ON TWO-STAGE MINIMAX PROBLEMS

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Minimax problems are considered whose admissible sets are given implicitly as the solution sets of another minimax problem. For the solution a parametric method is proposed. Special cases of it are extensions of Courant's exterior penalty method and Tihonov's regularization method of Nonlinear Programming to minimax problems.

In solving quadratic problems explicitly, a representation of modified best approximate solutions of linear equations in Hilbert spaces is given that extends results for the usual case.

1. Introduction. Let $X$ and $Y$ be not empty subsets of real linear topological Hausdorff spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively,

$$f: X \times Y \rightarrow \mathbb{R}, \quad g: X \times Y \rightarrow \mathbb{R}$$

be two real valued functions on $X \times Y$, and denote $X_f \times Y_f$ the solution set of the minimax problem $(X, Y, f)$, i.e.,

$$(x_0, y_0) \in X_f \times Y_f: \iff \bigwedge_{x \in X} \bigwedge_{y \in Y} f(x, y_0) \leq f(x_0, y_0) \leq f(x_0, y) .$$

Note that if $(x_1, y_1)$ and $(x_2, y_2)$ are in $X_f \times Y_f$ then also $(x_1, y_2) \in X_f \times Y_f$, being thus a product set.

Under the assumption that $X_f$ and $Y_f$ are not empty, we give the following

DEFINITION 1. A two-stage minimax problem, in the notation $\mathcal{M}_{g/f}$, is the minimax problem

$$\mathcal{M}_{g/f}: = (X_f, Y_f, g|X_f \times Y_f) .$$

Considering $\mathcal{M}_{g/f}$ as a two-person zero-sum game, it describes the following conflict situation: Two antagonists choose independently from each other $x \in X$, resp. $y \in Y$, and the first one gets from the second one the vector-payoff $(f(x, y), g(x, y)) \in \mathbb{R}^2$. The preference relation may be induced by the lexicographic order of $\mathbb{R}^2$:

$(x_1, y_1)$ is better than $(x_2, y_2)$ for the first (second) player, if $(f(x_1, y_1), g(x_1, y_1))$ is lexicographically greater (smaller) than $(f(x_2, y_2), g(x_2, y_2))$. If the players are cautious, they have to take as optimal strategies the components of a solution of $\mathcal{M}_{g/f}$, provided there exists one.

Many games are of this nature; for example (see §§3, 4 and 5
below) constrained games, where on the first stage the constraints have to be satisfied, or games, in which you are interested in optimal strategies of minimum (semi-) norm, like for instance in certain differential games, where the (semi-) norm represents the consumption of energy, which of course should be minimal among all optimal strategies.

A method for solving \( M_{g/f} \) that first produces the whole sets \( X_f \) and \( Y_f \), meets with great numerical difficulties. Therefore the following algorithm is of interest that solves \( M_{g/f} \) without computing \( X_f \) and \( Y_f \): Take an arbitrary real positive nullsequence \( \{r_n\}_{n\in\mathbb{N}} \subset \mathbb{R} \) and find a solution \( (x_n, y_n) \) of the problem \( (X, Y, f + r_n g) \), \( (n \in \mathbb{N}) \).

Under certain conditions the accumulation points of \( \{x_n\}_{n\in\mathbb{N}}, \{y_n\}_{n\in\mathbb{N}} \) (unique in some cases) build a solution of \( M_{g/f} \), as is shown below.

2. A solution algorithm for the general problem \( M_{g/f} \).

**DEFINITION 2.**

(a) A function \( f: X \rightarrow \mathbb{R} \) is called

(i) inf-compact, if \( \{x | x \in X, f(x) \leq c\}, c \in \mathbb{R}, \) is compact.

(ii) sup-compact, if \(-f\) is inf-compact.

(b) A function \( h: X \times Y \rightarrow \mathbb{R} \) is called \((x_0, y_0)\)-supinf-compact, for a fixed \( (x, y) \in X \times Y \), if \( h(x, \cdot) \) is inf-compact and \( h(\cdot, y) \) is sup-compact.

We say that a real function \( h(x, y) \) on \( X \times Y \) is u.s.c.-l.s.c., if \( h(x, y) \) is upper semi-continuous in \( x \) for each \( y \in Y \) and lower semi-continuous in \( y \) for each \( x \in X \).

For a real positive sequence

\[\{r_n\}_{n \in \mathbb{N}} \subset \mathbb{R}, \text{ with } r_n \longrightarrow +0 \text{ for } n \longrightarrow \infty,\]

let \( p_n \) be defined by

\[p_n: X \times Y \longrightarrow \mathbb{R} \quad p_n: (x, y) \longmapsto f(x, y) + r_n g(x, y), \quad (n \in \mathbb{N}).\]

**THEOREM 1.** Under the conditions

(i) \( X \) and \( Y \) are convex and closed.

