

Pacific Journal of Mathematics

RANK₂ p -GROUPS, $p > 3$, AND CHERN CLASSES

KAHTAN ALZUBAIDY

RANK₂ P -GROUPS, $P > 3$, AND CHERN CLASSES

KAHTAN ALZUBAIDY

In this paper, the integral cohomology ring of a Blackburn's type III rank₂ p -group ($p > 3$) (the rank of a p -group is the rank of a maximal elementary abelian subgroup) is computed and the even dimensional generators are expressed in terms of Chern classes of certain group representations. Then this group satisfies Atiyah's conjecture on the coincidence of topological and algebraic filtrations defined on the complex representation ring of the group.

Let G be any finite group and $R(G)$ the complex representation ring of G . There is a convergent spectral sequence $\{E_r^{i,j}; 2 \leq r \leq \infty\}$ such that

$$E_2^{i,\text{even}} = H^i(G, \mathbf{Z}), \quad E_2^{i,\text{odd}} = 0, \quad \text{and} \quad E_2^{i,j} = R_i^{\text{top}}(G)/R_{i+1}^{\text{top}}(G)$$

where

$$R(G) = R_0^{\text{top}}(G) \supseteq R_1^{\text{top}}(G) \supseteq \dots \supseteq R_{2k-1}^{\text{top}}(G) = R_{2k}^{\text{top}}(G) \supseteq R_{2k+1}^{\text{top}} = \dots$$

is a topologically defined even filtration on $R(G)$. $R(G)$ can be given an algebraic filtration by using the Grothendick operations γ^i ; thus $R_{2k}^r(G)$ is the subgroup generated by monomials $\gamma^{n_1}(\xi_1) \dots \gamma^{n_r}(\xi_r)$, $n_1 + \dots + n_r \geq k$ and ξ_1, \dots, ξ_r elements of the augmentation ideal of $R(G)$. The definition is completed by $R_0^r(G) = R(G)$ and $R_{2k-1}^r(G) = R_{2k}^r(G)$. $R(G)$ is a filtered ring with respect to both filtrations, $R_{2k}^r(G) \subseteq R_{2k}^{\text{top}}(G)$ for all k , and the equality holds for $k = 0, 1$, and 2 [2]. Atiyah conjectured that $R_{2k}^{\text{top}}(G) = R_{2k}^r(G)$, $k \geq 0$ and showed that a group G satisfies this conjecture if the even dimensional subring $H^{\text{even}}(G, \mathbf{Z})$ of the integral cohomology ring $H^*(G, \mathbf{Z})$ is generated by Chern classes of representations of the group G . Though the alternating group on four elements A_4 is a counter example [13], a long standing conjecture is that the two filtrations coincide when G is a finite p -group.

Rank₂ p -groups, $p > 3$, are classified by N. Blackburn [8, staz 14.4] as follows;

- I: Metacyclic p -groups.
- II: $G = \langle A, B, C: A^p = B^p = C^{p^{n-2}} = [A, C] = [C, B] = 1, [B, A] = C^{p^{n-3}} \rangle$.
- III: $G = \langle A, B, C: A^p = B^p = C^{p^{n-2}} = [B, C] = 1, [A, C^{-1}] = B, [B, A] = C^{sp^{n-3}} \rangle$ where $n \geq 4$ and $s = 1$ or some quadratic nonresidue mod p .

In [11] and [12], C.B. Thomas shows that $H^{\text{even}}(G, \mathbf{Z})$ of some split metacyclic p -groups and Blackburn type II groups are generated by Chern classes, and hence they satisfy Atiyah's conjecture. He conjectured that a similar result holds for the remaining rank₂ p -groups, $p > 3$. This would be the best possible result, since there is a 4-dimensional generator of $H^*(3\mathbf{Z}_p, \mathbf{Z})$ which can not come from representations [9, Proposition 4.2]. For a metacyclic p -group in general the conjecture is proved by the author [1]. In this paper the conjecture is proved for Blackburn type III p -groups. The method used is mainly computational and the main result is given as follows:

THEOREM 9.

