

Pacific Journal of Mathematics

HAUSDORFF MEASURE, BMO, AND ANALYTIC FUNCTIONS

ROBERT P. KAUFMAN

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Besicovitch's theorem on removable singularities is extended to function of class BMO. The extended d theorem admits a converse.

Let W be an open set in the plane, and the class $BMO(W)$ be defined as follows: a measurable function f on W is BMO if to each ball B contained in W there is a constant $c=c(B)$ so that $\iint_B |f(x, y) - c(B)| dx dy \leq Am(B)$, with a constant $A = A(f)$. For any complex function f on W , $S(f)$ is the set of points at which f fails to admit a complex derivative; $S(f)$ in general is neither open nor closed, but is in fact a Borel set.

THEOREM (a). *Let $f \in BMO(W)$ and suppose that $S(f)$ has 1-dimensional Hausdorff measure 0. Then there is a function f_1 , holomorphic on W , equal to f on $W - S(f)$.*

(b) *Let S be a compact set of positive 1-measure. Then there is a function g , analytic off S , of class $BMO(\mathbb{R}^2)$ with Taylor expansion $g(z) = z^{-1} + \dots$ at infinity.*

Proof of (a). This relies on Theorem 1 of [3] and the following variant form of Vitali's covering theorem: if a sequence of open balls $(B(a_i, r_i))_i^\infty$ covers a bounded set E , then it contains a disjoint collection $(B(a_j, r_j))_j^\infty$ such that $\bigcup_j B(a_j, 3r_j)$ covers E . Let V be a bounded subset of W and $\varepsilon > 0$; we construct coverings of $V \cap S(f)$ and $V - S(f)$ separately. Inasmuch as $S(f)$ has 1-measure 0 we can cover it with balls $B(a_i, r_i)$ such that $B(a_i, 2r_i) \subseteq W$ and $\sum r_i < \varepsilon$. For each point z in $V - S(f)$ we can find a number $r(z) > 0$ so that $B(z, 6r(z)) \subseteq W$ and $|f(w) - f(z) - (z - w)f'(z)| < \varepsilon r(z)$ when $w \in B(z, 6r(z))$. The collection $B(z, r(z))$ contains a disjoint sequence $B(z_j, r(z_j))$ such that $\bigcup B(z_j, 3\pi r(z_j)) \supseteq V - S(f)$; by the disjointness, $\sum 9\pi r^2(z_j) \leq 9m(V)$. Using the fact that constants are analytic, we see that the conditions of [3, p. 108] are fulfilled, so that $f = f_1$ a.e., for some function f_1 holomorphic on W . Then $f - f_1$ is differentiable on $S - S(f)$, and so $f = f_1$ there.

An easy improvement can be obtained from [3], namely, the constant $c(B)$ in the definition of $BMO(W)$ can be replaced by a polynomial (depending on B).

Proof of (b). By a theorem of Frostman [2, p. 7], S carries a

probability measure μ , such that $\mu(B(z, r)) \leq cr$ for every ball B of radius $r > 0$. Let

$$g(z) = \int (z - \zeta)^{-1} \mu(d\zeta)$$

so that g is analytic off S and $g(z) = z^{-1} + \dots$. To prove that $g \in \text{BMO}(R^2)$ we choose a ball $B = B(w, r)$, set $B^* = B(w, 2r)$ and $C = R^2 - B^*$. Let

$$g_1(z) = \iint_{B^*} (z - \zeta)^{-1} \mu(d\zeta)$$

$$g_2(z) = g(z) - g_1(z).$$

Now

$$\iint_B |g_1(z)| dx dy \leq \mu(B^*) \sup \iint_B |z - \zeta|^{-1} dx dy$$

$$= 2\pi r \mu(B^*) \leq 4\pi cr^2.$$

Further

$$\iint_B |g_1(z) - g_1(w)| dx dy \leq \iint_C \int_B |(z - \zeta)^{-1} - (w - \zeta)^{-1}| dx dy \mu(d\zeta).$$

From the inequality $|\zeta - w| > 2r$ ($\zeta \in C$), we find that the inner integral doesn't exceed $(4\pi/3) \cdot r^3 |\zeta - w|^{-2}$, and the entire integral is at most

$$(4\pi/3) \cdot r^3 \int_C |w - \zeta|^{-2} \mu(d\zeta)$$

$$\leq (4\pi/3) \cdot r^3 c (2r)^{-1} = 2\pi/3 \cdot cr^2.$$

Hence

$$\iint_B |g(z) - g_2(w)| dx dy \leq 3cm(B).$$

REMARKS. (i) A Borel set S of positive 1-dimensional measure contains a compact set S_0 of the same kind (Besicovitch) [2, p. 11].

(ii) Besicovitch proved (a) for bounded functions; for continuous f_1 he proved the sufficiency of the hypothesis that $S(f)$ have σ -finite 1-dimensional measure. Combining his method for this variant, with the one presented above for BMO, we can replace continuity of f by VMO (vanishing mean oscillation), that is

$$\iint_B |f(x, y) - c(B)| dx dy \leq m(B)\varepsilon(m(B)), \text{ where } \varepsilon(0+) = 0.$$

(iii) The variant just mentioned also admits a converse; to explain this we observe that if the probability measure μ figuring in the proof of (b) has the stronger property that $\mu(B(z, r)) \leq r\varepsilon(r)$ with $\varepsilon(0+) = 0$, then the function g is VMO(R^2). We use the

following theorem [4]; a Borel set S , not of σ -finite 1-dimensional measure, contains a compact set S_0 , with positive Hausdorff measure for a measure function $h(u) = u\varepsilon(u)$; by Frostman's theorem S_0 then carries a probability measure μ with the stronger property needed to improve VMO to BMO.

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Richard A. Boyce , Irreducible representations of finite groups of Lie type through block theory and special conjugacy classes	253
Robert Jay Daverman and Dennis J. Garity , Intrinsically $(n - 2)$ -dimensional cellular decompositions of E^n	275
Juan Ferrera , Spaces of weakly continuous functions	285
William George Frederick , μ -theta functions	293
Christopher George Gibson and T. D. Ward , On stratifying pairs of linear mappings	329
Stanley Joseph Gurak , Minimal polynomials for Gauss circulants and cyclotomic units	347
Joachim Georg Hartung , On two-stage minimax problems	355
Robert P. Kaufman , Hausdorff measure, BMO, and analytic functions	369
Neal I. Koblitz , p -adic analog of Heine's hypergeometric q -series	373
Kurt Kreith , Picone-type theorems for hyperbolic partial differential equations	385
Nicholas J. Kuhn , The geometry of the James-Hopf maps	397
Donald Michael Redmond , Explicit formulae for a class of Dirichlet series	413
J. R. Respass and Elliott Ward Cheney, Jr. , Best approximation problems in tensor-product spaces	437
Allen Ross Schweinsberg , The operator equation $AX - XB = C$ with normal A and B	447
Hans-Willi Sieberg and Guentcho Svetoslavov Skordev , Fixed point index and chain approximations	455
Kondagunta Sundaresan , Geometry and nonlinear analysis in Banach spaces	487