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**INTERPOLATION IN STRONGLY LOGMODULAR ALGEBRAS**

RAHMAN MAHMOUD YOUNIS

## INTERPOLATION IN STRONGLY LOGMODULAR ALGEBRAS

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Let  $A$  be a strongly logmodular subalgebra of  $C(X)$ , where  $X$  is a totally disconnected compact Hausdorff space. For  $s$  a weak peak set for  $A$ , define  $A_s = \{f \in C(X): f|_s \in A|_s\}$ . We prove the following:

**THEOREM 1.** Let  $s$  be a weak peak set for  $A$ . If  $b$  is an inner function such that  $b|_s$  is invertible in  $A|_s$  then there exists a function  $F$  in  $A \cap C(X)^{-1}$  such that  $F = \bar{b}$  on  $s$ .

**THEOREM 2.** Let  $s$  be a weak peak set for  $A$ . If  $U \in C(X)$ ,  $|U| = 1$  on  $s$  and  $\text{dist}(U, A_s) < 1$ , then there exists a unimodular function  $\tilde{U}$  in  $C(X)$  such that  $\tilde{U} = U$  on  $s$  and  $\text{dist}(\tilde{U}, A) < 1$ .

1. **Introduction.** The purpose of this paper is to prove certain properties related to strongly logmodular algebras.

In their study of Local Toeplitz operators, Clancey and Gosselin [3] established one of these properties in a special case ( $H^\infty$ ) under a highly restrictive condition. In [7], the author proved this property for  $H^\infty$  without any condition.

In the present paper, we obtain this and another property for arbitrary strongly logmodular algebras. The proofs in [3] and [7] use special properties of  $H^\infty$  that are not shared by arbitrary strongly logmodular algebra. In the present work we use new techniques.

Let  $A$  be a strongly logmodular subalgebra of  $C(X)$ , where  $X$  is a totally disconnected compact Hausdorff space. If  $s$  is a weak peak set for  $A$ , define  $A_s = \{f \in C(X): f|_s \in A|_s\}$ . The main results of this work are: Theorem 3.2. Let  $s$  be a weak peak set for  $A$ , and let  $b$  be an inner function such that  $b|_s$  is invertible in  $A|_s$ . Then there exists a function  $F$  in  $A \cap C(X)^{-1}$  such that  $F = \bar{b}$  on  $s$ .

**THEOREM 3.1.** Let  $s$  be a weak peak set for  $A$ , and let  $u$  be in  $C(X)$  such that  $|u| = 1$  on  $s$  and  $\text{dist}(u, A_s) < 1$ . There exists a unimodular function  $\tilde{u}$  in  $C(X)$  such that  $\tilde{u} = u$  on  $s$  and  $\text{dist}(\tilde{u}, A) < 1$ .

2. **Preliminaries.** Let  $X$  be a compact Hausdorff space. We denote by  $C(X)[C_{\mathbb{R}}(X)]$  the space of continuous complex [real] valued functions on  $X$ . The algebra  $C(X)$  is a Banach algebra under the supremum norm  $\|f\|_\infty = \sup\{|f(x)|: x \in X\}$ .

Let  $A$  be a function subalgebra of  $C(X)$ . A subset  $S$  of  $X$  is

said to be a peak set for  $A$  if there exists  $f$  in  $A$  such that  $f = 1$  on  $S$  and  $|f| < 1$  off  $S$ . A set  $S$  is a weak peak set for  $A$  if  $S$  is an arbitrary intersection of peak sets for  $A$ . Let  $A^{-1}$  denote the group of invertible elements in  $A$  and  $\log |A^{-1}| = \{\log |f| : f \in A^{-1}\}$ .

A function algebra  $A$  is called a strongly logmodular subalgebra of  $C(X)$  if  $\log |A^{-1}|$  is equal to  $C_{\mathbb{R}}(X)$ . The reader is referred to [2] and [4] for many of the basic properties of weak peak sets and additional properties of function algebra and to [5] and [1] for discussions concerning strongly logmodular algebras.

Let  $A$  denote a fixed closed subalgebra of  $C(X)$  which contains the constants. Let  $B$  be a closed subalgebra of  $C(X)$  which contains  $A$ . We define  $B_1$  to be the closed subalgebra of  $C(X)$  generated by  $A$  and  $\{f^{-1} : f \in A \cap B^{-1}\}$ . It is clear that  $A \subset B_1 \subset B \subset C(X)$ . If  $B = B_1$ , then  $B$  is called a Douglas algebra.

A function  $b$  in  $A$  is called an inner function if  $|b| = 1$ . For a strongly logmodular algebra  $A$  on  $X$ , there is a useful characterization of  $B_1$  in [1, p. 8], which says that  $B_1$  is the closed subalgebra generated by  $A$  and  $\{\bar{b} \in B : b \text{ is an inner function}\}$ .