$H^*(G, \mathbf{Z}) = \mathbf{Z}[\alpha; \mu; \gamma_1, \dots, \gamma_{p-1}; \chi_1, \dots, \chi_{p-2}; \xi, \xi']$ where $\deg \alpha = 2$, $\deg \mu = 3$, $\deg \gamma_i = 2i$, $\deg \chi_i = 2i + 2$, $\deg \xi = \deg \xi' = 2p$ with the relations: $p\alpha = p\mu = sp^{n-3}\gamma_i = p\chi_i = p^{n-1}\xi = p^2\xi' = 0$, $\alpha^p = 0$, $\alpha\gamma_i = \alpha\chi_i = 0$, $\mu^2 = 0$, $\mu\gamma_i = \mu\chi_i = 0$, $\gamma_i\gamma_j = 0$, and $\chi_i\chi_j = 0$ for all i, j .

The method of computation used depends mainly on constructing a free action of the group G on a product of two spheres to determine the order of certain cohomology groups of G together with the method used by G. Lewis to compute the integral cohomology ring of a non-abelian group of order p^3 and exponent p . Lewis' method is based on the calculation of the E_2 terms of spectral sequences of two group extensions and the calculation of E_∞ terms by certain exact sequences of the restriction and corestriction maps. The reader is referred to [9] for the details of the method. $H^{\text{even}}(G, \mathbf{Z})$ is expressed in terms of Chern classes by using a special Riemann-Roch formula [12].

Preliminaries. The group G can be given by either of the following two extensions:

$$(1) \quad 1 \longrightarrow H \longrightarrow G \longrightarrow \mathbf{Z}_p \langle \bar{A} \rangle \longrightarrow 1.$$

Where $H = \mathbf{Z}_p \langle B \rangle + \mathbf{Z}_{p^{n-1}} \langle C \rangle$ is a normal abelian subgroup of index p in G , and

$$(2) \quad 1 \longrightarrow G^1 \longrightarrow G \longrightarrow \mathbf{Z}_p \langle \bar{A} \rangle + \mathbf{Z}_{p^{n-1}} \langle \bar{C} \rangle \longrightarrow 1$$

where $G^1 = \mathbf{Z}_p \langle B \rangle + \mathbf{Z}_p \langle C^{sp^{n-3}} \rangle$ is the commutator subgroup of G . The group G is isomorphic to the group $G' = \langle X, Y, Z: X^{p^{n-2}} = Y^p = [Y, Z] = 1, Z^p = X^{sp}, [X, Z] = Y, [X, Y] = X^{p^{n-3}} \rangle$ where $n \geq 4$ and $s = 1$ or some quadratic non-residue mod p [3, p. 145]. The isomorphism from G' onto G is given by: $X \leftrightarrow AC$, $Y \leftrightarrow B^{-1}$, and $Z \leftrightarrow C$.

$$\begin{aligned}
 X^p &\cong A^p C^p B^{1+2+\dots+(p-1)} C^{p^{n-3}+2p^{n-3}+\dots+(p-1)p^{n-3}} = C^p \cong Z^p \\
 XZ &\cong ACC = CA^{-1}BC = CACB^{-1} \cong ZXY \text{ and } XY \cong ACB^{-1} \\
 &= AB^{-1}C^{-1} = B^{-1}ACC^{p^{n-3}} \cong YX^{1+p^{n-3}}.
 \end{aligned}$$

If s is a quadratic nonresidue mod p , the isomorphism can similarly be defined by: $X \mapsto AC$, $Y \mapsto B^{-s}$, and $Z \mapsto C^s$.

PROPOSITION 1.

G' and hence G acts freely on the product of two spheres $S^{2p-1} \times S^{2p-1}$.

Proof. Let $\lambda: Y \mapsto e^{2\pi i/p} = a, Z \mapsto 1$ and $\lambda': Y \mapsto 1, Z \mapsto e^{2\pi i/p^{n-2}} = b$ be two 1-dimensional representations of the normal abelian subgroup $\langle Y, Z \rangle$ of index p in G' . The direct sum of the induced representations $i_i \lambda$ and $i_i \lambda'$ defines an action of the group G' on the product of two spheres $S^{2p-1} \times S^{2p-1}$. $1 \otimes 1, \bar{X} \otimes 1, \dots, \bar{X}^{p-1} \otimes 1$ forms a basis for the induced modules associated with $i_i \lambda$ and $i_i \lambda'$. By [5, p. 75] the induced representations are explicitly given as follows:

$$i_i \lambda(X) = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad i_i \lambda(Y) = \begin{bmatrix} a & & & & \\ & \cdot & \circ & & \\ & & \cdot & & \\ \circ & & & \cdot & \\ & & & & a \end{bmatrix}, \quad i_i \lambda(Z) = \begin{bmatrix} 1 & & & & \\ & a^{-1} & & & \\ & \cdot & \circ & & \\ & & \cdot & & \\ \circ & & & \cdot & \\ & & & & a^{-p+1} \end{bmatrix},$$

and

$$i_i \lambda'(X) = \begin{bmatrix} 0 & 0 & \dots & 0 & b^p \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}, \quad i_i \lambda'(Y) = \begin{bmatrix} 1 & b^{-1} & & & \\ & \cdot & \circ & & \\ & & \cdot & & \\ \circ & & & \cdot & \\ & & & & b^{-p+1} \end{bmatrix}, \quad i_i \lambda'(Z) = \begin{bmatrix} b & & & & \\ & \cdot & \circ & & \\ & & \cdot & & \\ \circ & & & \cdot & \\ & & & & b \end{bmatrix}.$$

Let $g \in G'$ be any element. Then $g = Z^i Y^j X^k$ where $0 \leq i < p^{n-2}$ and $0 \leq j, k < p$. The action of G' on the first and second sphere is given by:

$$g(x_1, \dots, x_p) = (a^j x_{p-k+1}, a^{j-i} x_{p-k+2}, \dots, a^{j-(p-1)i} x_{p-k})$$

and

$$g(x_1, \dots, x_p) = (b^{k p-i} x_{p-k+1}, a^{-j} b^{(k-1)p-i} x_{p-k+2}, \dots, a^{-(p-1)j} b^{-i} x_{p-k})$$

respectively for every point $(x_1, \dots, x_p) \in S^{2p-1}$. Any element $g \in G'$ which acts freely on $S^{2p-1} \times S^{2p-1}$ must equal to the identity. Thus G' and hence G acts freely on $S^{2p-1} \times S^{2p-1}$. □

The group G acts on the sphere $S^{2p-1} = S^1 * \dots * S^1$ (p -fold join)

by the induced representation of $C \mapsto e^{2\pi i/p^{n-2}}, B \mapsto 1$. By [9, § 6.2] we have the following complex $C'(S^{2p-1}) = \{C'_0 \leftarrow C'_1 \leftarrow \dots \leftarrow C'_{p-1} \leftarrow \dots \leftarrow C'_{2p-1}\}$ where C'_i is a G -free module except for $i = 0, 1, p - 1$, and $2p - 1$. $C'_0 = \mathbf{Z}(G/\langle B \rangle)$. $C'_1 = \mathbf{Z}G/\langle B \rangle \oplus F$, $C'_{p-1} = \mathbf{Z}G/\langle A \rangle \oplus F$, and $C'_{2p-1} = \mathbf{Z}G/\langle A \rangle \oplus F$ for some free G -module F . Consider $0 \leftarrow \mathbf{Z} \leftarrow C'_0 \leftarrow \dots \leftarrow C'_{2p-1} \leftarrow \mathbf{Z} \leftarrow 0$ and let K, L, M, N , and R be the image-kernels at $C'_0, C'_1, C'_{p-2}, C'_{p-1}$ and C'_{2p-1} respectively. Applying the Tate Cohomology to the resulting exact sequences we get the following exact sequences for i odd:

$$\begin{aligned} 0 &\longrightarrow H^i(G, M) \longrightarrow H^{i+1}(G, N) \longrightarrow H^{i+1}(\langle A \rangle, \mathbf{Z}) \longrightarrow H^{i+1}(G, M) \\ &\qquad\qquad\qquad \longrightarrow H^{i+2}(G, N) \longrightarrow 0 \\ 0 &\longrightarrow H^i(G, R) \longrightarrow H^{i+1}(G, \mathbf{Z}) \longrightarrow H^{i+1}(\langle A \rangle, \mathbf{Z}) \longrightarrow H^{i+1}(G, M) \\ &\qquad\qquad\qquad \longrightarrow H^{i+2}(G, N) \longrightarrow 0 \\ 0 &\longrightarrow H^{i+1}(G, \mathbf{Z}) \longrightarrow H^{i+1}(G, K) \longrightarrow H^{i+1}(\langle B \rangle, \mathbf{Z}) \longrightarrow H^{i+1}(G, \mathbf{Z}) \\ &\qquad\qquad\qquad \longrightarrow H^{i+2}(G, N) \longrightarrow 0 \\ 0 &\longrightarrow H^i(G, K) \longrightarrow H^{i+1}(G, L) \longrightarrow H^{i+1}(\langle B \rangle, \mathbf{Z}) \longrightarrow H^{i+1}(G, K) \\ &\qquad\qquad\qquad \longrightarrow H^{i+2}(G, L) \longrightarrow 0 \end{aligned}$$

and $H^i(G, L) \cong H^{i+p-3}(G, M)$, $H^i(G, N) \cong H^{i+p-1}(G, R)$ for all i by dimensional shifting. Similarly, there are exact sequences for i even. Then

$$\begin{aligned} |H^{i+2}(G, \mathbf{Z})| &\leq |H^{i+1}(G, R)| = |H^{i-p+2}(G, N)| \leq p |H^{i-2p+4}(G, K)| \\ &\leq p |H^{i-2p+3}(G, \mathbf{Z})| \leq p^2 |H^{i-2p+2}(G, \mathbf{Z})|. \end{aligned}$$

Thus the following lemma holds

LEMMA 2.

$$|H^{j+2p}(G, \mathbf{Z})| \leq p^2 |H^j(G, \mathbf{Z})| \text{ for all } j. \quad \square$$

Integral cohomolog rings: Consider the spectral sequence of extension (1).

$$E_2^{i,j} = H^i(\mathbf{Z}_p \langle \bar{A} \rangle, H^j(H, \mathbf{Z})).$$

$H^*(H, \mathbf{Z}) = P[\beta, \gamma] \otimes E[\mu]$ where $\deg \beta = \deg \gamma = 2$, $\deg \mu = 3$, and $p\beta = sp^{n-2}\gamma = p\mu = 0$ [1]. β and γ are maximal generators corresponding to $:B \mapsto 1/p, C \mapsto 0$ and $:C \mapsto 1/sp^{n-2}, B \mapsto 0$ respectively. The action of the group $\mathbf{Z}_p \langle \bar{A} \rangle$ on $H^*(H, \mathbf{Z})$ induced by A is given by:

$$\begin{aligned} \beta &\longmapsto \beta + sp^{n-3}\gamma, \gamma \longmapsto \gamma + \beta, \text{ and } \mu \longmapsto \mu. \\ E_2^{*,0} &= H^*(\mathbf{Z}_p \langle \bar{A} \rangle, \mathbf{Z}) = P[\alpha] \end{aligned}$$

where $\deg \alpha = 2$ and $p\alpha = 0$. α is a maximal generator corresponding to $\bar{A} \mapsto 1/p$. $E_2^{0,*} = H^*(H, \mathbf{Z})^{\mathbf{Z}_p\langle \bar{A} \rangle}$ the invariant elements:

$$\begin{aligned} \gamma_1 &= p\gamma, p^2\gamma, \dots, p^{n-3}\gamma; \gamma_2 = p\gamma^2, p^2\gamma^2, \dots, p^{n-3}\gamma^2; \dots; \\ \gamma_p &= p\gamma^p, p^2\gamma^p, \dots, p^{n-3}\gamma^p; \beta^2; \beta^3; \dots; \beta^p; \gamma^p - \gamma\beta^{p-1}; \mu. \end{aligned}$$

PROPOSITION 3. *The low dimensional cohomology groups are*

$$\begin{aligned} H^2(G, \mathbf{Z}) &\cong \mathbf{Z}_{p^{n-3}} \times \mathbf{Z}_p, H^3(G, \mathbf{Z}) \cong \mathbf{Z}_p, \text{ and } H^4(G, \mathbf{Z}) \\ &\cong \mathbf{Z}_{p^{n-3}} \times \mathbf{Z}_p \times \mathbf{Z}_p. \end{aligned}$$