(ii) \( f \) and \( g \) are u.s.c.-l.s.c., and \( g \) is bounded above in \( x \) for each \( y \in Y \) and bounded below in \( y \) for each \( x \in X \).

(iii) There exists a (fixed) \( (x_0, y_0) \in X_f \times Y_f \) such that \( g \) is \((x_0, y_0)\)-supinf-compact.

(iv) \( p_n \) is quasi-concave-convex, \( (n \in \mathbb{N}).\)

we have

(v) \( (X, Y, p_n) \) has a solution \( (x_n, y_n), (n \in \mathbb{N}).\)
(vi) \( \{x_n\}_{n \in \mathbb{N}} \) and \( \{y_n\}_{n \in \mathbb{N}} \) have cluster points \( \hat{x} \) and \( \hat{y} \), respectively, and each \((\hat{x}, \hat{y})\) solves \( \mathcal{M}_{g,f} \).

(vii) \( \lim_{n \to \infty} p_n(x_n, y_n) = f(\hat{x}, \hat{y}) \).

(viii) \( \lim_{n \to \infty} (p_n(x_n, y_n) - f(x_0, y_0))/r_n = g(\hat{x}, \hat{y}) \).

**Proof.** The sum of two u.s.c., (l.s.c.), functions on a closed set is u.s.c., (l.s.c.), and so by (ii) \( p_n = f + r_n g, \) \( (n \in \mathbb{N}) \), is u.s.c-l.s.c.

For \( n \in \mathbb{N} \) and \( c \in \mathbb{R} \) we have

\[
\begin{align*}
\{ y | y \in Y, \ p_n(x_0, y) \leq c \} = & \{ y | y \in Y, \ r_n g(x_0, y) \leq c - \inf_{y \in Y} f(x_0, y) \} \leq \{ y | y \in Y, g(x_0, y) \leq \frac{1}{r_n} (c - f(x_0, y_0)) \},
\end{align*}
\]

the last set is compact by (iii), and so \( p_n(\cdot, \cdot) \) is inf-compact. Similarly, \( p_n(\cdot, y_0) \) is sup-compact. Applying now Theorem 1 of Hartung [5], we get the existence of a saddle point \((x_n, y_n)\) of \( p_n \) over \( X \times Y, \) \( (n \in \mathbb{N}) \). For all \( x_f \in X_f \) and \( y_f \in Y_f \) we then get, with \( n \in \mathbb{N} \),

\[
\begin{align*}
(1) \quad [f(x_f, y_n) + r_n g(x_f, y_n)] - f(x_f, y_n) \leq p_n(x_n, y_n) - f(x_f, y_f) \\
\quad \leq [f(x_n, y_f) + r_n g(x_n, y_f)] - f(x_n, y_f),
\end{align*}
\]
or

\[
(2) \quad r_n g(x_f, y_n) \leq p_n(x_n, y_n) - f(x_f, y_f) \leq r_n g(x_n, y_f).
\]

Putting \( x_f = x_0, \ y_f = y_0 \), (2) gives because of (ii)

\[
(3) \quad -\infty < r_n \inf_{y \in Y} g(x_0, y) \leq p_n(x_n, y_n) - f(x_0, y_0) \leq r_n \sup_{x \in X} g(x, y_0) < +\infty,
\]

and so

\[
(4) \quad p_n(x_n, y_n) \longrightarrow f(x_0, y_0), \text{ as } r_n \longrightarrow +0 \text{ for } n \longrightarrow \infty.
\]

Dividing in (2) by \( r_n \), we get

\[
(5) \quad g(x_0, y_n) \leq \sup_{x \in X} g(x, y_0), \inf_{y \in Y} g(x_0, y) \leq g(x_n, y_0),
\]

which by (iii) means that \( x_n, y_n \) are elements of compact sets independent of \( n \). Therefore \( \{x_n\}_{n \in \mathbb{N}}, \ \{y_n\}_{n \in \mathbb{N}} \) have cluster points \( \hat{x} \in X, \ \hat{y} \in Y \). Let \( \{x_{n_k}\} \) be a subnet of \( \{x_n\}_{n \in \mathbb{N}} \) converging to \( \hat{x} \). By (ii) and (4) it follows that

