SPACES OF WEAKLY CONTINUOUS FUNCTIONS

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In this paper we study some properties about the space of weakly continuous functions on bounded sets of a Banach space $E$: $C_{wb}(E)$. We study the relation between $C_{wb}(E)$ and $C_{wbu}(E)$ (weakly uniformly continuous functions on bounded sets). And we give the following characterization: $C_{wbu}(E)$ is a barreled space if and only if $E$ is reflexive.

0. Notation and preliminaries. Throughout this paper $E$ will represent a real Banach space and $B_n$ the closed ball of radius $n$. The basic definitions of locally convex spaces and their properties are explained in [6]. We will say that a Banach space is weakly compactly generated (WCG), when it has a weakly compact total subset. Both separable and reflexive spaces are particular cases of WCG spaces. For further information, see [2].

For the topological concepts that are used, we will follow [5]. We will say that a completely regular topological space is realcompact when each $\varepsilon$-ultrafilter with the countable intersection property has an nonempty intersection. A subset of a topological space will be relatively pseudocompact when every real-valued continuous function defined on the space is bounded on the subset.

We will define the $bw$-topology on $E$ as the finest which agrees with the weak topology on bounded subsets of $E$. A subset will be $bw$-closed (respectively $bw$-open) if and only if it is weakly closed (respectively relatively open) when it is restricted to each $B_n$.

If $X$ is a topological space, $C(X)$ will represent the space of real-valued continuous functions on $X$. Except for when indicating the opposite, we will give $C(X)$ the compact-open topology, defined by the family of semi-norms

$$P_K(f) = \sup_{x \in K} |f(x)|$$

when $K$ ranges over the compact subsets of $X$. $C_{wb}(E)$ will represent the space of real functions which are weakly continuous when restricted to the bounded subsets of $E$. $C_{wbu}(E)$ will be the space of real functions which are weakly uniformly continuous when restricted to the bounded subsets of $E$.

$C_{wbu}(E) \subseteq C_{wb}(E)$, if we give the topology of the uniform convergence on weakly compact subsets to both, we will have that $C_{wbu}(E)$ is a subspace of the locally convex space $C_{wb}(E)$.

1. The space $C_{wb}(E)$. The space $E$ endowed with the $bw$-topo-
logy will be represent by $X$. It is evident that $C_{wb}(E)$ coincides with $C(X)$ as sets. On the other hand, the weak compacts of $E$ and the compacts of $X$ are the same (by being bounded). Therefore, both spaces are topologically isomorphic.

We are concerned with studying the properties of $C_{wb}(E)$, for which we need the following lemma:

**Lemma 1.1.** If $E$ is a weakly normal space, then $X$ is normal and hence completely regular.

**Proof.** If $E$ is a weakly normal space, then for every $n$, $B_n$ endowed with the weak restricted topology is normal.

Let $C$ and $F$ be closed subsets of $X$, $C \cap F = \emptyset$.

$C_n = C \cap B_n$, $F_n = F \cap B_n$. $C_1$ and $F_1$ are weakly closed. By Urysohn's lemma, we have $f_i: B_i \rightarrow [0, 1]$ weakly continuous function such that:

$$f_i(C_i) = \{0\} \quad \text{and} \quad f_i(F_i) = \{1\}$$

will be $f_i^*: B_1 \cup C_2 \cup F_2 \rightarrow [0, 1]$ defined by

$$f_i^*|_{B_1} = f_i, \quad f_i^*(C_2) = \{0\}, \quad f_i^*(F_2) = \{1\}.$$

This function is weakly continuous and it is defined on a weakly closed subset of $B_2$, therefore, by Tietze's theorem, it can be extended to another function $f_2: B_2 \rightarrow [0, 1]$ weakly continuous and such that

$$f_2(C_2) = \{0\} \quad \text{and} \quad f_2(F_2) = \{1\}.$$

We define by induction $f_n: B_n \rightarrow [0, 1]$ weakly continuous such that:

$$f_n(C_n) = \{0\}, \quad f_n(F_n) = \{1\}, \quad \text{and} \quad f_n|_{B_{n-1}} = f_{n-1}.$$

We define $f(x) = f_n(x)$ if $x \in B_n$.

