Pacific Journal of Mathematics

SPACES OF WEAKLY CONTINUOUS FUNCTIONS

Juan Ferrera

Vol. 102, No. 2 February 1982

PACIFIC JOURNAL OF MATHEMATICS
Vol. 102, No. 2, 1982

SPACES OF WEAKLY CONTINUOUS FUNCTIONS

JUAN FERRERA

In this paper we study some properties about the space of weakly continuous functions on bounded sets of a Banach space E: $C_{wb}(E)$. We study the relation between $C_{wb}(E)$ and $C_{wbu}(E)$ (weakly uniformly continuous functions on bounded sets). And we give the following characterization: $C_{wbu}(E)$ is a barreled space if and only if E is reflexive.

0. **Notation and preliminaries.** Throughout this paper E will represent a real Banach space and B_n the closed ball of radius n. The basic definitions of locally convex spaces and their properties are explained in [6]. We will say that a Banach space is weakly compactly generated (WCG), when it has a weakly compact total subset. Both separable and reflexive spaces are particular cases of WCG spaces. For further information, see [2].

For the topological concepts that are used, we will follow [5]. We will say that a completely regular topological space is realcompact when each z-ultrafilter with the countable intersection property has an nonempty intersection. A subset of a topological space will be relatively pseudocompact when every real-valued continuous function defined on the space is bounded on the subset.

We will define the bw-topology on E as the finest which agrees with the weak topology on bounded subsets of E. A subset will be bw-closed (respectively bw-open) if and only if it is weakly closed (respectively relatively open) when it is restricted to each B_n.

If X is a topological space, $C(X)$ will represent the space of real-valued continuous functions on X. Except for when indicating the opposite, we will give $C(X)$ the compact-open topology, defined by the family of semi-norms

$$P_K(f) = \operatorname*{Sup}_{x \in K} |f(x)|$$

when K ranges over the compact subsets of X. $C_{wb}(E)$ will represent the space of real functions which are weakly continuous when restricted to the bounded subsets of E. $C_{wbu}(E)$ will be the space of real functions which are weakly uniformly continuous when restricted to the bounded subsets of E.

$C_{wbu}(E) \subset C_{wb}(E)$, if we give the topology of the uniform convergence on weakly compact subsets to both, we will have that $C_{wbu}(E)$ is a subspace of the locally convex space $C_{wb}(E)$.

1. **The space $C_{wb}(E)$.** The space E endowed with the bw-topo-

logy will be represent by X. It is evident that $C_{wb}(E)$ coincides with $C(X)$ as sets. On the other hand, the weak compacts of E and the compacts of X are the same (by being bounded). Therefore, both spaces are topologically isomorphic.

We are concerned with studying the properties of $C_{wb}(E)$, for which we need the following lemma:

LEMMA 1.1. *If E is a weakly normal space, then X is normal and hence completely regular.*

Proof. If E is a weakly normal space, then for every n, B_n endowed with the weak restricted topology is normal.

Let C and F be closed subsets of X, $C \cap F = \varnothing$.

$C_n = C \cap B_n$, $F_n = F \cap B_n$. C_1 and F_1 are weakly closed. By Urysohn's lemma, we have $f_1\colon B_1 \to [0, 1]$ weakly continuous function such that:

$$f_1(C_1) = \{0\} \quad \text{and} \quad f_1(F_1) = \{1\}$$

will be $f_2^*\colon B_1 \cup C_2 \cup F_2 \to [0, 1]$ defined by

$$f_2^*|_{B_1} = f_1,\ f_2^*(C_2) = \{0\},\ f_2^*(F_2) = \{1\} .$$

This function is weakly continuous and it is defined on a weakly closed subset of B_2, therefore, by Tietze's theorem, it can be extended to another function $f_2\colon B_2 \to [0, 1]$ weakly continuous and such that

$$f_2(C_2) = \{0\} \quad \text{and} \quad f_2(F_2) = \{1\} .$$

We define by induction $f_n\colon B_n \to [0, 1]$ weakly continuous such that:

$$f_n(C_n) = \{0\},\ f_n(F_n) = \{1\},\ \text{and}\ f_n|_{B_{n-1}} = f_{n-1} .$$

We define $f(x) = f_n(x)$ if $x \in B_n$.