\[
(6) \quad f(\hat{x}, y) \geq \lim_{n_k \to \hat{x}} \sup_{x_{n_k}} f(x_{n_k}, y) \\
\quad \geq \lim_{n_k \to \hat{x}} \sup (p_{n_k}(x_{n_k}, y_{n_k}) - r_{n_k} g(x_{n_k}, y)) \\
\quad \geq \lim_{n_k \to \hat{x}} \sup (p_{n_k}(x_{n_k}, y_{n_k}) - r_{n_k} \sup_{x \in X} g(x, y)) \\
\quad \geq f(x_0, y_0), \text{ for all } y \in Y, i.e., \hat{x} \in X_f,
\]
and analogously, \( \hat{y} \in Y_f \). Let now \( \hat{y} \) be a cluster point of the subnet \( \{ y_{nk} \} \) of \( \{ y_n \} \) existing by (5), and \( \{ y_{nk_i} \} \) a subnet of it converging to \( \hat{y} \). Then of course \( x_{nk_i} \to \hat{x} \), and

\[
(7) \quad (\hat{x}, \hat{y}) \in X_f \times Y_f .
\]

From (2) we get, since \( f(x, y_f) = \text{const} = f(x_0, y_0) \) for \( (x, y_f) \in X_f \times Y_f \),

\[
(8) \quad \sup_{x \in X_f} g(x, y) \leq \frac{p_n(x_n, y_n) - f(x_0, y_0)}{\tau_n} \leq \inf_{y \in Y_f} g(x_n, y) .
\]

The functions \( x \mapsto \inf_{y \in Y_f} g(x, y) \) and \( y \mapsto \sup_{x \in X_f} g(x, y) \) are u.s.c., resp. l.s.c., and thus (8) yields

\[
(9) \quad \sup_{x \in X_f} g(x, \hat{y}) \leq \liminf_{y \in Y_f} \sup_{x \in X_f} g(x, y_{nk_i}) \leq \limsup_{y \in Y_f} \inf_{x \in X_f} g(x_{nk_i}, y) \leq \inf_{y \in Y_f} g(\hat{x}, y) ,
\]

which gives

\[
(10) \quad g(\hat{x}, \hat{y}) \leq \sup_{x \in X_f} g(x, y_*) \leq \inf_{y \in Y_f} g(\hat{x}, y) \leq g(\hat{x}, \hat{y}) ,
\]

i.e., \( (\hat{x}, \hat{y}) \) is a saddle point of \( g/X_f \times Y_f \). Similarly, \( \hat{y} \) is a saddle point component of \( g/X_f \times Y_f \), and so (vi) is shown. The statement (vii) now follows from (4). Let

\[
\begin{align*}
{b}_n &= \sup_{x \in X_f} g(x, y_n) , \quad {c}_n = \inf_{y \in Y_f} g(x_n, y) , \\
b &= \liminf_{n \to \infty} {b}_n , \quad \text{and} \quad c = \limsup_{n \to \infty} {c}_n ,
\end{align*}
\]

and \( \{ b_n \}_{n \in \mathbb{N}}, \{ c_n \}_{n \in \mathbb{N}} \) be sequences converging to \( b \) and \( c \), respectively. The corresponding \( y_{ns} \) and \( x_{ns} \) are contained in compact sets by (5), and thus there exist subnets \( \{ y_{ns} \} \) and \( \{ x_{ns} \} \) converging resp. to a \( y^* \in Y_f \) and an \( x^* \in X_f \). Then of course \( b_{ns} \) is converging to \( b \) and \( c_{ns} \) to \( c \), and we get from (8)

\[
(11) \quad \sup_{x \in X_f} g(x, y^*) \leq \liminf_{y \to y^*} \sup_{x \in X_f} g(x, y_{ns}) \leq \liminf_{n \to \infty} \sup_{x \in X_f} g(x, y_n) \leq \liminf_{n \to \infty} \frac{p_n(x_n, y_n) - f(x_0, y_0)}{\tau_n} \leq \limsup_{n \to \infty} \frac{p_n(x_n, y_n) - f(x_0, y_0)}{\tau_n} .
\]
\[ \leq \lim_{n \to \infty} \sup_{y \in Y_f} \inf_{x_{n,f}} g(x_n, y) \leq \lim_{n \to \infty} \sup_{y \in Y_f} \inf_{x_{n,f}} g(x_{n,f}, y) \leq \inf_{y \in Y_f} g(x^*, y) , \]

which gives (viii).