Proof. $H^2(G, \mathbf{Z}) \cong \text{Hom}(G/G^1, \mathbf{Q}/\mathbf{Z}) \cong \mathbf{Z}_{p^{n-3}} \times \mathbf{Z}_p$ where \mathbf{Q} is the field of rationals [4]. By spectral sequence of extension (1) $H^2(G, \mathbf{Z})$ is generated by α and γ_1 . Let $\text{Res}: H^*(G, \mathbf{Z}) \rightarrow H^*(H, \mathbf{Z})$ and $\text{Cor}: H^*(H, \mathbf{Z}) \rightarrow H^*(G, \mathbf{Z})$ be the restriction and corestriction homomorphisms. $\text{Cor}(\text{Res}(\alpha) \cdot \gamma) = \alpha \text{Cor}(\gamma) = 0$ since $\text{Res}_2(\alpha) = 0$. $\text{Res}(\text{Cor} \gamma) = (1 + A + \dots + A^{p-1})\gamma = p\gamma + (1 + 2 + \dots + p - 1)\beta + (sp^{n-3} + \dots + sp^{n-2} - 1)\gamma = p\gamma$. Therefore $\gamma_1 = \text{Cor}(\gamma)$ and $\alpha\gamma_1 = 0$. Similarly, $\gamma_i = \text{Cor}(\gamma^i)$ and $\alpha^i\gamma_i = 0$ for $1 \leq i < p$. By Res - Cor sequences [9, p. 504 (5')]

$$0 \longrightarrow H^2(H, \mathbf{Z})_A \longrightarrow T^3 \longrightarrow H^3(H, \mathbf{Z})^A \longrightarrow 0$$

is exact. $|H^2(H, \mathbf{Z})_A| = p^{n-3}$ and $|H^3(H, \mathbf{Z})^A| = p$. Then $|T^3| = p^{n-3} \times p$. $0 \rightarrow H^3(G, \mathbf{Z}) \rightarrow T^3 \xrightarrow{\tau} H^2(G, \mathbf{Z}) \xrightarrow{\cup \alpha} H^4(G, \mathbf{Z})$ is exact [9, p. 504 (4')] $|I_m \tau| = |\text{Ker} \cup \alpha| = p^{n-3}$ since $\alpha\gamma_1 = 0$. $|H^3(G, \mathbf{Z})| = |T^3|/|\text{Im} \tau| = p$. Therefore $H^3(G, \mathbf{Z}) \cong \mathbf{Z}_p$ and generated by μ since

$$\text{Res}_3: H^3(G, \mathbf{Z}) \longrightarrow H^3(H, \mathbf{Z})$$

is an epimorphism. The following diagrams is commutative and the top row is exact [9, p. 504 (4)].

$$\begin{array}{ccccccc} H^2(H, \mathbf{Z}) & \xrightarrow{\text{Cor}} & H^2(G, \mathbf{Z}) & \longrightarrow & H^3(G, K) & \xrightarrow{\theta} & H^3(H, \mathbf{Z}) \xrightarrow{\text{Cor}} H^3(G, \mathbf{Z}) \\ \cong \downarrow & & & & \downarrow \cong & & \\ \text{Hom}(H, \mathbf{Q}/\mathbf{Z}) & \xrightarrow{\text{Cor}} & \text{Hom}(\langle \bar{A}, \bar{C} \rangle, \mathbf{Q}/\mathbf{Z}) & & & & \end{array}$$

where $K = \text{Ker} \{ \mathbf{Z}_p\langle A \rangle \rightarrow \mathbf{Z} \}$. $\text{Cor}: H^3(H, \mathbf{Z}) \rightarrow H^3(G, \mathbf{Z})$ is zero since $\text{Cor} \mu = \text{Cor Res} \mu = p\mu = 0$. $|\text{Im Cor}_2| = p^{n-3}$ since $\text{Cor} \gamma = \gamma_1$ and $\text{Cor} \beta = 0$ because $\text{Cor}(\text{Res}(\alpha) \cdot \beta) = \alpha \text{Cor} \beta = 0$. Then $|H^3(G, K)| = |\text{Im} \theta| \cdot |H^2(G, \mathbf{Z})|/|\text{Im Cor}_2| = p \times p$. The following sequence is exact

$$H^3(G, \mathbf{Z}) \xrightarrow{\text{Res}} H^3(H, \mathbf{Z}) \longrightarrow H^3(G, K) \longrightarrow H^4(G, \mathbf{Z}) \xrightarrow{\text{Res}} H^4(H, \mathbf{Z}).$$

