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THE LEFSCHETZ NUMBER AND BORSUK-ULAM THEOREMS

DANIEL H. GOTTLIEB

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Let M be a manifold, with or without boundary, which is dominated by a finite complex. Let G be a finite group which acts faithfully and freely on M . Let $f: M \rightarrow M$ be a G -map. Let A_f denote the Lefschetz number of f and let $o(G)$ denote the order of G . The main result states, under the conditions above, that $o(G)$ divides A_f . Even in the case of compact M this result was not widely known. We use Wall's finiteness obstruction theory to extend the result from compact M to finitely dominated M .

The remainder of the paper is devoted to various easy applications of the result. In Theorem 5 we assume that $\pi_i(M)$ is finitely generated for all $i > 1$. Then we show that if $\pi_1(M)$ has torsion, $\pi_*(M)$ cannot be only torsion.

In Theorem 6, we have a connected Lie group L acting on M and f is an L -map. We show that the orbit map $\omega: L \rightarrow M$ induces the trivial homomorphism on fundamental groups if $A_f \neq 0$. This implies that the action of L on M can be lifted to any regular covering space.

We show that any linear transformation $T: R^n \rightarrow R^n$ which commutes with the based free action of a finite group G of order greater than 2 must have a non-negative determinant (Theorem 8).

Then we come to the Borsuk-Ulam type results. We consider maps $f: (C^{n+1} - 0) \rightarrow C^n$. A primitive k -root of unity ξ gives rise to a free Z_k -action on C^n . We show that the equation $\sum_{i=0}^{k-1} \bar{\xi}^i f(\xi^i x) = 0$ always has a solution $x \in C^{n+1} - 0$. This result gives various conditions on the degeneracy of the images of the orbit of the Z_k action in C^n . In particular, we show that if $f: S^n \rightarrow R^r$ and if $n \geq r(p-1)$, then some orbit of the Z_p -action must be mapped into a point. The proof uses the equation above and Vandermonde determinants.

2. Free actions and the Lefschetz number. A manifold M (or space) is *dominated* by a finite complex K if there exists maps $f: M \rightarrow K$ and $g: K \rightarrow M$ such that $g \cdot f$ is homotopic to the identity of M . We will need various facts about finitely dominated spaces in order to prove the result that $o(G)$ divides A_f for noncompact M . It is easily shown that this is true for compact M . We use the theory of C.T.C. Wall, [8], to extend to the noncompact case.

LEMMA 1. *Let M be a finitely dominated manifold. The orbit*

space M/G is a finitely dominated manifold.

Proof. Consider the universal principal G -bundle $G \rightarrow E_G \rightarrow B_G$. Since G acts on M , we may replace the fibre to obtain the bundle $M \rightarrow M_G \rightarrow B_G$. Since G acts freely on M , we know that M_G is a homotopy equivalent to M/G . Since G is finite and acts freely on M , we see that $M \rightarrow M/G$ is a covering, so M/G is a manifold, and hence it is finite dimensional.

We know from Wall's work [8] that a space is finitely dominated if and only if it is homotopy equivalent to a finite dimensional CW -complex and also homotopy equivalent to a possibly different CW -complex of finite type (i.e., a complex whose n -skeletons are finite complexes for all n). Thus M is homotopy equivalent to a complex of finite type. Since M_G is the total space of a fibration whose base and fibre are of finite type, M_G is homotopy equivalent to a complex of finite type [4; Lemma 1.1]. Also, by the first paragraph, M_G is homotopy equivalent to M/G which is a finite dimensional complex. So M/G is finitely dominated.

REMARK. The above lemma follows from Lemma 5.5 in [2].

Now we come to the main result.