We have that $f$ is continuous on $X$, $f(C) = \{0\}$ and $f(F) = \{1\}$. Hence $X$ is normal.

Unfortunately we have not a general result, eliminating the hypotheses of weak normality, which affirms that $X$ is always completely regular; which is necessary for the study of $C(X)$. Nevertheless, if $E$ endowed with the coarser topology that makes the functions of $C_{wb}(E)$ continuous, is represented by $\tilde{X}$, we achieve that the above space is completely regular, and we have that the following inclusions are continuous:

$$X \longrightarrow \tilde{X} \longrightarrow (E, \sigma(E, E'))$$
giving equality to the first inclusion if and only if $X$ is completely regular.

We proceed to study the properties of $C_{wb}(E)$. In the first place we will see when it is a bornological space. According to Nachbin-Shirota's theorem [7,8], this would be equivalent to $X$ being realcompact. We have the following statement:

**Theorem 1.2.** If $E$ is weakly normal, then $C_{wb}(E)$ is bornological if and only if $E$ is weakly realcompact.

**Proof.** Simply by noting the fact that the $B_n$ balls with weak restricted topology are realcompacts, it follows that $X$ (respectively $E$) is realcompact (respectively weakly realcompact), we will do the proof for $E$, but with light modifications serving for $X$.

Let $\{U_a\}_{a \in A}$ be a $z$-ultrafilter. Each $U_a = f_a^{-1}(0)$ with $f_a: E \to \mathbb{R}$ weakly continuous.

We have that for every index sequence $(\alpha_n) \in A$, $\bigcap_{n=1}^{\infty} U_{\alpha_n} \neq \emptyset$.

(1) There is $n_0$ such that $U_\alpha \cap B_{n_0} \neq \emptyset$ for every $\alpha \in A$.

If it were not like this, for each $n \in \mathbb{N}$ we would have $\alpha_n \in A$ such that $U_{\alpha_n} \cap B_n = \emptyset$. With which we would have that $\bigcap_{n=1}^{\infty} U_{\alpha_n} = \emptyset$, failing the countable intersection property.

Therefore, for every $n \geq n_0 \{U_\alpha \cap B_n\}_{\alpha \in A}$ is a filter basis in $B_n$.

(2) There exists $n_1 \geq n_0$ in such a way that the filter basis $\{U_\alpha \cap B_{n_1}\}_{\alpha \in A}$ has the countable intersection property.

Supposing the above fails: for every $n \geq n_0$ there would be $\{\alpha_{n,m}\}_{m \in \mathbb{N}}$ index sequence in such a way that $\bigcap_{n=1}^{\infty} (U_{\alpha_{n,m}} \cap B_n) = \emptyset$. The countable family $\{U_{\alpha_{n,m}}\}_{n,m \in \mathbb{N}, n \geq n_0}$ has an empty intersection contrary to the countable intersection property.

(3) $\{U_\alpha \cap B_{n_1}\}_{\alpha \in A}$ is the basis of a $z$-filter with the countable intersection property in $B_{n_1}$, because $f_\alpha|_{B_{n_1}}$ is a continuous function on $B_{n_1}$ endowed with the weak topology restricted, for every $\alpha \in A$.

(4) $\{U_\alpha \cap B_{n_1}\}_{\alpha \in A}$ is a basis for a $z$-ultrafilter.