**Corollary 1.** If we have for some \( (x_i, y_i) \in X_f \times Y_f, (i = 1, 2) \), and for \( c \in R^2 \) that the level sets

\[
\{x \mid x \in X, f(x, y_i) \geq c_1, g(x, y_2) \leq c_2\}, \\
\{y \mid y \in Y, f(x_i, y) \leq c_1, g(x, y) \leq c_2\}
\]

are compact and \( g \) satisfies the boundedness condition of (ii), we can take instead of \( g \) the function

\[ \bar{g}(x, y) := f(x, y_i) + f(x_i, y) + g(x, y), \]

which is \( (x, y) \)-sup inf-compact, and

\[ \bar{g}/X_f \times Y_f = g/X_f \times Y_f + \text{const}. \]

**Proof.** We show that \( \bar{g}(\cdot, y_2) \) is sup-compact. For \( c \in R \) and \( x \in X \) we have:

\[ \bar{g}(x, y_2) \geq c \implies (g(x, y_2) \geq c - f(x, y_2) - \max_{x \in X} f(x, y_i), \]

and

\[ f(x, y_i) \geq c - f(x_i, y_2) - \sup_{x \in X} g(x, y_2). \]

**Definition 3.** Let \( U \) be a convex subset of a real normed linear space, then a function \( h : U \to R \) is called uniformly quasi-convex, if there exists a continuous isotonic function \( \delta : [0, \infty) \to [0, \infty) \) with \( \delta(0) = 0, \delta(t) > 0 \) for \( t > 0 \), such that for all \( u_1, u_2 \in U \)

\[ h\left( \frac{1}{2}(u_1 + u_2) \right) \leq \max \{ h(u_1), h(u_2) \} - \delta(\| u_1 - u_2 \|). \]

Similarly, \( h \) is uniformly quasi-concave, if \((-h)\) is uniformly quasi-convex.

**Theorem 2.** If in addition to (i), (ii), (iv) of Theorem 1, \( X \) and \( Y \) are reflexive Banach spaces, \( X_f \) and \( Y_f \) are not empty, and \( g \) is uniformly quasi-concave-convex, then

\[ (X, Y, p_n) \text{ has a solution } (x_n, y_n), (n \in N), \{x_n\}_{n \in N} \text{ and } \{y_n\}_{n \in N} \text{ converge (strongly) to an } \hat{x} \in X \text{ and } \hat{y} \in Y, \]

resp., and \((\hat{x}, \hat{y})\) is the solution of \( \mathcal{M}_{g,f} \).
Proof. Let \( x_f \in X_f \) be fixed, then by Definition 3 there exists a continuous isotonic function \( \delta_{x_f} : [0, \infty) \to [0, \infty) \) with \( \delta_{x_f}(t) = 0 \Leftrightarrow t = 0 \), such that for all \( y \in Y \) and \( y_f \in Y_f \)

\[
\delta_{x_f}(\|y - y_f\|) \leq \max \{ g(x_f, y), g(x_f, y_f) \} - g(x_f, \frac{1}{2}(y + y_f)).
\]

For \( c \in \mathbb{R} \) we have

\[
g(x_f, y) \leq c \iff \|y - y_f\| \leq \delta_{x_f}^{-1}\left(\max_{y \in Y} \{c, g(x_f, y)\}\right) - \inf_{y \in Y} g\left(x_f, \frac{1}{2}(y + y_f)\right),
\]

and so the level set

\[
T_{x_f}^\ast = \{y \in Y, g(x_f, y) \leq c\}
\]

is bounded.

\( g(x_f, \cdot) \) is l.s.c. and quasi-convex, and thus \( T_{x_f}^\ast \) is convex and closed, hence weakly compact, and so \( g(x_f, \cdot) \) is weakly inf-compact, for all \( x_f \in X_f \). Similarly, \( g(\cdot, y) \) is weakly sup-compact, for all \( y \in Y_f \). Herewith all conditions of Theorem 1 are fulfilled in the weak topology, and we get the existence of a solution \( (x_n, y_n) \) of \((X, Y, p_n), (n \in \mathbb{N})\). Since \( g \) is uniformly quasi-concave-convex, there exists a unique solution \((\hat{x}, \hat{y})\) of \( \mathcal{M}_{A,f} \), and so the whole sequences \( \{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}} \) are converging weakly to \( \hat{x} \) and \( \hat{y} \), respectively.

Putting in (12) \( x_f = \hat{x}, y = y_n \) and \( y_f = \hat{y} \), we get with (8)

\[
\delta_\hat{x}(\|y_n - \hat{y}\|) \leq \max \left\{ \frac{p_n(x_n, y_n) - f(x_f, y_f)}{r_n}, g(\hat{x}, \hat{y}) \right\} - g\left(\hat{x}, \frac{1}{2}(y_n + \hat{y})\right).
\]

(13)

\[
1/2(y_n + \hat{y}) \to \hat{y}, \text{ for } n \to \infty, \text{ g(x, \cdot) is weakly l.s.c., and so (13) yields by using (viii) of Theorem 1}
\]

(14)

\[
\limsup_{n \to \infty} \delta_\hat{x}(\|y_n - \hat{y}\|) \leq g(\hat{x}, \hat{y}) - g(\hat{x}, \hat{y}),
\]

which gives the strong convergence of \( \{y_n\}_{n \in \mathbb{N}} \) to \( \hat{y} \). Analogously the strong convergence of \( \{x_n\}_{n \in \mathbb{N}} \) to \( \hat{x} \) follows.