THEOREM 2. *Let M be a finitely dominated manifold, and let G act freely and faithfully on M and $f: M \rightarrow M$ be a G -map. Then $o(G)$ divides A_f .*

Proof. First we show the result for a compact manifold M with a finite group G acting freely on M and a G -map $f: M \rightarrow M$. We obtain a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{f} & M \\ \downarrow & & \downarrow \\ M/G & \xrightarrow{\bar{f}} & M/G \end{array}$$

where the vertical maps are covering projections and \bar{f} is induced by f . Now M/G is a finite CW -complex and we may adjust \bar{f} by a homotopy so that it has finitely many isolated fixed points. Now f restricted to a fibre over a fixed point maps the fibre onto itself and it either leaves all the points fixed or leaves no points fixed. Thus if x is a fixed point of f then all the points in the fibre, which is an orbit of G , are fixed, and they all must have the same fixed point index. The fibre contains $o(G)$ points, so $o(G)$ must divide A_f .

Now suppose that M is a finitely dominated manifold. It follows from Wall's theory [8] that the cartesian product of M with a finite complex, whose Euler-Poincare number is zero, must be a homotopy equivalent to a finite complex. Thus we have a compact manifold K and homotopy equivalences $h: K \rightarrow S^3 \times (M/G)$ and $h^{-1}: S^3 \times (M/G) \rightarrow K$. We let G act on $S^3 \times M$ by letting $g(s, m) = (s, g(m))$. Then G acts freely on $S^3 \times M$ and gives rise to a diagram

$$\begin{array}{ccc} \tilde{K} & \xrightarrow{\tilde{h}} & S^3 \times M \\ \downarrow p & & \downarrow \\ K & \xrightarrow{h} & S^3 \times (M/G) \end{array}$$

where $\tilde{K} \rightarrow K$ is in the pull back of the G -bundle $S^3 \times M \rightarrow S^3 \times (M/G)$. So $\tilde{K} \rightarrow K$ is a covering space and \tilde{h} is a G -map. Also there is some lifting \tilde{h}^{-1} of h^{-1} which is a G -map and $\tilde{h} \circ \tilde{h}^{-1}$ is homotopic to the identity by equivariant homotopies.

Now let $F: S^3 \times M \rightarrow S^3 \times M$ be given by $F(s, m) = (s_0, f(m))$ where $s_0 \in S^3$ is a base point. F is a G -map and so the composition $\tilde{h}^{-1}F\tilde{h}: \tilde{K} \rightarrow \tilde{K}$ is a G -map. Now $A_{\tilde{h}^{-1}F\tilde{h}} = A_F = A_f$ and by the theorem for compact complexes $o(G)$ divides $A_{\tilde{h}^{-1}F\tilde{h}}$ and hence it divides A_f .

REMARK. An earlier version of this theorem was proved in [1; Theorem 4] by transfer methods for G a cyclic group and M homotopy equivalent to a finite complex. Nakaoka in [7] developed a transfer based on the coincidence number $A_{f,g}$ for maps f and g from M to itself where M is a compact manifold. Using a result related to these transfers he succeeded in showing that $o(G)$ divides $A_{f,g}$ where f and g are G -maps. This result gives Theorem 2 in the compact case. The compact case was originally proved by G. Hirsch [3a].

COROLLARY 3. $o(G)$ divides $\chi(M)$, the Euler-Poincare number.

COROLLARY 4. If $f: M \rightarrow M$ is a G -map where G acts freely and is finite, then f cannot be homotopic to a constant map.

3. The fundamental group. At this time, despite our great knowledge of homotopy groups, we do not possess a single example of a finite complex, with nontrivial higher homotopy groups, whose homotopy groups are completely known. The following result illustrates a difficulty. It generalizes the famous result that if π has torsion, then $K(\pi, 1)$ cannot be a finite complex.

THEOREM 5. *Assume that M is a manifold such that $\pi_i(M)$ is finitely generated for all $i > 1$. Then if some nontrivial element of $\pi_1(M)$ has finite order, some element of $\pi_i(M)$ does not have finite order for some $i > 1$. (Here we need not assume that M has finite type.)*

Proof. Consider the universal cover \tilde{M} of M . Now \tilde{M} has finite dimension and $\pi_i(\tilde{M}) \cong \pi_i(M)$ is finitely generated for $i > 1$, and $\pi_1(\tilde{M}) = 0$. Hence it follows from Wall's theory that \tilde{M} is homotopy equivalent to a finite complex. Now $\pi_1(M)$ acts freely on \tilde{M} and if it contains an element of finite order, then a finite cyclic group G acts freely on \tilde{M} and $o(G)$ divides $\chi(\tilde{M})$ by Corollary 3.

