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**ON THE IMAGE OF THE GENERALIZED GAUSS MAP OF A
COMPLETE MINIMAL SURFACE IN \mathbb{R}^4**

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The generalized Gauss map of an immersed oriented surface M in R^4 is the map which associates to each point of M its oriented tangent plane in $G_{2,4}$, the Grassmannian of oriented planes in R^4 . The Grassmannian $G_{2,4}$ is naturally identified with Q_2 , the complex hyperquadric

$$\left\{ [z_1, z_2, z_3, z_4] \mid \sum_{k=1}^4 z_k^2 = 0 \right\} \text{ in } P^3(C).$$

The normalized Fubini-Study metric on $P^3(C)$ with holomorphic curvature 2 induces an invariant metric on $Q_2 \cong G_{2,4}$, which corresponds exactly to the metric on the canonical representation of $S^2(1/\sqrt{2}) \times S^2(1/\sqrt{2})$ in R^6 as $\{X \in R^6 \mid x_1^2 + x_2^2 + x_3^2 = (1/2), x_4^2 + x_5^2 + x_6^2 = (1/2)\}$. The product representation above allows us to associate with any map g in Q_2 two canonical projections g_1, g_2 . In the case where g is complex analytic map defined on some Riemann surface S_0 , the projections g_1, g_2 are complex analytic also. Detailed treatment can be found in the recent work of Hoffman and Osserman.

The study of the image of the Gauss map of a complete minimal surface in R^3 was motivated in one way to generalize a classical theorem of S. Bernstein [1], and was initiated by Osserman [7, 8, 9]. And the value distribution of the generalized Gauss map of a complete minimal surface in R^4 , due to the product representation of Q_2 , can therefore be studied in a similar manner. In fact, results treating the case in R^3 have been extended to that in R^4 by Chern [3], Osserman [9], Hoffman and Osserman [5]. Very recently, Xavier [10] has made a remarkable improvement in the study of the case in R^3 . Therefore it's quite natural to extend it to the case in R^4 , which will be shown in the following theorem.

THEOREM 1. *Let S be a complete minimal surface in R^4 with g its generalized Gauss map and g_1, g_2 , the corresponding projections. Then S must be a plane if*

- (i) both g_1 and g_2 omit more than 6 points, or
- (ii) one projection is constant and the other omits more than 4 points.

Proof. Let S be given by

$$(1) \quad X: S_0 \longrightarrow R^4$$

where S_0 is a Riemann surface. Its generalized Gauss map can be expressed by

$$(2) \quad g = [\phi_1(\zeta), \phi_2(\zeta), \phi_3(\zeta), \phi_4(\zeta)]$$

where

$$(3) \quad \phi_k(\zeta) = 2 \frac{\partial x_k}{\partial \zeta}$$

with ζ a local complex parameter. And the projection g_1, g_2 are expressed by

$$(4) \quad g_1 = \frac{\phi_3 + i\phi_4}{\phi_1 - i\phi_2}, \quad g_2 = \frac{\phi_3 - i\phi_4}{\phi_1 - i\phi_2}.$$

The induced metric is given by

$$(5) \quad ds^2 = \frac{1}{4} |f|^2 (1 + |g_1|^2)(1 + |g_2|^2) |d\zeta|^2$$

where $f(\zeta) = \phi_1 - i\phi_2$. For detailed explanation, see Osserman [9].

Without loss of generality, we may assume S_0 to be simply connected. Combining our hypothesis in (i), (ii) with the Koebe uniformization theorem and the Picard's theorem, we may assume further that S_0 is the unit disk $D = \{\zeta \in \mathbb{C} \mid |\zeta| < 1\}$.

A crucial lemma used by Xavier [10] can be adapted easily in our case as:

LEMMA. Let $g_1: D \rightarrow \mathbb{C} - \{0, a\} (a \neq 0)$, $g_2: D \rightarrow \mathbb{C} - \{0, b\} (b \neq 0)$ be holomorphic functions. Then

$$\int_D \prod_{j=1}^2 \left[\frac{|g'_j|}{|g_j|^\alpha + |g_j|^{2-\alpha}} \right]^p d\bar{z} dz < \infty$$

for any $\alpha = 1 - 1/k$, $k \in \mathbb{Z}^+$ and $0 \leq p < 1/2$, where $\zeta = \xi + i\eta$.

Now we proceed our proof. Suppose S is not a plane. Under the hypothesis in (i) or (ii), we may assume that both g_1 and g_2 are holomorphic.

For the case (i), suppose g_1 omits a_1, \dots, a_s in \mathbb{C} and g_2 omits b_1, \dots, b_s in \mathbb{C} . Consider the function

$$(6) \quad h = g'_1 g'_2 f^{-2/p} \prod_{i=1}^s (g_1 - a_i)^{-\alpha} \prod_{j=1}^s (g_2 - b_j)^{-\alpha},$$

where $\alpha = 1 - 1/k$ with $10/11 \leq \alpha < 1$ and $p = 5/12\alpha$.

For the case (ii), suppose g_1 constant and g_2 omits b_1, \dots, b_s in \mathbb{C} . And consider the function

$$(7) \quad h = g_2' f^{-2/p} \prod_{j=1}^4 (g_2 - b_j)^{-\alpha},$$

where $\alpha = 1 - 1/k$ with $10/11 \leq \alpha < 1$ and $p = 3/4\alpha$.

In both cases, using the same arguments in [10], we can see that from one hand, essentially due to a theorem of Yau [11, Th. 1].

$$(8) \quad |h| \notin L^p(S_0)$$

and from the other hand, by direct calculation, we get

$$(9) \quad |h| \in L^p(S_0)$$

which is impossible. \square

Next we shall extend a theorem of Gackstatter [6] on complete abelian minimal surfaces in \mathbf{R}^3 to those in \mathbf{R}^4 .

