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## **A COMMUTATIVITY THEOREM FOR RINGS WITH DERIVATIONS**

**B. FELZENSZWALB AND ANTONIO GIAMBRUNO**

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**Let  $R$  be a prime ring with no nonzero nil ideals and suppose that  $d$  is a derivation of  $R$  such that  $d(x^n) = 0$ ,  $n = n(x) \geq 1$ , for all  $x \in R$ . It is shown that either  $d = 0$  or  $R$  is an infinite commutative domain of characteristic  $p \neq 0$  and  $p \nmid n(x)$  if  $d(x) \neq 0$ .**

Let  $R$  be an associative ring. Recall that an additive mapping  $d$  of  $R$  into itself is a derivation if  $d(xy) = d(x)y + xd(y)$  for all  $x, y \in R$ .

In [2] it was shown that if  $R$  is a prime ring and  $d$  is a derivation of  $R$  such that  $d(x^n) = 0$  for all  $x \in R$ , where  $n \geq 1$  is a fixed integer, then either  $d = 0$  or  $R$  is an infinite commutative domain of characteristic  $p \neq 0$  where  $p \nmid n$ . Moreover, the following question was raised:

If  $R$  is a ring with no nonzero nil ideals and  $d$  is a derivation of  $R$  such that  $d(x^n) = 0$ ,  $n = n(x) \geq 1$ , for all  $x \in R$ , can we conclude that  $R$  must be rather special or  $d = 0$ ?

If  $d$  is an inner derivation (i.e., if there exists an element  $a \in R$  such that  $d(x) = ax - xa$ ) Herstein's hypercenter theorem [3] asserts that under the above conditions  $d$  must be zero. This is not always the case for arbitrary derivations. Take for instance a commutative domain  $A$  of characteristic  $p \neq 0$  and let  $d$  be the usual derivation on the polynomial ring  $A[X]$ ; then  $d(f^n) = 0$  for all  $f \in A[X]$ , but  $d \neq 0$ .

We shall prove the following

**THEOREM.** *Let  $R$  be a prime ring with no nonzero nil ideals and let  $d$  be a derivation of  $R$  such that*

$$d(x^n) = 0, \quad n = n(x) \geq 1, \quad \text{for all } x \in R.$$

*Then either  $d = 0$  or  $R$  is an infinite commutative domain of characteristic  $p \neq 0$  and  $p \nmid n(x)$  if  $d(x) \neq 0$ .*

For primitive rings the above theorem was proved in [2]; however the proof we give here is independent.

Notice that the conclusion of the theorem is false if one removes the assumption of primeness. In fact, let  $R = A[X] \oplus M_2(A)$  where  $A$  is a commutative domain of characteristic  $p \neq 0$  and  $M_2(A)$  is the ring of  $2 \times 2$  matrices over  $A$ . Let  $d$  be the derivation of  $R$  defined

as follows:  $d$  is the usual derivation on the polynomial ring  $A[X]$  and  $d = 0$  on  $M_2(A)$ . Then  $R$  has no nil ideals,  $d(r^p) = 0$  for all  $r \in R$ , but  $d \neq 0$  and  $R$  is not commutative.

We begin with a slight generalization of a result of Posner [4, Lemma 3].

**LEMMA.** *Let  $R$  be a prime ring with a derivation  $d \neq 0$  and let  $U$  be a nonzero ideal of  $R$ . If  $d(u)u = ud(u)$ , for all  $u \in U$ , then  $R$  is commutative.*

*Proof.* Let  $u, v \in U$ ; since  $d(u)u = ud(u)$ ,  $d(v)v = vd(v)$  and  $d(u+v)(u+v) = (u+v)d(u+v)$  we get

$$(1) \quad d(u)v + d(v)u = ud(v) + vd(u).$$

Thus, since  $u$  and  $uv$  lie in  $U$ , arguing as above we have that

$$d(u)uv + d(uv)u = ud(uv) + uvd(u) = ud(u)v + u(ud(v) + vd(u)).$$

Hence, from (1) and the fact that  $d(u)u = ud(u)$  it follows that  $d(uv)u = u(d(u)v + d(v)u)$ . In other words,  $d(u)(vu - uv) = 0$  for all  $u, v \in U$ . From this we obtain

