d\tilde{s}^2 \) are those finitely many points \( p_1, \ldots, p_r \) in \( \overline{\tilde{M}} \) where \( d\tilde{s}^2 = \infty \).

By a rotation of \( S \), we may assume that

1. both \( g_1 \) and \( g_2 \) have only simple poles, and they don't have poles in common,
2. the poles of \( g_1, g_2 \) don't fall into the boundary points \( p_1, \ldots, p_r \), and hence,
(iii) at each pole of \( g \) or \( g_2 \), \( f \) must have a simple zero,
(iv) \( f \) has no other zeros, and
(v) at each \( p_j \), \( f \) must have a pole of order \( m_j \geq 1 \).

Now suppose \( g \) is an \( N_1 \)-sheet and \( g_2 \) is an \( N_2 \)-sheet branching covering.
Then by the Riemann relation for the differential \( \frac{f d\zeta}{\zeta} \), we have

\[
(N_1 + N_2) - \sum_{j=1}^{r} m_j = 2\gamma - 2
\]

where \( \gamma \) is the genus of \( \bar{M} \).

And by the Riemann relation for \( g_1 \) and \( g_2 \), in case of non-constant, we have

\[
\sum_{a_1} (l_1 - 1) - 2N_1 = 2\gamma - 2
\]
(12)
\[
\sum_{a_2} (l_2 - 1) - 2N_2 = 2\gamma - 2
\]
(13)

where \( \sum_{a_1} (l_1 - 1) \) and \( \sum_{a_2} (l_2 - 1) \) are the total branching orders of \( g_1 \) and \( g_2 \) respectively.

Now suppose \( \mathcal{S} \) is nonflat, i.e., \( g_1, g_2 \) can’t both be constant, and that

(a) \( g_1 \) omits 5 values \( a_1, \ldots, a_5 \), \( g_2 \) omits 4 values \( b_1, \ldots, b_4 \), and neither one is constant. Then clearly

\[
g_1^{-1}\{a_\nu \mid 1 \leq \nu \leq 5\} \subset \{p_1, \ldots, p_r\},
\]
(14)
\[
g_2^{-1}\{b_\mu \mid 1 \leq \mu \leq 4\} \subset \{p_1, \ldots, p_r\}.
\]
(15)

And (12), (13) can be written as

\[
\sum_{a_1} (l_1 - 1) + 3N_1 = 2\gamma - 2 + \sum_{a_\nu} 1,
\]
(16)
\[
\sum_{a_2} (l_2 - 1) + 2N_2 = 2\gamma - 2 + \sum_{b_\mu} 1.
\]
(17)

Comparing with (11), we get

\[
2 \sum_{j=1}^{r} m_j < \sum_{a_\nu} 1 + \sum_{b_\mu} 1
\]
(18)

which contradicts (14) and (15).

(b) \( g_1 \) constant and \( g_2 \) omits 4 points \( b_1, \ldots, b_4 \). Clearly (15) and (17) still hold with \( N_1 = 0, N_2 > 0 \). From (11), (17), we have

\[
\sum_{j=1}^{r} m_j < \sum_{b_\mu} 1
\]
(19)

which contradicts (15).

\[\square\]

**Corollary.** If (a) \( g_1 \) omits exactly 4 points and \( g_2 \) omits exactly
4 points, or
(b) \(g_1\) constant and \(g_2\) omits exactly 3 points, then (a) \(r = 4, m_j = 1\) or (b) \(r = 3, m_j = 1\), respectively. Further, in neither case \(S\) can have flat points.

Proof. Note that \(p\) is a flat point of \(S\) if and only if \(g_1'(p) = 0\) and \(g_2'(p) = 0\). In case (a) comparing (11) with
\[
\sum_{g_1 = a_\nu} (l_1 - 1) + 2N_1 = 2\gamma - 2 + \sum_{g_1 = a_\nu} 1
\]
and (17), we get \(r = 4, m_j = 1, l_1 = 1, l_2 = 1\).