Now if $\pi_i(M) \cong \pi_i(\tilde{M})$ were all torsion groups for $i > 1$, then by the Hurewicz theorem mod finite groups, the homology groups of $H_i(\tilde{M})$ would be finite for all i . Thus $\chi(\tilde{M}) = 1$ so $o(G) = 1$ which contradicts the hypothesis that G is not trivial.

Now we turn to the action of a Lie group L on a compact manifold M with or without boundary. There is the orbit map or evaluation map $\omega: L \rightarrow M$ given by evaluating $l \in L$ at a base point $m_0 \in M$. Thus $\omega: l \mapsto l(m_0)$. We are concerned with the induced homomorphism $\omega_*: \pi_1(L) \rightarrow \pi_1(M)$.

THEOREM 6. *Let L be a connected Lie group acting on a compact manifold M with or without boundary. Suppose there is an equivariant map $f: M \rightarrow M$ such that $\Lambda_f \neq 0$. Then $\omega_*: \pi_1(L) \rightarrow \pi_1(M)$ is trivial.*

Proof. Every element $\alpha \in \pi_1(L)$ can be represented by a homomorphism $S^1 \rightarrow L$. Thus we need only prove the theorem in the case of $L = S^1$.

We shall show that the S^1 action must have a fixed point and this will give us $\omega_* = 0$. There is an element $t \in S^1$ such that $\{t^n \mid \text{all } n \in \mathbf{Z}\}$ is dense in S^1 . There is a sequence $s_p \in S^1$ for p a prime such that s_p converges to t and s_p has order p . Since only a finite number of primes divide Λ_f , all but a finite number of s_p must have a fixed point x_p by Theorem 2. Now the sequence $\{x_p\}$ must have a cluster point x since M is compact, thus x must be a fixed point for t and hence a fixed point for all of S^1 .

COROLLARY 7. *With the hypotheses above, the action of L on M can be lifted to any regular covering \tilde{M} of M .*

Proof. By Corollary 11 of [3], $\omega_* = 0$ is sufficient.

REMARK. Theorem 6 is related to Theorem 5 of [1]. Using the first paragraph of the proof of Theorem 6 and Theorem 5 of [1] we can eliminate the compactness condition on M . Theorem 6 and the result from [1] that $\omega_*: \tilde{H}_*(L) \rightarrow \tilde{H}_*(M)$ is zero for integral homology under the hypothesis of Theorem 6 are interesting to compare.

4. **Borsuk-Ulam theorems.** In this section we study spheres and complex and real vector spaces. The classical Borsuk-Ulam theorem involves a continuous map $f: (R^{n+1} - 0) \rightarrow R^n$ and states that some orbit of the action $x \mapsto -x$ on $R^{n+1} - 0$ is mapped onto a single point of R^n . We investigate the situation of a map $(C^{n+1} - 0) \xrightarrow{f} C^n$ and ask if an orbit of the action $x \mapsto \xi x$ is mapped onto a single point.

But before we begin with the Borsuk-Ulam theorems we prove a different result on the determinant of a linear transformation. We say that a finite group G acts based freely on R^n if it acts faithfully on R^n and freely on $R^n - 0$. We say a linear transformation $T: R^n \rightarrow R^n$ commutes with the action of G if T commutes with every element $g \in G$.

THEOREM 8. *If G acts based freely on R^n and if T is a linear transformation which commutes with the action of G , then $\det T \geq 0$ or $o(G) = 1$ or 2 .*

Proof. If $\det T \neq 0$, then T is an isomorphism and we can regard $T: R^n - 0 \rightarrow R^n - 0$. Now $A_T = 1 + (-1)^{n-1} \deg T$ and $\deg T = \det T / |\det T|$. Assume $o(G) > 2$. Then n must be even since otherwise $\chi(R^n - 0) = 2$ and $o(G)$ divides $\chi(R^n - 0)$.