$$0 = d(u)(vxu - uvx) = d(u)v(xu - ux)$$

for all  $u, v \in U$  and  $x \in R$ . Since  $R$  is a prime ring we conclude that  $U = Z(U) \cap K$  where  $Z(U)$  is the center of  $U$  and  $K = \{u \in U \mid d(u) = 0\}$ . If  $U = K$  then the primeness of  $R$  forces  $d = 0$ , a contradiction; hence  $U = Z(U)$  is commutative and, so,  $R$  is commutative.

We now prove the theorem stated above

*Proof of the Theorem.* To prove the theorem it is enough to show that if  $d \neq 0$ , then  $R$  is commutative. In fact, if this is the case, then  $nx^{n-1}d(x) = d(x^n) = 0$ ,  $n = n(x) \geq 1$ , for all  $x \in R$ . Since  $d \neq 0$  it follows that  $R$  is of characteristic  $p \neq 0$  (and  $p \nmid n(x)$  if  $d(x) \neq 0$ ); thus,  $d(x^p) = px^{p-1}d(x) = 0$  for all  $x \in R$ . If  $R$  is finite then  $R$  is a field and all its elements are  $p$ th powers, forcing  $d = 0$ ; hence  $R$  is infinite.

We also note that given  $x, y \in R$ , there exists  $k \geq 1$  such that  $d(x^k) = d(y^k) = 0$ . In fact it is enough to consider  $k = nm$  where  $d(x^n) = 0$  and  $d(y^m) = 0$ .

Henceforth we assume  $d \neq 0$ . Our object is to show that  $R$  is commutative.

Let  $J$  be the Jacobson radical of  $R$ . Suppose first that  $J \neq 0$ . We shall prove that  $d(x)x = xd(x)$ , for all  $x \in J$ , by Lemma 1 the

result will follow.

Let  $x \in J$  and  $y \in R$ ; let  $n \geq 1$  be such that

$$d((1+x)^{-1}y^n(1+x)) = d(y^n) = 0.$$

Then,

$$d((1+x)(1+x)^{-1}y^n(1+x)) = d(y^n + y^n x) = y^n d(x).$$

On the other hand,

$$\begin{aligned} d((1+x)(1+x)^{-1}y^n(1+x)) \\ &= d((1+x)^{-1}y^n(1+x) + x(1+x)^{-1}y^n(1+x)) \\ &= d(x)(1+x)^{-1}y^n(1+x). \end{aligned}$$

Therefore,

$$d(x)(1+x)^{-1}y^n(1+x) = y^n d(x).$$

Multiplying this last equality from the right by  $(1+x)^{-1}$ , we get

$$d(x)(1+x)^{-1}y^n = y^n d(x)(1+x)^{-1}.$$

Thus  $d(x)(1+x)^{-1}$  commutes with some power of every element in  $R$  and so  $d(x)(1+x)^{-1}$  is in the hypercenter of  $R$ . By [3], since  $R$  has no nil ideals, the hypercenter of  $R$  coincides with the center of  $R$ . Hence  $d(x)(1+x)^{-1}$  is central and so, on commuting it with  $x$ , we obtain  $d(x)x = xd(x)$ . This establishes the theorem when  $J \neq 0$ .

*Thus we may assume, henceforth, that  $R$  is a semisimple ring.*

We claim that  $R$  has no zero-divisors. In fact, let  $a \neq 0$  in  $R$  and let  $\lambda = \{y \in R/ya = 0\}$ . If  $y \in \lambda$  and  $x \in R$ , there exists  $n \geq 1$  such that

$$d((ax + axy)^n) = d((ax)^n) = 0.$$

Since  $ya = (axy)^2 = 0$  it follows that

$$(ax)^n d(y) = d((ax)^n y) = 0.$$

This says that  $d(y)$  annihilates on the right a suitable power of every element in the right ideal  $aR$ . By [1], since  $R$  is semisimple, we have  $aRd(y) = 0$ . Hence, since  $R$  is prime and  $a \neq 0$ , we conclude that  $d(y) = 0$ . In other words,  $d$  vanishes on  $\lambda$ , a left ideal of  $R$ . By the primeness of  $R$ , it is easy to check that this forces  $d = 0$ , unless  $\lambda = 0$ . Thus,  $R$  has no zero-divisors.