3. The exterior penalty method for constrained minimax problems. Let \( A \) and \( B \) be subsets of \( X \) and \( Y \), resp., then we consider the constrained minimax problem

\[
(A, B, g).
\]
In [5] we give for this problem an interior penalty method, which works only if $A$ and $B$ have interior points, but if this is the case, it needs for convergence some sup inf-compactness of $g$ only over the sets $A$ and $B$, which especially is given, if $A$ and $B$ are compact.

If $A$ and $B$ have no interior points, we propose a sequential method approximating a solution of $(A, B, g)$ from the exterior in $X$ and $Y$ of the admissible sets, which is profitable, if the boundaries of $X$ and $Y$ are numerically less complicated than the boundaries of $A$ and $B$, which is especially the case, when $X$ and $Y$ are the whole spaces.

The penalty functions

$$P_A: X \rightarrow \mathbb{R}, P_B: Y \rightarrow \mathbb{R}$$

are assumed to have the properties

$$P_A(x) = \begin{cases} 0 & \text{for } x \in A \\ > 0 & \text{for } x \in X \setminus A \end{cases}, \quad P_B(y) = \begin{cases} 0 & \text{for } y \in B \\ > 0 & \text{for } y \in Y \setminus B \end{cases}.$$ 

Putting

$$f := P_B - P_A,$$

we get

$$X_f = A, \quad Y_f = B, \quad f/X_f \times Y_f = 0,$$

and

$$p_n = P_B - P_A + r_n g, \text{ with } r_n \rightarrow +0, \text{ for } n \rightarrow \infty, (n \in \mathbb{N}).$$

**Theorem 3.** If $A$ and $B$ are convex and closed, and the conditions (i), (ii), (iii), (iv) of Theorem 1 are fulfilled, then $(X, Y, p_n)$ has a solution $(x_n, y_n), (n \in \mathbb{N}), \{x_n\}_{n \in \mathbb{N}}, \{y_n\}_{n \in \mathbb{N}}$ have cluster points $\hat{x}, \hat{y}$, resp., solving $(A, B, g)$,

$$\lim_{n \to \infty} P_A(x_n) = 0, \quad \lim_{n \to \infty} P_B(y_n) = 0,$$

and

$$\lim_{n \to \infty} g(x_n, y_n) + \frac{1}{r_n}(P_B(y_n) - P_A(x_n)) = g(\hat{x}, \hat{y}).$$

**Proof.** By Theorem 1 we get the existence of a solution $(x_n, y_n)$ of $(X, Y, p_n), (n \in \mathbb{N})$, and for $x \in A, y \in B$

$$r_n g(x, y_n) + P_B(y_n) \leq p_n(x_n, y_n) \leq r_n g(x_n, y) - P_A(x_n),$$

or

$$-\infty < r_n \inf_{y \in Y} g(x, y) + P_B(y_n) \leq p_n(x_n, y_n) \leq r_n \sup_{x \in X} g(x, y) - P_A(x_n) < +\infty,$$
which yields with (4)

\[
0 \leq \limsup_{n \to \infty} P_B(y_n) \leq \lim_{n \to \infty} p_n(x_n, y_n) = 0 \leq \liminf_{n \to \infty} (-P_A(x_n)) \\
\leq -\limsup_{n \to \infty} P_A(x_n) \leq 0 .
\]

Since \( P_B \geq 0, P_A \geq 0 \), that gives

\[
\lim_{n \to \infty} P_A(x_n) = 0, \lim_{n \to \infty} P_B(y_n) = 0 .
\]

The remaining assertions follow from Theorem 1.

Corollary 1 and Theorem 2 then give a refined method. If for example \( A \) is given by

\[
A = \{x \in X, \ G_i(x) = 0, (i = 1, \cdots, m_i), \ G_j(x) \leq 0, (j = m_i + 1, \cdots, m)\}
\]

for some real valued functions \( G_i \) on \( X \), \( (i = 1, \cdots, m) \), we can take as a penalty function for instance

\[
P_A(x): = \sum_{i=1}^{m_1} (G_i(x))^2 + \sum_{i=m_1+1}^{m} \max [0, G_i(x)]^2 ,
\]

which is differentiable, when the \( G_i \) are.

