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Let A be a strongly logmodular subalgebra of $C(X)$, where X is a totally disconnected compact Hausdorff space. For s a weak peak set for A, define $A_S = \{f \in C(X): f|_S \in A|_S\}$. We prove the following:

THEOREM 1. Let s be a weak peak set for A. If b is an inner function such that $b|_S$ is invertible in $A|_S$ then there exists a function F in $A \cap C(X)^{-1}$ such that $F = \bar{b}$ on s.

THEOREM 2. Let s be a weak peak set for A. If $U \in C(X)$, $|U| = 1$ on s and $\operatorname{dist}(U, A_S) < 1$, then there exists a unimodular function $\tilde{U}$ in $C(X)$ such that $\tilde{U} = U$ on s and $\operatorname{dist}(\tilde{U}, A) < 1$.

1. Introduction. The purpose of this paper is to prove certain properties related to strongly logmodular algebras.

In their study of Local Toeplitz operators, Clancey and Gosselin [3] established one of these properties in a special case (H^∞) under a highly restrictive condition. In [7], the author proved this property for H^∞ without any condition.

In the present paper, we obtain this and another property for arbitrary strongly logmodular algebras. The proofs in [3] and [7] use special properties of H^∞ that are not shared by arbitrary strongly logmodular algebra. In the present work we use new techniques.

Let A be a strongly logmodular subalgebra of $C(X)$, where X is a totally disconnected compact Hausdorff space. If s is a weak peak set for A, define $A_S = \{f \in C(X): f|_S \in A|_S\}$. The main results of this work are: Theorem 3.2. Let s be a weak peak set for A, and let b be an inner function such that $b|_S$ is invertible in $A|_S$. Then there exists a function F in $A \cap C(X)^{-1}$ such that $F = \bar{b}$ on s.

THEOREM 3.1. *Let s be a weak peak set for A, and let u be in $C(X)$ such that $|u| = 1$ on s and* $\operatorname{dist}(u, A_S) < 1$. *There exists a unimodular function $\tilde{u}$ in $C(X)$ such that $\tilde{u} = u$ on s and* $\operatorname{dist}(\tilde{u}, A) < 1$.

2. Preliminaries. Let X be a compact Hausdorff space. We denote by $C(X)[C_R(X)]$ the space of continuous complex [real] valued functions on X. The algebra $C(X)$ is a Banach algebra under the supremum norm $\|f\|_\infty = \sup\{|f(x)|: x \in X\}$.

Let A be a function subalgebra of $C(X)$. A subset S of X is said to be a peak set for A if there exists f in A such that $f = 1$ on S and $|f| < 1$ off S. A set S is a weak peak set for A if S is an arbitrary intersection of peak sets for A. Let A^{-1} denote the group of invertible elements in A and $\log|A^{-1}| = \{\log|f|: f \in A^{-1}\}$.

A function algebra A is called a strongly logmodular subalgbra of $C(X)$ if $\log|A^{-1}|$ is equal to $C_R(X)$. The reader is referred to [2] and [4] for many of the basic properties of weak peak sets and additional properties of function algebra and to [5] and [1] for discussions concerning strongly logmodular algebras.

Let A denote a fixed closed subalgebra of $C(X)$ which contains the constants. Let B be a closed subalgebra of $C(X)$ which contains A. We define B_1 to be the closed subalgebra of $C(X)$ generated by A and $\{f^{-1}: f \in A \cap B^{-1}\}$. It is clear that $A \subset B_1 \subset B \subset C(X)$. If $B = B_1$, then B is called a Douglas algebra.

A function b in A is called an inner function if $|b| = 1$. For a strongly logmodular algebra A on X, there is a useful characterization of B_1 in [1, p. 8], which says that B_1 is the closed subalgebra generated by A and $\{\bar{b} \in B: b \text{ is an inner function}\}$.

3. The main result. Throughout this section, A will denote a fixed strongly logmodular algebra on X, where X is a compact, totally disconnected Hausdorff space. Examples of such algebras can be found in [5] and [6].