If not, it would be $Z \in B_{n_1}$ zero in $B_{n_1}$, that is, $Z = f^{-1}(0)$ with $f$ weakly continuous on $B_{n_1}$, in such a way that $Z \cap U_\alpha \cap B_{n_1} \neq \emptyset$ for every $\alpha \in A$, but $Z$ not containing any $U_\alpha \cap B_{n_1}$. But since $B_{n_1}$ is a weakly closed subset of $E$, by normality there exists $\tilde{f}: E \to \mathbb{R}$ weakly continuous, such that $\tilde{f}|_{B_{n_1}} = f$. Furthermore, $\tilde{f}^{-1}(0) = \tilde{Z}$ will be a zero of $E$ (with the weak topology), and $Z = \tilde{Z} \cap B_{n_1}$. But $\tilde{Z} \cap U_\alpha \neq \emptyset$ for every $\alpha \in A$, hence $\tilde{Z} = U_{\alpha_0}$ for some $\alpha_0$ by being $z$-ultrafilter, then $Z = U_{\alpha_0} \cap B_{n_1}$.

Just as by hypothesis $B_{n_1}$ is realcompact, it follows that $\bigcap_{\alpha \in A} U_\alpha \cap B_{n_1} \neq \emptyset$ and thus $\bigcap_{\alpha \in A} U_\alpha \neq \emptyset$. Hence $E$ is weakly realcompact.

Since $\tilde{f}$ is also continuous on $X$, it can also be inferred that $X$
is realcompact.

We have then that $\bar{X} = X$ is realcompact if and only if $E$ is weakly realcompact, and that $C_{wb}(E')$ is bornological if and only if $E$ is weakly realcompact.

Weakly realcompact Banach spaces are described in [1]. Nevertheless, there exists weakly realcompact spaces which are not weakly normal, as in the case of $l^\infty$ [1, p. 12]. In any case, the class of weakly normal and weakly realcompact spaces is wide; particularly every WCG space is weakly Lindelöf [9] and thus is in our hypotheses.

The following statement gives a partial answer to the problem for the case of not necessarily normal spaces.

**Theorem 1.3.** If $E$ is the dual of a separable space, then $C_{wb}(E')$ is bornological; in particular $C_{wb}(l^\infty)$ is bornological.

**Proof.** Let $E = F'$ be. $\{x_n\}_{n \in \mathbb{N}}$ a dense subset of $F$. We define:

$$f: \bar{X} \longrightarrow \mathbb{R}^N$$

$$x' \longrightarrow ((x_n, x'))_n .$$

This map is one to one because $\{x_n\}_{n \in \mathbb{N}}$ separates points of $E$ by being dense in $F$. Furthermore, $f$ is continuous by being $f = \tilde{f} \circ i$, being $i: \bar{X} \rightarrow E$ continuous and $\tilde{f}: E \rightarrow \mathbb{R}^N$ continuous, given that $\tilde{f}$ composed with $p_n: \mathbb{R}^N \rightarrow \mathbb{R}$ is $x_n$. Since $\mathbb{R}^N$ is realcompact and all its subsets are as well (by the points being $G_\delta$-sets), we have, because of [4], that $\bar{X}$ is realcompact and $C_{wb}(E)$ is bornological.

Finally we are going to see that $\bar{X}$ is a NS-space. That is, every relatively pseudocompact and closed subset is compact. Through [7] we achieve that $C_{wb}(E')$ is always barreled.

**Proposition 1.4.** $C_{wb}(E')$ is barreled.

**Proof.** Let $K \subset \bar{X}$ be a relatively pseudocompact and closed subset. For all $x' \in E'$, $x'(K)$ is bounded, hence $K$ is weakly bounded and thus bounded. Furthermore, $K \subset B_n$ for some $n$, from where it follows that $K$ is weakly closed. As on the other hand each weakly continuous function on $E$ is continuous on $\bar{X}$, we have that $K$ is weakly relatively pseudocompact. Also through [10] we achieve that $K$ is weakly compact and thus compact in $\bar{X}$.