4. A regularization algorithm for finding saddle points. To solve a minimax problem \((X, Y, f)\) you often have to take algorithms which need for convergency the solution to be unique, as for example the Arrow-Hurwicz-Uzawa gradient methods [1] (like the Lagrangeian method for convex programming) or the successive approximation method of Dem'janov [3]. Therefore, if this is not the case, we approximate \( f \) by a sequence of regularized functions, which have this missing property. Theorem 2 offers many possibilities for doing this. In the method we choose, the unique saddle points of the sequential functions are converging to the saddle point of \( f \) with minimum norm, which is of particular interest in certain problems. We don't need compactness conditions and thus \( f \) can be a Lagrange function of an ordinary convex program. Let \( \mathcal{H} \) and \( \mathcal{V} \) be real Hilbert spaces, \( \langle \cdot, \cdot \rangle \) denoting the inner product define the norm, \( \| \cdot \| : = \langle \cdot, \cdot \rangle^{1/2} \), resp., and \( \mathcal{H} \times \mathcal{V} \) may be provided with the induced norm.

Then we define for a real positive nullsequence \( \{r_n\}_{n \in \mathbb{N}} \) the regularized functionals

\[
p_n(x, y): = f(x, y) + r_n(\langle y, y \rangle - \langle x, x \rangle), (n \in \mathbb{N}) .
\]

**Theorem 4.** Let \( X \) and \( Y \) be convex and closed, \((X, Y, f)\) solv-
able, and $f$ be u.s.c-l.s.c. and concave-convex, then

$$(X, Y, p_n)$$ has a unique solution $(x_n, y_n)$, $(n \in \mathbb{N})$,

$$\hat{x} = \lim_{n \to \infty} x_n$$ and $$\hat{y} = \lim_{n \to \infty} y_n$$ exist, and $(\hat{x}, \hat{y})$ is

the solution of $(X, Y, f)$ with minimum norm.

**Proof.** By the parallelogram law the function

$$g(x, y) = \langle y, y \rangle - \langle x, x \rangle$$

is strictly concave-convex and uniformly quasi-concave-convex. Then $p_n(x, y)$ has these properties, too, and the saddle points of $p_n$ are uniquely determined. The rest of the assertions follow from Theorem 2.

5. An explicit solution of quadratic minimax problems. Let $\mathcal{H}$ and $\mathcal{V}$ be real Hilbert spaces as in §4, and $X = \mathcal{H}, Y = \mathcal{V}$. Then we consider the quadratic functionals

$$F(x, y) = \langle x, Px \rangle - 2\langle x, c \rangle + 2\langle x, Ly \rangle + \langle y, Qy \rangle - 2\langle d, y \rangle,$$

$$G(x, y) = \langle x, Sx \rangle + \langle y, Ty \rangle,$$

where $c \in X$, $d \in Y$; $P$ and $S$ are self-adjoint negative semidefinite linear operators on $X$, $Q$ and $T$ are self-adjoint positive semidefinite linear operators on $Y$, $L$ is a linear operator of $Y$ into $X$ and all operators are bounded, and the two stage minimax problem

(1) $$\mathcal{M}_{G/F} = \mathcal{M}_{G/F}(c, d).$$

$$\langle x, -Sx \rangle$$ and $$\langle y, Ty \rangle$$ are seminorms to the power two, representing for instance in differential games often the consumption of energy, which should be minimal among the optimal strategies of $(X, Y, F)$.

Defining now a linear and bounded operator $\begin{pmatrix} P & L \\ L^* & Q \end{pmatrix} = : A$ by

$$A : (x, y) \mapsto (P x + L y, L^* x + Q y),$$

we assume that $(c, d) \in R(A)$, and (as it can be seen by putting the derivatives of $F(x, y)$ with respect to $x$ and $y$ equal to zero) this is a necessary and sufficient condition for the solution set of $(X, Y, F')$ to be not empty, which then is given by

$$X_F \times Y_F = \{(x, y) | (x, y) \in X \times Y, A(x, y) = (c, d)\}.$$

Let $A$ be normally solvable ($R(A)$ is closed), then the element of $X_F \times Y_F$ with minimum norm is
where $A^+$ denotes the pseudoinverse (e.g., Holmes [6], p. 220). Note that $A^+w = A^+\text{Proj}_{R(A^+)}w$, for $w \in X \times Y$, and $R(A^+) \perp N(A)$. With $p_n(x, y) = F(x, y) + r_nG(x, y)$, $r_n \in R$, $r_n \to +0$, for $n \to \infty$, $(n \in N)$,

$$B = \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix},$$
and $A_n = A + r_nB$

the solution set of $(X, Y, p_n)$ is

$\{(x, y) \in X \times Y | A_n(x, y) = (c, d), (n \in N)\}$.