We go on with the final steps of the proof by showing that if  $R$  is a domain then  $R$  is commutative. As before it is enough to show that  $d(x)x = xd(x)$  for all  $x \in R$ .

Let  $x \neq 0$  in  $R$  and let  $A = C_R(x^n)$  be the centralizer of  $x^n$  in  $R$ , where  $n \geq 1$  is such that  $d(x^n) = 0$ . If  $a \in A$ , then

$$0 = d(ax^n - x^na) = d(a)x^n - x^nd(a) ;$$

that is,  $A$  is invariant under  $d$  and we may consider  $d$  as a derivation on  $A$ .

Now,  $A$  is a domain whose center,  $Z(A)$ , is nonzero for  $0 \neq x^n \in Z(A)$ . By localizing  $A$  at  $Z(A) \setminus \{0\}$  we obtain a domain  $Q \supset A$  whose center is a field containing  $x^n$ ; in particular,  $x$  is invertible in  $Q$ . As it is well known,  $d$  extends uniquely to a derivation on  $Q$  (which we shall also denote by  $d$ ) as follows:

$$d(az^{-1}) = d(a)z^{-1} + ad(z)z^{-2}, \quad a \in A, \quad z \in Z(A) \setminus \{0\}.$$

Moreover, by our basic hypothesis on  $d$ , we have that  $d(q^m) = 0$ ,  $m = m(q) \geq 1$ , for all  $q \in Q$ .

Let  $q \in Q$  and let  $m \geq 1$  be such that

$$d(q^m) = d(x^{-1}q^mx) = 0.$$

Then,

$$d(x)x^{-1}q^mx = d(x(x^{-1}q^mx)) = d(q^mx) = q^md(x).$$

Multiplying this equality from the right by  $x^{-1}$ , we obtain

$$d(x)x^{-1}q^m = q^md(x)x^{-1}.$$

In other words,  $d(x)x^{-1}$  lies in the hypercenter of  $Q$ . As before, by [3], it follows that  $d(x)x^{-1}$  lies in the center of  $Q$  and so, we conclude that  $d(x)x = xd(x)$ . This completes the proof of the theorem.

We finish with the following

**COROLLARY.** *Let  $R$  be a prime ring with no nonzero nil ideals. If  $d$  is a derivation of  $R$  such that  $d(u^n) = 0$ ,  $n = n(u) \geq 1$ , for all  $u \in U$ , where  $U$  is a nonzero ideal of  $R$ , then either  $d = 0$  or  $R$  is an infinite commutative domain of characteristic  $p \neq 0$  and  $p \mid n(u)$  if  $d(u) \neq 0$ .*

*Proof.* Suppose  $d \neq 0$ . Let

$$\delta(U) = \{u \in U \mid d^i(u) \in U, \text{ for all } i \geq 1\}.$$

Then,  $\delta(U)$  is an ideal of  $R$  invariant under  $d$ . Moreover, by hypothesis, some power of every element in  $U$  lies in  $\delta(U)$ . Since  $R$  has no nonzero nil ideals, we must have  $\delta(U) \neq 0$ .

Now, as an ideal of  $R$ ,  $\delta(U)$  is also a prime ring with no nonzero nil ideals. By the above theorem, the conclusion holds in  $\delta(U)$ . Since  $R$  is prime, the result follows.

## REFERENCES

1. B. Felzenszwalb, *On a result of Levitzki*, *Canad. Math. Bull.*, **21** (2) (1978), 241-242.
2. ———, *Derivations in prime rings*, *Proc. Amer. Math. Soc.*, **84** (1982), 16-20.
3. I. N. Herstein, *On the hypercenter of a ring*, *J. Algebra*, **36** (1975), 151-157.
4. E. C. Posner, *Derivations in prime rings*, *Proc. Amer. Math. Soc.*, **8** (1957), 1093-1100.

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