We have that f is continuous on X, $f(C) = \{0\}$ and $f(F) = \{1\}$. Hence X is normal.

Unfortunately we have not a general result, eliminating the hypotheses of weak normality, which affirms that X is always a completely regular; which is necessary for the study of $C(X)$. Nevertheless, if E endowed with the coarser topology that makes the functions of $C_{wb}(E)$ continuous, is represented by $\tilde{X}$, we achieve that the above space is completely regular, and we have that the following inclusions are continuous:

$$X \longrightarrow \tilde{X} \longrightarrow (E, \sigma(E, E'))$$

giving equality to the first inclusion if and only if X is completely regular.

We proceed to study the properties of $C_{wb}(E)$. In the first place we will see when it is a bornological space. According to Nachbin-Shirota's theorem [7.8], this would be equivalent to X being realcompact. We have the following statement:

THEOREM 1.2. *If E is weakly normal, then $C_{wb}(E)$ is bornological if and only if E is weakly realcompact.*

Proof. Simply by noting the fact that the B_n balls with weak restricted topology are realcompacts, it follows that X (respectively E) is realcompact (respectively weakly realcompact), we will do the proof for E, but with light modifications serving for X.

Let $\{U_\alpha\}_{\alpha\in A}$ be a z-ultrafilter. Each $U_\alpha = f_\alpha^{-1}(0)$ with $f_\alpha\colon E\to R$ weakly continuous.

We have that for every index sequence $(\alpha_n)\in A$, $\bigcap_{n=1}^\infty U_{\alpha_n} \neq \varnothing$.

(1) There is n_0 such that $U_\alpha \cap B_{n_0} \neq \varnothing$ for every $\alpha\in A$.

If it were not like this, for each $n\in N$ we would have $\alpha_n \in A$ such that $U_{\alpha_n}\cap B_n = \varnothing$. With which we would have that $\bigcap_{n=1}^\infty U_{\alpha_n} = \varnothing$, failing the countable intersection property.

Therefore, for every $n \geqq n_0$ $\{U_\alpha \cap B_n\}_{\alpha\in A}$ is a filter basis in B_n.

(2) There exists $n_1 \geqq n_0$ in such a way that the filter basis $\{U_\alpha \cap B_{n_1}\}_{\alpha\in A}$ has the countable intersection property.

Supposing the above fails: for every $n \geqq n_0$ there would be $\{\alpha_{n,m}\}_{m\in N}$ index sequence in such a way that $\bigcap_{m=1}^\infty (U_{\alpha_{n,m}} \cap B_n) = \varnothing$. The countable family $\{U_{\alpha_{n,m}}\}_{n,m\in N, n\geqq n_0}$ has an empty intersection contrary to the countable intersection property.

(3) $\{U_\alpha \cap B_{n_1}\}_{\alpha\in A}$ is the basis of a z-filter with the countable intersection property in B_{n_1}, because $f_\alpha|_{B n_1}$ is a continuous function on B_{n_1} endowed with the weak topology restricted, for every $\alpha\in A$.

(4) $\{U_\alpha \cap B_{n_1}\}_{\alpha\in A}$ is a basis for a z-ultrafilter.

If not, it would be $Z\in B_{n_1}$ zero in B_{n_1}, that is, $Z = f^{-1}(0)$ with f weakly continuous on B_{n_1}, in such a way that $Z\cap U_\alpha \cap B_{n_1} \neq \varnothing$ for every $\alpha\in A$, but Z not containing any $U_\alpha \cap B_{n_1}$. But since B_{n_1} is a weakly closed subset of E, by normality there exists $\tilde f\colon E\to R$ weakly continuous, such that $\tilde f|_{B n_1} = f$. Furthermore, $\tilde f^{-1}(0) = \tilde Z$ will be a zero of E (with the weak topology), and $Z = \tilde Z \cap B_{n_1}$. But $\tilde Z \cap U_\alpha \neq \varnothing$ for every $\alpha\in A$, hence $\tilde Z = U_{\alpha_0}$ for some α_0 by being z-ultrafilter, then $Z = U_{\alpha_0} \cap B_{n_1}$.