Let s be a subset of X which is a weak peak set for A. Define $A_S = \{f \in C(X): f|_S \in A|_S\}$. The algebra A_S is closed in $C(X)$ since $A|_S$ is closed in $C(X)|_S$. For u in $C(X)$, we define $\operatorname{dist}_S(u, A) = \inf\{\|u - h\|_S: h \in A\}$ and $\operatorname{dist}(u, A_S) = \inf\{\|u - h\|_\infty: h \in A_S\}$, where $\|u - h\|_S = \sup\{|u(x) - h(x)|: x \in S\}$. It is not difficult to see that $\operatorname{dist}(u, A_S) = \operatorname{dist}_S(u, A)$ for any u in $C(X)$.

Our main result is as follows:

THEOREM 3.1. *Let s be a weak peak set for A, and let u be in $C(X)$ such that $|u| = 1$ on s and* $\operatorname{dist}(u, A_S) < 1$. *Then there exists a unimodular function $\tilde{u}$ in $C(X)$ such that $\tilde{u} = u$ on s and* $\operatorname{dist}(\tilde{u}, A) < 1$.

In the special case of $A = H^\infty$ (the Hardy space of the unit circle) the above theorem appeared in [7] which answers a question raised in [3].

To prove Theorem 3.1, we need the following special case of [1, Theorem 4.1].

THEOREM A. *Let A be a strongly logmodular subalgebra of $C(X)$*

and J be an ideal in $C(X)$, where X is a totally disconnected compact Hausdorff space. Then the closure of $A+J$ is a Douglas algebra.

Theorem 3.1 follows from the following fact, which is interesting in its own right.

THEOREM 3.2. *Let s be a weak peak set for A, and let b be an inner function such that $b|_S$ is invertible in $A|_S$. Then there exists a function F in $A \cap C(X)^{-1}$ such that $F = \bar{b}$ on s.*

Proof. Step 1. There is a peak set $E \supset s$ such that $b|_E \in A_E^{-1}$. If not, there is a $\phi_E \in M(A_E)$ such that $\phi_E(b) = 0$. Since $M(A_E) \subset M(A)$, which is compact we can choose a convergent subnet $\phi'_E \to \phi$. Clearly $\phi \in M(A_S)$, and $\phi(b) = 0$ by continuity, contradicting $b|_S \in A_S^{-1}$.

Step 2. Let h peaks on s. Let $\phi \in M(A)$, $\phi(h) = 1$, and μ be the positive measure representing ϕ and supp μ be its support. Since $|h| \leqq 1$ and $\phi(h) = \int h d\mu = 1$, we have $h = 1$ on supp μ. Because $h = 1$ exactly on s, we have supp $\mu \subset s$. This shows that $\phi \in M(A_S)$. Since $b|_S \in A_S^{-1}$, $\phi(b) \neq 0$. Thus $1 - h$ and b have no common zeros on $M(A)$, and thus by [2, p. 27], there are $f, g \in A$ with $fb + g(1-h) = 1$.

Step 3. Fix $c > 2\|g\|_\infty$, where g is as in step (2). Let $E = \{x \in X: |1-h| < 1/6c\}$. There exists a clopen set W such that $s \subset W \subset E$. On the set $X \backslash W$ we have $|1-h| > \delta$, for some positive number δ. Let $g_1 = (c/2)\chi_W + (11/6 + c)(1/\delta)\chi_{X \backslash W}$. Certainly, $g_1 \in C(X)^{-1}$. Since A is strongly logmodular, there exists $G \in A^{-1}$ such that $\log|g_1| = \log|G|$. Thus $|G| = c/2$ on W and $|G| = (11/6 + c)(1/\delta)$ on $X \backslash W$.