Res_3 is an epimorphism and $|\text{Im Res}_4| = p^{n-3}$ since $\text{Res} \alpha = 0$. Then

$|H^4(G, \mathbf{Z})| = p^{n-3} \times p \times p$. Therefore $H^4(G, \mathbf{Z}) \cong \mathbf{Z}_{p^{n-3}}\langle\gamma_2\rangle + \mathbf{Z}_p\langle\alpha^2\rangle + \mathbf{Z}_p\langle\chi\rangle$ where χ is an additional generator. \square

Consider now the spectral sequence of extension (2).

$$E_2^{i,j} = H^i(\mathbf{Z}_p\langle\bar{A}\rangle + \mathbf{Z}_{p^{n-3}}\langle\bar{C}\rangle, H^j(G^1, \mathbf{Z})) .$$

$E_2^{*,0} = H^*(\mathbf{Z}_p \times \mathbf{Z}_{p^{n-3}}, \mathbf{Z}) = P[\alpha, \gamma] \otimes E[\delta]$ where $\deg \alpha = \deg \gamma = 2$, $\deg \delta = 3$, and $p\alpha = sp^{n-3}\gamma = p\delta = 0$. $E_2^{0,*} = H^*(G^1, \mathbf{Z})^{\langle\bar{A}, \bar{C}\rangle} = P[\beta^2, \beta^3, \dots; p^{n-3}\gamma]$.

The odd generators in the exterior part vanished since they are trivial under the action of $\langle\bar{A}, \bar{C}\rangle$. By comparing the two spectral sequences $\gamma^i \leftrightarrow \gamma_i$ for $1 \leq i < p$.

$$E_2^{*,2j} = H^*(\mathbf{Z}_p \times \mathbf{Z}_{p^{n-3}}, \mathbf{Z}_p \times \mathbf{Z}_p) \cong H^*(\mathbf{Z}_{p^{n-3}}, \mathbf{Z}_p \times \mathbf{Z}_p) \otimes H^*(\mathbf{Z}_p, \mathbf{Z}_p \times \mathbf{Z}_p)$$

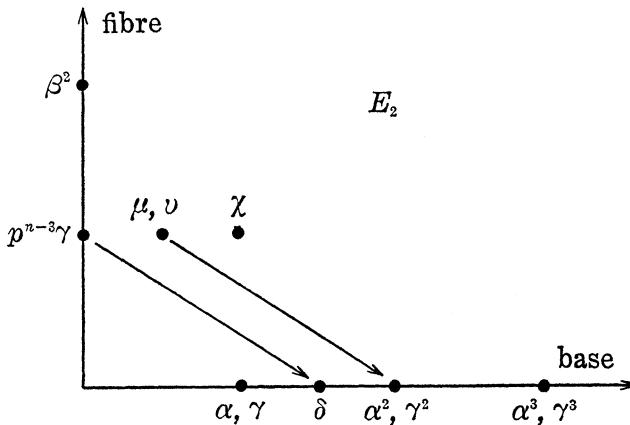
by Künneth formula. This induces a horizontal multiplication

$$\circ: E_2^{i,2j} \times E_2^{k,2j} \longrightarrow E_2^{i+k,2j}, j > 0$$

and

$$\beta: E_2^{i,j} \longrightarrow E_2^{i,j+2}$$

is monomorphism for $j \geq 2$ and isomorphism for $j > 0$ [4]. Let $\mu, \nu \in E_2^{1,2}$ be two independent generators. Then $\chi = \mu \circ \nu \in E_2^{2,2}$ by horizontal multiplication. Since the odd rows are zero, then $E_2 = E_3$. From the cohomology groups at the low dimensions $d_3(\alpha) = d_3(\gamma) = d_3(\mu) = d_3(\chi) = 0$. Others are easily deduced from the E_2 -diagram. Since $\gamma \leftrightarrow \gamma_1$, then $\alpha\gamma_1 = \delta\gamma_1 = \mu\gamma_1 = \nu\gamma_1 = \chi\gamma_1 = 0$. Then the additive structure of E_2 can be given are follows:



LEMMA 4.

$$E_2^{2i,0} = \mathbf{Z}_p\langle\alpha^i\rangle + \mathbf{Z}_{sp^{n-3}}\langle\gamma^i\rangle, E_2^{2i,4} = \mathbf{Z}_p\langle\chi\alpha^{i-1}\rangle + \mathbf{Z}_p\langle\beta^2\alpha^i\rangle ,$$

$$E_2^{2i,2} = \mathbb{Z}_p \langle \chi \alpha^{i-1} \rangle, E_2^{2,2} = \mathbb{Z}_p \langle \chi \rangle; E_2^{2i+1,0} = \mathbb{Z}_p \langle \delta \alpha^{i-1} \rangle, \\ E_2^{2i+1,2} = \mathbb{Z}_p \langle \alpha^i \mu \rangle + \mathbb{Z}_p \langle \alpha^i \nu \rangle, \text{ and } E_2^{*,2j+1} = 0 (j > 0).$$

The other terms are given by periodicity $E_2^{*,4} = E_2^{*,6} = \dots$. □

LEMMA 5. γ^p and β^p are universal cycles and hence $\beta^p: E_2^{i,j} \rightarrow E_2^{i,j+2p} \rightarrow$ is an isomorphism for $j > 0$.

Proof. By double cosets formula for the generalization of corestriction \mathcal{N} [6, Theorem 3]

$$\begin{aligned} \text{Res}_H \mathcal{N}(\gamma) &= \prod_{i=0}^{p-1} \left(\gamma - i\beta - \frac{1}{2}i(i-1)sp^{n-3}\gamma \right) \\ &= \prod_{i=0}^{p-1} (\gamma - i\beta) + \sum_{j=0}^{p-1} \left(\prod_{i=0}^{p-1} (\hat{\gamma} - i\beta) \frac{1}{2}j(j-1)sp^{n-3}\gamma \right) \\ &= \prod_{i=0}^{p-1} (\gamma - i\beta) = \gamma^p - \gamma\beta^{p-1} \end{aligned}$$

where $\hat{}$ means a deleted term. $\text{Res}_H \mathcal{N}(\beta) = \prod_{i=0}^{p-1} (\beta - isp^{n-3}\gamma) = \beta^p$. Therefore γ^p and β^p are universal cycles [9, Corollary II]. □

The additive structure of E_4 can now be given as follows:

LEMMA 6.

$$E_4^{2i,0} = \mathbb{Z}_p \langle \alpha^i \rangle + \mathbb{Z}_p \langle \gamma^i \rangle; E_4^{2,2j} = \mathbb{Z}_p \langle \chi \beta^{j-1} \rangle, j > 0; E_4^{2i,2j} = 0, \\ j \not\equiv 0(p), j > 0, i \neq 1; E_4^{i,2j} = 0, j \not\equiv 1(p), j > 1; E_4^{2i+1,2j} = 0, \\ j \not\equiv 0(P)j \neq 1, i > 0; E_4^{2i+1,2} = \mathbb{Z}_p \langle \alpha^i \mu \rangle;$$

and

$$E_4^{2i+1,2(p-1)} = \mathbb{Z}_p \langle \delta \alpha^{i-1} \beta^{p-1} \rangle.$$

The other terms are given by periodicity $E_4^{*,j} = E_4^{*,j+2p} = \dots$. □

Then $E_4 = E_\infty$ in dimensions $\leq 2p$.