Just as by hypothesis B_{n_1} is realcompact, it follows that $\bigcap_{\alpha\in A} U_\alpha \cap B_{n_1} \neq \varnothing$ and thus $\bigcap_{\alpha\in A} U_\alpha \neq \varnothing$. Hence E is weakly realcompact.

Since $\tilde f$ is also continuous on X, it can also be inferred that X

is realcompact.

We have then that $\tilde{X} = X$ is realcompact if and only if E is weakly realcompact, and that $C_{wb}(E)$ is bornological if and only if E is weakly realcompact.

Weakly realcompact Banach spaces are described in [1]. Nevertheless, there exists weakly realcompact spaces which are not weakly normal, as in the case of l^∞ [1, p. 12]. In any case, the class of weakly normal and weakly realcompact spaces is wide; particularly every *WCG* space is weakly Lindelöf [9] and thus is in our hypotheses.

The following statement gives a partial answer to the problem for the case of not necessarily normal spaces.

THEOREM 1.3. *If E is the dual of a separable space, then $C_{wb}(E)$ is bornological; in particular $C_{wb}(l^\infty)$ is bornological.*

Proof. Let $E = F'$ be. $\{x_n\}_{n\in N}$ a dense subset of F. We define:

$$\begin{aligned} f\colon \tilde{X} &\longrightarrow R^N \\ x' &\longrightarrow ((x_n, x'))_n . \end{aligned}$$

This map is one to one because $\{x_n\}_{n\in N}$ separates points of E by being dense in F. Furthermore, f is continuous by being $f = \tilde{f}\circ i$, being $i\colon \tilde{X}\to E$ continuous and $\tilde{f}\colon E\to R^N$ continuous, given that $\tilde{f}$ composed with $p_n\colon R^N\to R$ is x_n. Since R^N is realcompact and all its subsets are as well (by the points being G_δ-sets), we have, because of [4], that $\tilde{X}$ is realcompact and $C_{wb}(E)$ is bornological.

Finally we are going to see that $\tilde{X}$ is a *NS*-space. That is, every relatively pseudocompact and closed subset is compact. Through [7] we achieve that $C_{wb}(E)$ is always barreled.

PROPOSITION 1.4. *$C_{wb}(E)$ is barreled.*

Proof. Let $K\subset\tilde{X}$ be a relatively pseudocompact and closed subset. For all $x'\in E'$, $x'(K)$ is bounded, hence K is weakly bounded and thus bounded. Furthermore, $K\subset B_n$ for some n, from where it follows that K is weakly closed. As on the other hand each weakly continuous function on E is continuous on $\tilde{X}$, we have that K is weakly relatively pseudocompact. Also through [10] we achieve that K is weakly compact and thus compact in $\tilde{X}$.

2. The space $C_{wbu}(E)$. First of all, we will study the relationship between $C_{wbu}(E)$ and $C_{wb}(E)$.

PROPOSITION 2.1. *$C_{wbu}(E)$ is a dense subspace of $C_{wb}(E)$.*

Proof. Let f be a function of $C_{wb}(E)$. For every weakly compact subset K of E, $f|_K$ will be weakly continuous. Let $\tilde{f}_K$ be an extension to the Stone-Cech compactification of E endowed with the weak topology, $\beta(E)$, which will be uniformly continuous. Then $f_K = \tilde{f}_K|_E$ is weakly uniformly continuous on E. Then $f_K \in C_{wbu}(E)$ for every weakly compact subset K.

Obviously, $\{f_K\}_K$ is a net that converges uniformly on weakly compact subsets of E, to f.

PROPOSITION 2.2. *$C_{wbu}(E) = C_{wb}(E)$ if and only if E is reflexive.*

Proof. If E is reflexive, the equality holds because the balls are weakly compacts.