2. The space $C_{wbu}(E)$. First of all, we will study the relationship between $C_{wbu}(E')$ and $C_{wb}(E')$.

**Proposition 2.1.** $C_{wbu}(E)$ is a dense subspace of $C_{wb}(E')$. 
Proof. Let \( f \) be a function of \( C_{\omega b}(E) \). For every weakly compact subset \( K \) of \( E \), \( f|_K \) will be weakly continuous. Let \( \tilde{f}_K \) be an extension to the Stone-Cech compactification of \( E \) endowed with the weak topology, \( \beta(E) \), which will be uniformly continuous. Then \( \tilde{f}_K(f)|_K = \tilde{f}_K|_K \) is weakly uniformly continuous on \( E \). Then \( f_K \in C_{\omega b}(E) \) for every weakly compact subset \( K \).

Obviously, \( \{f_K\}_K \) is a net that converges uniformly on weakly compact subsets of \( E \), to \( f \).

**Proposition 2.2.** \( C_{\omega b}(E) = C_{\omega b}(E) \) if and only if \( E \) is reflexive.

Proof. If \( E \) is reflexive, the equality holds because the balls are weakly compacts.

Conversely given \( f \) weakly continuous on \( E \), we do have that \( f \in C_{\omega b}(E) = C_{\omega b}(E) \). Then \( f \) is weakly uniformly continuous on \( B_1 \), and consequently it is bounded on \( B_1 \), because it is totally bounded. Since \( f \) is weakly uniformly continuous on \( B_1 \), it follows that there exists a weak neighborhood of zero, \( V \), such that \( |f(x) - f(y)| < 1 \) provided that \( x - y \in V \) and \( x, y \in B_1 \). Since \( B_1 \) is weakly totally bounded, we infer that there exists \( x_1, \ldots, x_{n_0} \in B_1 \) such that

\[
B_1 \subset \bigcup_{i=1}^{n_0} \{x_i + V\}.
\]

Thus for every \( x \in B_1 \)

\[
|f(x)| \leq \max_{i=1, \ldots, n_0} \{|f(x_i)| + 1\}.
\]

This means that every weakly continuous function over \( E \) is bounded on \( B_1 \), hence \( B_1 \) is weakly relatively pseudocompact, and weakly closed. By [10] it follows that \( B_1 \) is weakly compact and thus \( E \) reflexive.

The proof of this proposition suggest that, if \( E \) is not reflexive, one method to find a function which belongs to \( C_{\omega b}(E) \) and does not belong to \( C_{\omega b}(E) \), it would be to find a weakly continuous function over \( E \) which is not bounded on \( B_1 \).

**Example 2.3.** If \( E \) is a nonreflexive separable space, the James-Klee theorem [2, p. 7] states there exists \( \phi \in E' \) which does not attain its norm.

We define the function

\[
f_\phi : B_1 \longrightarrow \mathbb{R} \text{ by } f_\phi(x) = \frac{1}{\|\phi\| - \phi(x)}.
\]

This function is weakly continuous on \( B_1 \), and is not bounded. Since \( E \) is a separable space, then it is \( WCG \) and therefore weakly normal.
Thus by Tietze's theorem there exists \( f \) weakly continuous on \( E \), which extends \( f_0 \), and which is not bounded on the unit ball; therefore it can not belong to \( C_{wbu}(E) \).

**Corollary 2.4.** \( C_{wbu}(E) \) is complete if and only if \( E \) is reflexive.

**Proof.** If \( E \) is reflexive, \( C_{wb}(E) = C_{wbu}(E) \) and also \( C_{wb}(E) \) is complete by [3]. Thus \( C_{wbu}(E) \) is complete.

Conversely, since \( C_{wbu}(E) \) is dense in \( C_{wb}(E) \), if it is complete, both spaces have to be the same and because of that it is reflexive.