If $S$ and $T$ are definite and normally solvable, then $\langle y, Ty \rangle^{1/2}$ and $\langle x, -Sx \rangle^{1/2}$ are representing norms equivalent to the given ones on $Y$ and $X$, respectively. So by Theorem 4 $(X, Y, p_n)$ has a unique solution

$$(x_n, y_n) = A_n^{-1}(c, d),$$

and

$$A^{+(S,T)}(c, d) = \lim_{n \to \infty} A_n^{-1}(c, d)$$

exists and is the solution of $M_{G/F}$. Since (2) holds for all $(c, d) \in R(A)$, we have

$$(3) \quad A_n^{-1} \longrightarrow A^{+(S,T)} \text{ (strongly), as } n \to \infty,$$

where $A^{+(S,T)}$, the solution operator of $M_{G/F}$, is a linear and bounded operator, because of Banach's inverse mapping theorem.

If $I$ denotes the identity on the spaces, resp., then $A^{+(I,I)} = A^+$. If $S$ and $T$ are not invertible, then $(X, Y, p_n)$ and $M_{G/F}$ are not uniquely solvable, in general. Then we are interested in the solutions of minimum norm.

The solution set $X_F \times Y_F$ of $A(x, y) = (c, d)$ is given by

$A^+(c, d) + N(A)$, with $A^+(c, d) \perp N(A)$.

Now if $(x, y) \in N(A)$, then

$$\langle x, Px \rangle + \langle x, Ly \rangle = 0$$

$$\langle y, Qy \rangle + \langle x, Ly \rangle = 0,$$

$$\langle x, Px \rangle \leq 0 \implies \langle x, Ly \rangle \geq 0, \langle y, Qy \rangle \geq 0 \implies \langle x, Ly \rangle \leq 0,$$

and so $\langle x, Ly \rangle = 0$ and $x \in N(P)$, $y \in N(Q)$. Thus

$$(4) \quad X_F \times Y_F = (x, y) + N(P) \times N(Q), \text{ for any } (x, y) \in X_F \times Y_F,$$

and
Let $\bar{P}$ be a self-adjoint negative semidefinite bounded linear operator on $X$ with

$$N(\bar{P}) \cap N(S) = N(P) \cap N(S), \quad \bar{P}/X = \text{const};$$

and $\bar{Q}$ be a self-adjoint positive semidefinite bounded linear operator on $Y$ with

$$N(\bar{Q}) \cap N(S) = N(Q) \cap N(S), \quad \bar{Q}/Y = \text{const};$$

Putting

$$\bar{S} = \bar{P} + S, \quad \bar{T} = \bar{Q} + T,$$

we have

$$N(\bar{S}) = N(P) \cap N(S), \quad N(\bar{T}) = N(Q) \cap N(T).$$

Let $\bar{S}$ and $\bar{T}$ be normally solvable, then (cf. Petryshyn [8])

$$\inf \{ \| \bar{S}x \| \mid x \in N(\bar{S}) \}, \quad \inf \{ \| \bar{T}y \| \mid y \in N(\bar{T}) \},$$

and so $\langle x, -\bar{S}x \rangle^{1/2}/N(\bar{S})^{-1}$, $\langle y, \bar{T}y \rangle^{1/2}/N(\bar{T})^{-1}$ are equivalent norms to the given ones, resp., restricted correspondingly. With $\bar{G}(x, y) = \langle x, \bar{S}x \rangle + \langle y, \bar{T}y \rangle$, $\bar{p}_n = F + r_\varepsilon \bar{G}$, $\bar{B} = \begin{pmatrix} \bar{S} & 0 \\ 0 & \bar{T} \end{pmatrix}$, the solution set of $(X, Y, \bar{p}_n)$ is

$$\bar{A}_n^+(c, d) + N(\bar{A}_n), \quad (n \in N).$$

Now $\bar{A}_n^+(c, d) \perp N(\bar{A}_n)$ and $N(\bar{A}_n) = N(S) \times N(T)$, thus $\bar{A}_n^+(c, d)$ solves $(N(\bar{S})^{-1}, N(\bar{T})^{-1}, \bar{p}_n)$, $(n \in N)$. Applying Theorem 4 to this problem, we get

$$\bar{A}^+_{\bar{T}, \bar{S}}(c, d) = \lim_{n \to \infty} \bar{A}_n^+(c, d)$$

solves uniquely

$$\mathcal{M}_{\bar{T}, \bar{S}},$$

where $\bar{F} = F/N(\bar{S})^{-1} \times N(\bar{T})^{-1}$. 