And in case (b) comparing (11) with
\[
\sum_{g_2 = b_\mu} (l_2 - 1) + N_2 = 2\gamma - 2 + \sum_{g_2 = b_\mu} 1
\]
and \(N_1 = 0\), we get \(r = 3, m_j = 1, l_2 = 1\).

For complete minimal surface with finite total curvature, it's known \([4]\) that \(M = \overline{M} - \{p_1, \cdots, p_r\}\) and \(m_j \geq 2\). Thus, Theorem 2 and corollary together give an alternative proof of

THEOREM 3 (Hoffman-Osserman \([5]\)). Let \(S\) be a complete minimal surface in \(\mathbb{R}^n\) with finite total curvature. Then \(S\) must be a plane if
(a) both \(g_1\) and \(g_2\) omit more than 3 points, or
(b) \(g_1\) constant and \(g_2\) omits more than 2 points.

REFERENCES

Received July 13, 1981. Work partially supported by FAPESP (BRAZIL), contract No. 11-Mate. 78/1105

Instituto de Matemática e Estatística
Universidade de São Paulo
São Paulo-Brasil
<table>
<thead>
<tr>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>S. Agou, Degré minimum des polynômes $f(\sum_{i=0}^{m} a_i X^{p^i})$ sur les corps finis de caractéristique $p &gt; m$</td>
<td>1</td>
</tr>
<tr>
<td>Chi Cheng Chen, On the image of the generalized Gauss map of a complete minimal surface in $\mathbb{R}^4$</td>
<td>9</td>
</tr>
<tr>
<td>Thomas Curtis Craven and George Leslie Csordas, On the number of real roots of polynomials</td>
<td>15</td>
</tr>
<tr>
<td>Allan L. Edelson and Kurt Kreith, Nonlinear relationships between oscillation and asymptotic behavior</td>
<td>29</td>
</tr>
<tr>
<td>B. Felzenszwalb and Antonio Giambruno, A commutativity theorem for rings with derivations</td>
<td>41</td>
</tr>
<tr>
<td>Richard Elam Heisey, Manifolds modelled on the direct limit of lines</td>
<td>47</td>
</tr>
<tr>
<td>Steve J. Kaplan, Twisting to algebraically slice knots</td>
<td>55</td>
</tr>
<tr>
<td>Jeffrey C. Lagarias, Best simultaneous Diophantine approximations. II. Behavior of consecutive best approximations</td>
<td>61</td>
</tr>
<tr>
<td>Masahiko Miyamoto, An affirmative answer to Glauberman’s conjecture</td>
<td>89</td>
</tr>
<tr>
<td>Thomas Bourque Muenzenberger, Raymond Earl Smithson and L. E. Ward, Characterizations of arboroids and dendritic spaces</td>
<td>107</td>
</tr>
<tr>
<td>William Leslie Pardon, The exact sequence of a localization for Witt groups. II. Numerical invariants of odd-dimensional surgery obstructions</td>
<td>123</td>
</tr>
<tr>
<td>Bruce Eli Sagan, Bijective proofs of certain vector partition identities</td>
<td>171</td>
</tr>
<tr>
<td>Kichi-Suke Saito, Automorphisms and nonselfadjoint crossed products</td>
<td>179</td>
</tr>
<tr>
<td>John Joseph Sarraille, Module finiteness of low-dimensional $PI$ rings</td>
<td>189</td>
</tr>
<tr>
<td>Gary Roy Spoar, Differentiable curves of cyclic order four</td>
<td>209</td>
</tr>
<tr>
<td>William Charles Waterhouse, Automorphisms of quotients of $TIGL(n_i)$</td>
<td>221</td>
</tr>
<tr>
<td>Leslie Wilson, Mapgerms infinitely determined with respect to right-left equivalence</td>
<td>235</td>
</tr>
<tr>
<td>Rahman Mahmoud Younis, Interpolation in strongly logmodular algebras</td>
<td>247</td>
</tr>
</tbody>
</table>