Thus $A_T = 1 - (\det T / |\det T|)$ and so A_T can either be 2 or 0. But it cannot be 2 since $o(G) > 2$ and divides A_T . Then $A_T = 0$ and hence $\det T > 0$.

Now we come to the main underlying result for the Borsuk-Ulam theorem.

THEOREM 9. *Let G be a finite group acting based freely on a vector space V and let W be a proper invariant subspace. Then any G -map $f: (V - 0) \rightarrow W$ must contain 0 in its image.*

Proof. Suppose 0 is not in the image of f . Then the composition $(V - 0) \rightarrow W - 0 \hookrightarrow V - 0$ is a G -map and it must be homotopic to a constant, so by Corollary 4 either G is trivial or $f(x) = 0$ for some $x \in V - 0$.

THEOREM 10. *Let ξ be a primitive k th root of unity. Let $f: \mathbf{C}^{n+1} - 0 \rightarrow \mathbf{C}^n$ be any continuous map. Then there is an $x \in \mathbf{C}^n - 0$ such that $\sum_{i=0}^{k-1} \bar{\xi}^i f(\xi^i x) = 0$.*

Proof. This follows from Theorem 9 after noting that $x \mapsto \xi x$ leads to a based free action of \mathbf{Z}_k on $\mathbf{C}^{n+1} - 0$ and that \mathbf{C}^n can be regarded as an invariant subspace and $F(x) = \sum_{i=1}^k \bar{\xi}^i f(\xi^i x)$ is an equivariant map.

Observe that we may let S^{2n+1} be the unit sphere in \mathbf{C}^{n+1} and Theorem 10 guarantees that $\sum_{i=1}^k \bar{\xi}^i f(\xi^i x) = 0$ for some $x \in S^{2n+1}$ and $f: S^{2n+1} \rightarrow \mathbf{C}^n$.

We will call the set $\{x, \xi x, \dots, \xi^{k-1} x\}$ a k -orbit on S^{2n+1} where ξ is a primitive k th root of unity.

COROLLARY 11. *Given k and $f: S^{2n+1} \rightarrow \mathbf{C}^n$, there is a k -orbit whose image lies in a $k - 2$ dimensional complex hyperplane.*

Proof. Choose x so that $\sum \bar{\xi}^i f(\xi^i x) = 0$. Let $x_i = f(\xi^i x)$. Consider the set of $k - 1$ vectors. $\{(x_1 - x_k), \dots, (x_{k-1} - x_k)\}$. These lie in a $k - 2$ dimensional subspace since they are linearly dependent since they satisfy $\sum_{i=1}^k \bar{\xi}^i (x_i - x_k) = 0$. Hence the translation by x_k of this subspace is a $k - 2$ dimensional hyperplane which contains the vectors x_1, \dots, x_k .

COROLLARY 12. *Given k and $f: S^{2n+1} \rightarrow \mathbf{R}^n$, there is a k -orbit whose image under f lies in a $k - 3$ dimensional real hyperplane.*

Proof. The equation $\sum_{i=1}^k \bar{\xi}^i f(\xi^i x) = 0$ gives two equations, one for the real part and one for the imaginary part.

COROLLARY 13. *A map $f: S^{2n+1} \rightarrow \mathbf{R}^n$ carries some 3-orbit to a point.*

COROLLARY 14. *A map $f: S^{2n+1} \rightarrow \mathbf{R}^n$ carries some 4-orbit into one or two points so the two pairs of antipodal points are each carried into a point.*

Proof. Let $x_j = f(\xi^j x)$. Then

$$-ix_1 - x_2 + ix_3 + x_4 = 0.$$

So $x_1 = x_3$ and $x_2 = x_4$.

THEOREM 15. *Suppose p is a prime and $f: S^n \rightarrow \mathbf{R}^r$. If $n \geq r(p - 1)$, then there is a p -orbit whose image is a single point.*

Proof. For $p = 2$ this is the classical Borsuk-Ulam theorem. For $p > 2$ we see that n must be odd and $r(p - 1)$ must be even so we may assume that $n > r(p - 1)$.