**3. The main result.** Throughout this section,  $A$  will denote a fixed strongly logmodular algebra on  $X$ , where  $X$  is a compact, totally disconnected Hausdorff space. Examples of such algebras can be found in [5] and [6].

Let  $s$  be a subset of  $X$  which is a weak peak set for  $A$ . Define  $A_s = \{f \in C(X) : f|_s \in A|_s\}$ . The algebra  $A_s$  is closed in  $C(X)$  since  $A|_s$  is closed in  $C(X)|_s$ . For  $u$  in  $C(X)$ , we define  $\text{dist}_s(u, A) = \inf \{\|u - h\|_s : h \in A\}$  and  $\text{dist}(u, A_s) = \inf \{\|u - h\|_\infty : h \in A_s\}$ , where  $\|u - h\|_s = \sup \{|u(x) - h(x)| : x \in s\}$ . It is not difficult to see that  $\text{dist}(u, A_s) = \text{dist}_s(u, A)$  for any  $u$  in  $C(X)$ .

Our main result is as follows:

**THEOREM 3.1.** *Let  $s$  be a weak peak set for  $A$ , and let  $u$  be in  $C(X)$  such that  $|u| = 1$  on  $s$  and  $\text{dist}(u, A_s) < 1$ . Then there exists a unimodular function  $\tilde{u}$  in  $C(X)$  such that  $\tilde{u} = u$  on  $s$  and  $\text{dist}(\tilde{u}, A) < 1$ .*

In the special case of  $A = H^\infty$  (the Hardy space of the unit circle) the above theorem appeared in [7] which answers a question raised in [3].

To prove Theorem 3.1, we need the following special case of [1, Theorem 4.1].

**THEOREM A.** *Let  $A$  be a strongly logmodular subalgebra of  $C(X)$*

and  $J$  be an ideal in  $C(X)$ , where  $X$  is a totally disconnected compact Hausdorff space. Then the closure of  $A + J$  is a Douglas algebra.

Theorem 3.1 follows from the following fact, which is interesting in its own right.

**THEOREM 3.2.** *Let  $s$  be a weak peak set for  $A$ , and let  $b$  be an inner function such that  $b|_s$  is invertible in  $A|_s$ . Then there exists a function  $F$  in  $A \cap C(X)^{-1}$  such that  $F = \bar{b}$  on  $s$ .*

*Proof.* *Step 1.* There is a peak set  $E \supset s$  such that  $b|_E \in A_E^{-1}$ . If not, there is a  $\phi_E \in M(A_E)$  such that  $\phi_E(b) = 0$ . Since  $M(A_E) \subset M(A)$ , which is compact we can choose a convergent subnet  $\phi'_E \rightarrow \phi$ . Clearly  $\phi \in M(A_s)$ , and  $\phi(b) = 0$  by continuity, contradicting  $b|_s \in A_s^{-1}$ .

*Step 2.* Let  $h$  peaks on  $s$ . Let  $\phi \in M(A)$ ,  $\phi(h) = 1$ , and  $\mu$  be the positive measure representing  $\phi$  and  $\text{supp } \mu$  be its support. Since  $|h| \leq 1$  and  $\phi(h) = \int h d\mu = 1$ , we have  $h = 1$  on  $\text{supp } \mu$ . Because  $h = 1$  exactly on  $s$ , we have  $\text{supp } \mu \subset s$ . This shows that  $\phi \in M(A_s)$ . Since  $b|_s \in A_s^{-1}$ ,  $\phi(b) \neq 0$ . Thus  $1 - h$  and  $b$  have no common zeros on  $M(A)$ , and thus by [2, p. 27], there are  $f, g \in A$  with  $fb + g(1 - h) = 1$ .

*Step 3.* Fix  $c > 2\|g\|_\infty$ , where  $g$  is as in step (2). Let  $E = \{x \in X: |1 - h| < 1/6c\}$ . There exists a clopen set  $W$  such that  $s \subset W \subset E$ . On the set  $X \setminus W$  we have  $|1 - h| > \delta$ , for some positive number  $\delta$ . Let  $g_1 = (c/2)\chi_W + (11/6 + c)(1/\delta)\chi_{X \setminus W}$ . Certainly,  $g_1 \in C(X)^{-1}$ . Since  $A$  is strongly logmodular, there exists  $G \in A^{-1}$  such that  $\log |g_1| = \log |G|$ . Thus  $|G| = c/2$  on  $W$  and  $|G| = (11/6 + c)(1/\delta)$  on  $X \setminus W$ .