From the identity $fb + g(1-h) = 1$, we have the following inequalities. On W: $|f| = |1 - g(1-h)| \geqq 1 - |g||1-h| \geqq 1 - c/2 \cdot 1/6c = 1 - 1/12 = 11/12$, and on X: $|f| \leqq 1 + |g||1-h| \leqq 1 + c/2 \cdot 2 = 1 + c$.

Let $F = f - G(1-h)$. Certainly, F is in A and $F = f = \bar{b}$ on s. Hence on W we have that

$$\begin{aligned} |F| &\geqq |f| - |G||1-h| \\ &\geqq 11/12 - c/2 \cdot 1/6c = 5/6 \end{aligned}$$

and

$$\begin{aligned} |F| &\geqq |G||1-h| - |f| \\ &\geqq (11/6 + c)(1/\delta) \cdot \delta - (1 + c) \\ &= 11/6 + c - 1 - c = 5/6 \quad \text{on} \quad X \backslash W . \end{aligned}$$

Thus $F \in A \cap C(X)^{-1}$. This ends the proof of the theorem.

Proof of Theorem 3.1. Without loss of generality we can assume that $|u| = 1$ on X. It is easy to see that $A_S = A + J$, where $J = \{f \in C(X): f(s) = 0\}$. Thus, by Theorem A, we have that A_S is a Douglas algebra. From the inequality, $\operatorname{dist}(u, A_S) < 1$, we have $\|u - g\bar{b}\|_\infty < 1$, for some g in A and some inner function b which is invertible in A_S. Consequently, $\operatorname{Re} \bar{u}\bar{b}g \geqq \delta_1 > 0$, for some positive number δ_1 ($\operatorname{Re} f$ denotes the real part of f). By Theorem 3.2, there exists F in $A \cap C(X)^{-1}$ such that $F = \bar{b}$ on s. Since $|F| \geqq \delta_2 > 0$, for some positive number δ_2, we have $\operatorname{Re} \bar{u}\bar{b}\bar{F}/|F|Fg = |F| \operatorname{Re} \bar{u}\bar{b}g \geqq \delta_1\delta_2 > 0$. Thus there exists a positive real number $R > 0$ such that $\|R - \bar{u}\bar{b}\bar{F}/|F|Fg\|_\infty < R$. Hence $\|1 - \bar{u}\bar{b}\bar{F}/|F|Fg/R\|_\infty < 1$. Set $\tilde{u} = ubF/|F|$; then $|\tilde{u}| = 1$, $\tilde{u} = u$ on s, and the last inequality shows that $\operatorname{dist}(\tilde{u}, A) < 1$. This ends the proof of the theorem.

The following corollary is a generalization of Theorem 3.2.

COROLLARY 3.3. *If s is a weak peak set for A and f in $C(X)$ such that $f|_S$ is invertible in $A|_S$, then there exists G in $A \cap C(X)^{-1}$ such that $G = f$ on s.*

Proof. The hypothesis that $f|_S$ is invertible in $A|_S$ shows that $f(x) \neq 0$ for all $x \in s$. Let W be a clopen set of X such that $f(x) \neq 0$ for all x in W. The function $f\chi_W + 1 - \chi_W \in C(X)^{-1}$, so we can write it in the form vg, where $v \in C(X)$, $|v| = 1$ and $g \in A^{-1}$. [This is possible because A is strongly logmodular]. Both the functions v and $\bar{v}$ are in A_S. By Theorem A there exists h in A and an inner function b which is invertible in A_S such that $\|v - h\bar{b}\|_\infty < 1$. Since $\bar{v}\bar{b} \in A_S$ and $\|1 - \bar{v}\bar{b}h\|_\infty < 1$, then by [2, p. 49] we have $\bar{v}\bar{b}h = e^{u_1}$ for some u_1 in A_S. By the definition of A_S, there exists u in A such that $u = u_1$ on s. Thus $v = \bar{b}he^{-u}$ on s. By Theorem 3.2 there exists $F = \bar{b}$ on s. Set $G = Fhe^{-u}g$, then G is the required function. This completes the proof of the corollary.

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