THEOREM 2. *Let S be a complete abelian minimal surface in \mathbf{R}^4 , and g its generalized Gauss map. Then S must be plane if either*

- (a) *one projection, say g_1 , omits more than 4 points and the other projection g_2 omits more than 3 points, or*
- (b) *g_1 is constant and g_2 omits more than 3 points.*

Proof. By a complete abelian minimal surface S in \mathbf{R}^4 . We mean that S can be constructed out of a meromorphic differential $f d\zeta$ and two meromorphic functions g_1, g_2 on a compact Riemann surface \bar{M} with the metric

$$ds^2 = \frac{|f|^2}{4} (1 + |g_1|^2)(1 + |g_2|^2) |d\zeta|^2$$

which never vanishes. And the construction is made in the sense of L. Bers [2] such that the immersion is given by the formula

$$(10) \quad x = \operatorname{Re} \int \frac{f}{2} (1 + g_1 g_2, i(1 - g_1 g_2), g_1 - g_2, -i(g_1 + g_2)) d\zeta$$

on a covering space M over $\bar{M} - \{p \mid ds^2(p) = \infty\}$ as long as (10) is well-defined. The boundary points to the metric ds^2 are those finitely many points p_1, \dots, p_r in \bar{M} where $ds^2 = \infty$.

By a rotation of S , we may assume that

- (i) both g_1 and g_2 have only simple poles, and they don't have poles in common,
- (ii) the poles of g_1, g_2 don't fall into the boundary points p_1, \dots, p_r , and hence,

- (iii) at each pole of g_1 or g_2 , f must have a simple zero,
- (iv) f has no other zeros, and
- (v) at each p_j , f must have a pole of order $m_j \geq 1$.

Now suppose g_1 is an N_1 -sheet and g_2 is an N_2 -sheet branching covering. Then by the Riemann relation for the differential $fd\zeta$, we have

$$(11) \quad (N_1 + N_2) - \sum_{j=1}^r m_j = 2\gamma - 2$$

where γ is the genus of \bar{M} .

And by the Riemann relation for g_1 and g_2 , in case of non-constant, we have

$$(12) \quad \sum_{g_1} (l_1 - 1) - 2N_1 = 2\gamma - 2$$

$$(13) \quad \sum_{g_2} (l_2 - 1) - 2N_2 = 2\gamma - 2$$

where $\sum_{g_1} (l_1 - 1)$ and $\sum_{g_2} (l_2 - 1)$ are the total branching orders of g_1, g_2 , respectively.

Now suppose S is nonflat, i.e., g_1, g_2 can't both be constant, and that

(a) g_1 omits 5 values a_1, \dots, a_5 , g_2 omits 4 values b_1, \dots, b_4 and neither one is constant. Then clearly

$$(14) \quad g_1^{-1}\{a_\nu \mid 1 \leq \nu \leq 5\} \subset \{p_1, \dots, p_r\},$$

$$(15) \quad g_2^{-1}\{b_\mu \mid 1 \leq \mu \leq 4\} \subset \{p_1, \dots, p_r\}.$$

And (12), (13) can be written as

$$(16) \quad \sum_{g_1 \neq a_\nu} (l_1 - 1) + 3N_1 = 2\gamma - 2 + \sum_{g_1 = a_\nu} 1,$$

$$(17) \quad \sum_{g_2 \neq b_\mu} (l_2 - 1) + 2N_2 = 2\gamma - 2 + \sum_{g_2 = b_\mu} 1.$$

Comparing with (11), we get

$$(18) \quad 2 \sum_{j=1}^r m_j < \sum_{g_1 = a_\nu} 1 + \sum_{g_2 = b_\mu} 1$$

which contradicts (14) and (15).

(b) g_1 constant and g_2 omits 4 points b_1, \dots, b_4 . Clearly (15) and (17) still hold with $N_1 = 0, N_2 > 0$. From (11), (17), we have

$$(19) \quad \sum_{j=1}^r m_j < \sum_{g_2 = b_\mu} 1$$

which contradicts (15). □

COROLLARY. *If (a) g_1 omits exactly 4 points and g_2 omits exactly*

4 points, or

(b) g_1 constant and g_2 omits exactly 3 points, then (a) $r = 4$, $m_j = 1$ or (b) $r = 3$, $m_j = 1$, respectively. Further, in neither case S can have flat points.

Proof. Note that p is a flat point of S if and only if $g'_1(p) = 0$ and $g'_2(p) = 0$. In case (a) comparing (11) with

$$\sum_{g_1 \neq a_\nu} (l_1 - 1) + 2N_1 = 2\gamma - 2 + \sum_{g_1 = a_\nu} 1$$

and (17), we get $r = 4$, $m_j = 1$, $l_1 \equiv 1$, $l_2 \equiv 1$.

And in case (b) comparing (11) with

$$\sum_{g_2 \neq b_\mu} (l_2 - 1) + N_2 = 2\gamma - 2 + \sum_{g_2 = b_\mu} 1$$

and $N_1 = 0$, we get $r = 3$, $m_j = 1$, $l_2 \equiv 1$. □

For complete minimal surface with finite total curvature, it's known [4] that $M = \bar{M} - \{p_1, \dots, p_r\}$ and $m_j \geq 2$. Thus, Theorem 2 and corollary together give an alternative proof of

THEOREM 3 (Hoffman-Osserman [5]). *Let S be a complete minimal surface in \mathbf{R}^4 with finite total curvature. Then S must be a plane if*

- (a) both g_1 and g_2 omit more than 3 points, or
- (b) g_1 constant and g_2 omits more than 2 points.

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