Denote by $Z$ the solution set of $\mathcal{M}_{\bar{T}, \bar{S}}$, and let $(x_1, y_1)$, $(x_2, y_2) \in Z$; then we have by (4) for all $(u, v) \in N(P) \times N(Q)$:

$$\langle u, \bar{S}^* \bar{S}(x_1 - x_2) \rangle = 0, \quad \langle v, \bar{T}^* \bar{T}(y_1 - y_2) \rangle = 0.$$

With (4) again $(x_1 - x_2) \in N(P)$, $(y_1 - y_2) \in N(Q)$, and so $(x_1 - x_2) \in N(\bar{S})$, $(y_1 - y_2) \in N(\bar{T})$, hence we have, with (8), the representation

$$Z = (x, y) + N(\bar{S}) \times N(\bar{Q}), \quad \text{for any } (x, y) \in Z.$$
Thus the element of $Z$ with minimum norm is given by the solution of $\mathcal{M}_{G/F}$. Because of (6), (7) there are

$$
\tilde{G}/X_F \times Y_F = G/X_F \times Y_F + \text{const},
$$

and $Z$ the solution set of $\mathcal{M}_{G/F}$, too. Then (9), (10) yield,

$$
(11) \quad A^{+(S,T)}(c, d) \text{ is the solution of } \mathcal{M}_{G/F} \text{ with minimum norm.}
$$

Since (9) holds for all $(c, d)$ in the range of $A$, we have just proved the

**Theorem 5.** Let with the definitions above $A$, $\tilde{S}$, $\tilde{T}$ be normally solvable, then there exists a linear and bounded operator

$$
A^{+(S,T)}: X \times Y \longrightarrow X \times Y
$$

such that for all $(c, d) \in R(A)$

$$
A^{+(S,T)}(c, d) \text{ is the minimum norm solution of the two stage minimax problem (1) } \mathcal{M}_{G/F}(c, d), \text{ and permits the representation}
$$

$$
(12) \quad A^{+(S,T)}(c, d) = \lim_{r \to 0} \left( P + r\tilde{S} \quad L \atop L^* \quad Q + r\tilde{T} \right)^+(c, d).
$$

If $N(\tilde{S}) = \{0\}$, $N(\tilde{T}) = \{0\}$, then on the right hand side in (12) we have ordinary inversion.

Conveniently one takes

$$
\tilde{S} = \begin{cases} S, & \text{if } N(S) = \{0\} \\ P + S, & \text{otherwise} \end{cases}, \quad \tilde{T} = \begin{cases} T, & \text{if } N(T) = \{0\} \\ Q + T, & \text{otherwise} \end{cases}
$$

6. A note on best approximate solutions of linear equations. Let $W$, $X$, $Y$ be real Hilbert spaces as above and

$$
C: X \longrightarrow Y, \quad D: X \longrightarrow W
$$

be continuous linear operators. We are given an element $y \in Y$ and the problem of finding an element $x \in X$ which solves the equation

$$
(1) \quad Cx = y.
$$

If $y \in R(C)$, there exists no solution of (1). Then we consider the problem of finding an element $x(y) \in X$ of minimum seminorm $\|Dx\|$ which gives a minimum value for the discrepancy $\|Cx - y\|$, $x \in X$. An element $x(y)$ with this property may be called a 'D-best approximate solution' of (1). In the case $D = I$ (= identity) usually $x(y)$
is called a ‘best approximate solution’ (e.g., Holmes [6], p. 214) or ‘pseudo-solution’ (e.g., Morozov [7]) of (1). In order to find a $D$-best approximate solution of (1) we have to solve the problem

$$\text{(2)} \quad \text{minimize } \{ \langle x, D^*Dx \rangle \mid \langle x, C^*Cx \rangle - 2\langle x, C^*y \rangle = \text{min}, \ x \in X \}.$$  

Applying now Theorem 5 to this special two stage problem (2) we get

THEOREM 6. If $C, C^*C + D^*D$ are normally solvable, then there exists a continuous linear operator

$$C^{+\nu}: Y \longrightarrow X$$

$$y \longmapsto C^{+\nu}y,$$

such that

for all $y \in Y$ $C^{+\nu}y$ is the $D$-best approximate solution to $Cx = y$ of minimum norm, $(x \in X),$

and

$$\text{(3)} \quad C^{+\nu} = \lim_{r \to 0} (C^*C + r\tilde{D})^+C^*,$$

where

$$\tilde{D} = \begin{cases} D^*D, & \text{if } N(D) = \{0\} \\ C^*C + D^*D, & \text{otherwise} \end{cases}$$

If $N(\tilde{D}) = \{0\}$, then on the right hand side of (3) we have ordinary inversion, and especially for $D = I$ we get

$$\text{(4)} \quad C^{+I} \equiv C^+ = \lim_{r \to 0} (C^*C + rI)^{-1}C^*,$$

a representation given for instance by Morozov [7].

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