From the identity  $fb + g(1 - h) = 1$ , we have the following inequalities. On  $W$ :  $|f| = |1 - g(1 - h)| \geq 1 - |g||1 - h| \geq 1 - c/2 \cdot 1/6c = 1 - 1/12 = 11/12$ , and on  $X$ :  $|f| \leq 1 + |g||1 - h| \leq 1 + c/2 \cdot 2 = 1 + c$ .

Let  $F = f - G(1 - h)$ . Certainly,  $F$  is in  $A$  and  $F = \bar{b}$  on  $s$ . Hence on  $W$  we have that

$$\begin{aligned} |F| &\geq |f| - |G||1 - h| \\ &\geq 11/12 - c/2 \cdot 1/6c = 5/6 \end{aligned}$$

and

$$\begin{aligned} |F| &\geq |G||1 - h| - |f| \\ &\geq (11/6 + c)(1/\delta) \cdot \delta - (1 + c) \\ &= 11/6 + c - 1 - c = 5/6 \quad \text{on } X \setminus W. \end{aligned}$$

Thus  $F \in A \cap C(X)^{-1}$ . This ends the proof of the theorem.

*Proof of Theorem 3.1.* Without loss of generality we can assume that  $|u| = 1$  on  $X$ . It is easy to see that  $A_s = A + J$ , where  $J = \{f \in C(X): f(s) = 0\}$ . Thus, by Theorem A, we have that  $A_s$  is a Douglas algebra. From the inequality,  $\text{dist}(u, A_s) < 1$ , we have  $\|u - g\bar{b}\|_\infty < 1$ , for some  $g$  in  $A$  and some inner function  $b$  which is invertible in  $A_s$ . Consequently,  $\text{Re } \bar{u}b\bar{g} \geq \delta_1 > 0$ , for some positive number  $\delta_1$  ( $\text{Re } f$  denotes the real part of  $f$ ). By Theorem 3.2, there exists  $F$  in  $A \cap C(X)^{-1}$  such that  $F = \bar{b}$  on  $s$ . Since  $|F| \geq \delta_2 > 0$ , for some positive number  $\delta_2$ , we have  $\text{Re } \bar{u}\bar{b}\bar{F}/|F|Fg = |F| \text{Re } \bar{u}\bar{b}g \geq \delta_1\delta_2 > 0$ . Thus there exists a positive real number  $R > 0$  such that  $\|R - \bar{u}\bar{b}\bar{F}/|F|Fg\|_\infty < R$ . Hence  $\|1 - \bar{u}\bar{b}\bar{F}/|F|Fg/R\|_\infty < 1$ . Set  $\tilde{u} = \bar{u}\bar{b}\bar{F}/|F|Fg$ ; then  $|\tilde{u}| = 1$ ,  $\tilde{u} = u$  on  $s$ , and the last inequality shows that  $\text{dist}(\tilde{u}, A) < 1$ . This ends the proof of the theorem.

The following corollary is a generalization of Theorem 3.2.

**COROLLARY 3.3.** *If  $s$  is a weak peak set for  $A$  and  $f$  in  $C(X)$  such that  $f|_s$  is invertible in  $A|_s$ , then there exists  $G$  in  $A \cap C(X)^{-1}$  such that  $G = f$  on  $s$ .*

*Proof.* The hypothesis that  $f|_s$  is invertible in  $A|_s$  shows that  $f(x) \neq 0$  for all  $x \in s$ . Let  $W$  be a clopen set of  $X$  such that  $f(x) \neq 0$  for all  $x$  in  $W$ . The function  $f\chi_w + 1 - \chi_w \in C(X)^{-1}$ , so we can write it in the form  $vg$ , where  $v \in C(X)$ ,  $|v| = 1$  and  $g \in A^{-1}$ . [This is possible because  $A$  is strongly logmodular]. Both the functions  $v$  and  $\bar{v}$  are in  $A_s$ . By Theorem A there exists  $h$  in  $A$  and an inner function  $b$  which is invertible in  $A_s$  such that  $\|v - h\bar{b}\|_\infty < 1$ . Since  $\bar{v}\bar{b} \in A_s$  and  $\|1 - \bar{v}\bar{b}h\|_\infty < 1$ , then by [2, p. 49] we have  $\bar{v}\bar{b}h = e^{u_1}$  for some  $u_1$  in  $A_s$ . By the definition of  $A_s$ , there exists  $u$  in  $A$  such that  $u = u_1$  on  $s$ . Thus  $v = \bar{b}he^{-u}$  on  $s$ . By Theorem 3.2 there exists  $F = \bar{b}$  on  $s$ . Set  $G = Fhe^{-u}g$ , then  $G$  is the required function. This completes the proof of the corollary.

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