Conversely given f weakly continuous on E, we do have that $f \in C_{wb}(E) = C_{wbu}(E)$. Then f is weakly uniformly continuous on B_1 and consequently it is bounded on B_1, because it is totally bounded. Since f is weakly uniformly continuous on B_1, it follows that there exists a weak neighborhood of zero, V; such that $|f(x) - f(y)| < 1$ provided that $x - y \in V$ and $x, y \in B_1$. Since B_1 is weakly totally bounded, we infer that there exists $x_1, \cdots, x_{n_0} \in B_1$ such that

$$B_1 \subset \bigcup_{i=1}^{n_0} \{x_i + V\} .$$

Thus for every $x \in B_1$

$$|f(x)| \leqq \operatorname*{Max}_{i=1,\cdots,n_0} \{|f(x_i)| + 1\} .$$

This means that every weakly continuous function over E is bounded on B_1, hence B_1 is weakly relatively pseudocompact, and weakly closed. By [10] it follows that B_1 is weakly compact and thus E reflexive.

The proof of this proposition suggest that, if E is not reflexive, one method to find a function which belongs to $C_{wb}(E)$ and does not belong to $C_{wbu}(E)$, it would be to find a weakly continuous fuuction over E which is not bounded on B_1.

EXAMPLE 2.3. If E is a nonreflexive separable space, the James-Klee theorem [2, p. 7] states there exists $\phi \in E'$ which does not attain its norm.

We define the function

$$f_0 \colon B_1 \longrightarrow R \text{ by } f_0(x) = \frac{1}{\|\phi\| - \phi(x)} .$$

This function is weakly continuous on B_1, and is not bounded. Since E is a separable space, then it is WCG and therefore weakly normal.

Thus by Tietze's theorem there exists f weakly continuous on E, which extends f_0, and which is not bounded on the unit ball; therefore it can not belong to $C_{wbu}(E)$.

COROLLARY 2.4. *$C_{wbu}(E)$ is complete if and only if E is reflexive.*

Proof. If E is reflexive, $C_{wb}(E) = C_{wbu}(E)$ and also $C_{wb}(E)$ is complete by [3]. Thus $C_{wbu}(E)$ is complete.

Conversely, since $C_{wbu}(E)$ is dense in $C_{wb}(E)$, if it is complete, both spaces have to be the same and because of that it is reflexive.

THEOREM 2.5. *$C_{wbu}(E)$ is barrelled if and only if E is reflexive.*

Proof. If E is reflexive $C_{wbu}(E) = C_{wb}(E)$ and consequently barrelled. Conversely we consider the following diagram:

$$\begin{array}{ccccccc} B_1'' & \longrightarrow & B_2'' & \longrightarrow & \cdots & \longrightarrow & X'' \\ \uparrow{\scriptstyle i_1} & & \uparrow{\scriptstyle i_2} & & & & \uparrow{\scriptstyle i} \\ B_1 & \longrightarrow & B_2 & \longrightarrow & \cdots & \longrightarrow & X\,. \end{array}$$

B_n'' are endowed with the weak star topology restricted. X'' will be the inductive limit of the spaces B_n''.

i is continuous because when composed with the inclusions $j_n\colon B_n \to X$ it follows that $i \circ j_n = j_n^* \circ i_n$, been $j_n^*\colon B_n'' \to X''$ the canonical inclusion in the inductive limit; obviously $j_n^* \circ i_n$ is continuous because i_n is also continuous. i is one to one and $i(X)$ is dense in X''.

Let us consider the map restriction $\phi\colon C(X'') \to C(X)$ $f \to f \circ i$.

(1) We have that $\phi(C(X'')) = C_{wbu}(E)$. Let us see it.