**Theorem 2.5.** \( C_{wbu}(E) \) is barrelled if and only if \( E \) is reflexive.

**Proof.** If \( E \) is reflexive \( C_{wbu}(E) = C_{wb}(E) \) and consequently barrelled. Conversely we consider the following diagram:

\[
\begin{array}{c}
B''_1 \rightarrow B''_2 \rightarrow \cdots \rightarrow X'' \\
\uparrow i_1 & \uparrow i_2 & \uparrow i \\
B_1 \rightarrow B_2 \rightarrow \cdots \rightarrow X.
\end{array}
\]

\( B_n'' \) are endowed with the weak star topology restricted. \( X'' \) will be the inductive limit of the spaces \( B_n'' \).

\( i \) is continuous because when composed with the inclusions \( j_n: B_n \rightarrow X \) it follows that \( i \circ j_n = j_n^* \circ i_n \), being \( j_n^*: B_n'' \rightarrow X'' \) the canonical inclusion in the inductive limit; obviously \( j_n^* \circ i_n \) is continuous because \( i_n \) is also continuous. \( i \) is one to one and \( i(X) \) is dense in \( X'' \).

Let us consider the map restriction \( \phi: C(X'') \rightarrow C(X) \); \( f \rightarrow f \circ i \).

(1) We have that \( \phi(C(X'')) = C_{wbu}(E) \). Let us see it.

If \( f \in C_{wbu}(E) \), it follows that \( f_n = f \mid_{B_n}: B_n \rightarrow R \) is weakly uniformly continuous. Then, by density, it can be extended to \( \tilde{f}_n: B_n'' \rightarrow R \) uniformly continuous, on the other hand,

\[
\tilde{f}_n \mid_{B_n' \ast} = \tilde{f}_{n-1} \text{ because } (\tilde{f}_n \mid_{B_n'}) \mid_{B_n} = (\tilde{f}_n \mid_{B_n}) \mid_{B_n-1} = f_n \mid_{B_n-1} = f_{n-1}
\]

and

\[
\tilde{f}_{n-1} \mid_{B_n-1} = f_{n-1}.
\]

Then both functions are exactly the same over a dense part of \( B_n'' \) thus they are the same.

It can be defined \( \tilde{f}: X'' \rightarrow R \) continuous by \( \tilde{f}(x) = f_n(x) \) if \( x \in B_n'' \). Obviously \( \tilde{f} \mid_X = f \), thus \( f \in \phi(C(X'')) \).

Conversely if \( f \in \phi(C(X'')) \) it follows that there exists \( \tilde{f} \in C(X'') \) such that \( f = \tilde{f} \mid_X \); but \( \tilde{f}_n = \tilde{f} \mid_{B_n} \) is continuous on \( B_n'' \) which is compact. Then \( \tilde{f}_n \) is uniformly continuous. Therefore \( f_n = f \mid_{B_n} \) is equal
to $f_n |_{E_n}$ and because of that, weakly uniformly continuous.

(2) $\phi$ is linear.

(3) $\phi$ is one to one because $i(X)$ is dense in $X''$.

(4) $\phi$ is continuous because the continuity of $i: X \to X''$ implies that the compacts of $X$ are compacts of $X''$.

On the other hand, the space $X''$ is a countable union of compacts and therefore the topology of $C(X'')$ is given by a countable family of semi-norms. Thus $C(X'')$ is metrizable. Because of the definition, $X''$ is a $k$-space and therefore $C(X'')$ is complete [11]; then $C(X'')$ is a Frechet space.

Since $C_{wbu}(E)$ is barreled it follows that $f$ is a topological isomorphism, applying the Open Mapping Theorem. Then it can be inferred that $C_{wbu}(E)$ is complete and therefore $E$ is reflexive.

**Corollary 2.6.** If $E$ is a Banach space, the following are equivalent:

(1) $E$ is reflexive
(2) $C_{wbu}(E)$ is a Frechet space
(3) $C_{wbu}(E)$ is a Ptak space
(4) $C_{wbu}(E)$ is complete
(5) $C_{wbu}(E)$ is barreled
(6) $C_{wbu}(E) = C_{wb}(E)$.

**References**


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