Let $n = 2k - 1$. Then we extend $f: S^{2k-1} \rightarrow R^r$ radially to a map $g: C^k \rightarrow R^r$ by letting $g(z) = \|z\| f(z/\|z\|)$. Then we double the dimension of the range and domain and define a map $G: C^{2k} \rightarrow C^r$ by setting $G(z_1, z_2) = g(z_1) + ig(z_2)$ where $(z_1, z_2) \in C^k \oplus C^k \cong C^{2k}$. Now we define a map $F: (C^{2k} - 0) \rightarrow C^{r(p-1)}$ by letting $F(z) = (G(z), G(z)^2, \dots, G(z)^{p-1})$ where we understand that a vector $v = (v_1, \dots, v_r)$ raised to the j th power is the vector $v^j = (v_1^j, \dots, v_r^j)$.

Now since $2k > r(p - 1)$ (because $n \geq r(p - 1)$ by hypothesis), we can apply Theorem 10 which guarantees a solution to the equation

$$\sum_{j=0}^{p-1} \bar{\xi}^j F(\bar{\xi}^j x) = 0 .$$

Let $x_j = G(\bar{\xi}^j x)$ and $x_j^k = (G(\bar{\xi}^j x))^k$. Then we obtain the vector equations

$$\begin{aligned} \sum_{j=0}^{p-1} \bar{\xi}^j x_j &= 0 \\ \sum_{j=0}^{p-1} \bar{\xi}^j x_j^2 &= 0 \\ &\vdots \\ \sum_{j=0}^{p-1} \bar{\xi}^j x_j^{p-1} &= 0 . \end{aligned}$$

These equations are vector equations since x_j^k is a vector. If we look at, say, the first coordinates of the vectors $\{x_j^k\}$ and use x_j^k to denote its own first coordinate then the set of equations can be written in matrix form as

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_{p-1} \\ & & \vdots & \\ x_1^{p-1} & \cdots & x_{p-1}^{p-1} \end{pmatrix} \begin{pmatrix} 1 \\ \bar{\xi} \\ \vdots \\ \bar{\xi}^{p-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} .$$

Now the matrix gives the Vandermonde matrix and

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & \cdots & x_{p-1} \\ \vdots & & \\ x_1^{p-1} & \cdots & x_{p-1}^{p-1} \end{vmatrix} = \prod_{i < j} (x_j - x_i) .$$

Then an easy induction argument shows that the row echelon form of the Vandermonde matrix consists of rows composed of entries which are 0's or 1's. We have the equation

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ & \vdots & & \vdots & & \vdots \\ & & & & & \vdots \end{pmatrix} \begin{pmatrix} 1 \\ \xi^{\bar{1}} \\ \vdots \\ \xi^{\bar{p-1}} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since p is a prime and ξ is a primitive p th root of unity the only sum of these roots which equals zero is the sum of them all, $1 + \xi + \cdots + \xi^{p-1} = 0$. Thus the row echelon form must be

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \\ 0 & \cdots & \cdots & 0 \end{pmatrix}$$

which can only happen if $x_1 = x_2 = \cdots = x_{p-1}$. Thus all the 1st coordinates of the vectors x_1, \cdots, x_{p-1} agree, and similarly all the j th coordinates agree. So

$$G(x) = G(\xi x) = \cdots = G(\xi^{p-1}x) \text{ for some } x \in C^{2k} - 0.$$

This implies that

$$g(z) = g(\xi z) = \cdots = g(\xi^{p-1}z)$$

for some $z \in C^k - 0$ and since g is a radial extension of f we see that f maps an orbit to a single point.

REMARK. Theorem 15 was proved by Munkholm [5] and Nakaoka [6]. In both cases their methods apply in a more general situation. On the other hand Corollary 14 is new and is different from Mumkholm's and Nakaoka's work in that it considers an action of a cyclic group of order 4 and 4 is not a prime.

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