If $f \in C_{wbu}(E)$, it follows that $f_n = f|_{B_n}\colon B_n \to R$ is weakly uniformly continuous. Then, by density, it can be extended to $\tilde{f}_n\colon B_n'' \to R$ uniformly continuous, on the other hand,

$$\tilde{f}_n|_{B_{n-1}''} = \tilde{f}_{n-1} \text{ because } (\tilde{f}_n|_{B_{n-1}''})|_{B_{n-1}} = (\tilde{f}_n|_{B_n})|_{B_{n-1}} = f_n|_{B_{n-1}} = f_{n-1}$$

and

$$\tilde{f}_{n-1}|_{B_{n-1}} = f_{n-1}\,.$$

Then both functions are exactly the same over a dense part of B_{n-1}'' thus they are the same.

It can be defined $\tilde{f}\colon X'' \to R$ continuous by $\tilde{f}(x) = f_n(x)$ if $x \in B_n''$. Obviously $\tilde{f}|_X = f$, thus $f \in \phi(X''))$.

Conversely if $f \in \phi(C(X''))$ it follows that there exists $\tilde{f} \in C(X'')$ such that $f = \tilde{f}|_X$; but $\tilde{f}_n = \tilde{f}|_{B_n'}$ is continuous on B_n'' which is compact. Then $\tilde{f}_n$ is uniformly continuous. Therefore $f_n = f|_{B_n}$ is equal to $\tilde{f}_n|_{B_n}$ and because of that, weakly uniformly continuous.

(2) ϕ is linear.

(3) ϕ is one to one because $i(X)$ is dense in X''.

(4) ϕ is continuous because the continuity of $i: X \to X''$ implies that the compacts of X are compacts of X''.

On the other hand, the space X'' is a countable union of compacts and therefore the topology of $C(X'')$ is given by a countable family of semi-norms. Thus $C(X'')$ is metrizable. Because of the definition, X'' is a k-space and therefore $C(X'')$ is complete [11]; then $C(X'')$ is a Frechet space.

Since $C_{wbu}(E)$ is barreled it follows that f is a topological isomorphism, applying the Open Mapping Theorem. Then it can be inferred that $C_{wbu}(E)$ is complete and therefore E is reflexive.

COROLLARY 2.6. *If E is a Banach space, the following are equivalent:*

(i) *E is reflexive*

(ii) *$C_{wbu}(E)$ is a Frechet space*

(iii) *$C_{wbu}(E)$ is a Ptak space*

(iv) *$C_{wbu}(E)$ is complete*

(v) *$C_{wbu}(E)$ is barreled*

(vi) *$C_{wbu}(E) = C_{wb}(E)$.*

REFERENCES

1. H. H. Corson, *The weak topology of a Banach space*, Trans. Amer. Math. Soc , **101** (1961), 1-15.

2. J. Diestel, *Geometry of Banach Spaces*, Selected Topics. Lect. Notes in Math., 485 Springer-Verlag, New York, 1975.

3. J. Ferrera J. Gomez and J. L. G. Llavona, *On completion of spaces of weakly continuous, functions*, to appear in J. London Math. Soc.

4. L. Gilman and M. Jerison, *Rings of Continuous Functions*, Van Norstrand, 1960.

5. J. Kelley, *General Topology*, Springer.

6. G. Köthe, *Topological Vector Spaces I*, Springer-Verlag, New York, 1969.

7. L. Nachbin, *Topological vector spaces of continuous functions*, Proc. Nat. Acad. Sci. U.S.A., **40** (1954), 471-474.

8. T. Shirota, *On locally convex vector spaces of continuous functions*, Proc. Japan Acad., **30** (1954), 294-298.

9. M. Talagrand, *Sur une conjecture de H. H. Corson*, Bull. Sci. Math., (2), **99** (1975), 211-212.

10. M. Valdivia, *Some new results on weak compactness*, J. Functional Anal., **24** (1977), 1-10.

11. S. Warner, *The topology of compact convergence on continuous function spaces*, Duke Math. J., **25**, 265-282.

Received June 17, 1981.

UNIVERSIDAD COMPLUTENSE
CIUDAD UNIVERSITARIA
MADRID-3, SPAIN

Mathematical parers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd unmbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966, Regular subscription rate: $114.00 a year (6 Vol., 12 issues). Special rate: $57.00 a year to individual members of supporting institution.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address shoud be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 102, No. 2 February, 1982