LEMMA 7.

$$|H^{2p}(G, \mathbb{Z})| = p^{n+1}.$$

Proof. G acts freely on the product of the two spheres $S^{2p-1} \times S^{2p-1}$ by Proposition 1. Then by [10, Corollary 2.7] the following sequence is exact:

$$\begin{aligned} 0 \longrightarrow H^{2p-1}(G, \mathbb{Z}) \longrightarrow H^{2p}(G, \mathbb{Z}) \longrightarrow \mathbb{Z}_{p^n} \times \mathbb{Z}_{p^n} \longrightarrow H^{2p}(G, \mathbb{Z}) \\ \longrightarrow H^{2p-1}(G, \mathbb{Z}) \longrightarrow 0. \end{aligned}$$

Since $H^{2p-1}(G, \mathbb{Z}) = \mathbb{Z}_p \langle \alpha^{p-2} \mu \rangle$ by the previous spectral sequence,

then $|H^{2p}(G, \mathbf{Z})| = p^{n+1}$. □

By Res-Cor sequence

$$0 \longrightarrow \mathbf{Z}_{p^{n-2}} \longrightarrow H^{2i}(H, \mathbf{Z}) \longrightarrow H^{2i}(G, K) \longrightarrow H^{2i+1}(G, \mathbf{Z}) \longrightarrow \mathbf{Z}_p \longrightarrow 0$$

is exact where $K = \text{Ker}\{\mathbf{Z}\langle A \rangle \rightarrow \mathbf{Z}\}$. $|H^{2i}(H, \mathbf{Z})| = p^{i+n-2}$ and $|H^{2i+1}(G, \mathbf{Z})| = p$. Therefore $|H^{2i}(G, K)| = p^i$. If $\text{Cor}_{2i} = 0$, then $0 \rightarrow H^{2i-1}(G, \mathbf{Z}) \rightarrow H^{2i}(G, K) \rightarrow H^{2i}(H, \mathbf{Z}) \rightarrow 0$ is exact. Therefore $|H^{2i}(G, K)| = p^{i+n-1}$ which is a contradiction. Then $\text{Cor}(\beta^i) \neq 0$ for $2 \leq i < p$. Similarly, we can prove the following:

LEMMA 8. $\text{Cor}(\beta^i) \neq 0$ for $2 \leq i \leq p$ and $\text{Cor}(\gamma^p) \neq 0$. □

Let $\xi = \mathcal{N}(\gamma)$ and $\xi' = \mathcal{N}(\beta)$ $\text{Res}_H \mathcal{N}(\gamma) = \gamma^p - \gamma\beta^{p-1}$ and $\text{Res}_H \mathcal{N}(\beta) = \beta^p$. $\text{Cor Res } \mathcal{N}(\gamma) = p\mathcal{N}(\gamma) = \text{Cor}(\gamma^p) \neq 0$ and $\text{Cor Res } \mathcal{N}(\beta) = p\mathcal{N}(\beta) = \text{Cor}(\beta^p) \neq 0$. Therefore $\mathcal{N}(\gamma)$ and $\mathcal{N}(\beta)$ have orders p^{n-1} and p^2 respectively and are elements in $H^{2p}(G, \mathbf{Z})$. Since $|H^{2p}(G, \mathbf{Z})| = p^{n+1}$ by Lemma 7, then $\alpha^p = 0$ in $H^*(G, \mathbf{Z})$.

Let $\chi_i = \text{Cor}(\beta^{i+1})$, $1 \leq i < p - 1$. χ_i is not a polynomial in α and γ since $\alpha \text{Cor}(\beta^p) = 0$ and $\text{Res Cos}(\beta^p) = 0$. Therefore $H^{2i+2}(G, \mathbf{Z}) = \mathbf{Z}_p\langle \chi_i \rangle + \mathbf{Z}_p\langle \alpha^{i+1} \rangle + \mathbf{Z}_{sp^{n-3}}\langle \gamma^{i+1} \rangle$.

By using $\text{Cor}(\text{Res a.b}) = \text{a. Cor } b$, we have $\alpha\chi_i = \mu\chi_i = \chi_i\chi_j = 0$ and $\gamma_i\chi_j = 0$ since $\text{Res } \chi_i = 0$. If $\gamma_i\gamma_j = e \alpha^{i+j}$, then $\alpha\gamma_i\gamma_j = e\alpha^{i+j+1} = 0$. Then $e = 0$ and hence $\gamma_i\gamma_j = 0$. Thus we have:

THEOREM 9. *The integral cohomology ring $H^*(G, \mathbf{Z}) = \mathbf{Z}[\alpha; \mu; \gamma_1, \dots, \gamma_{p-1}; \chi_1, \dots, \chi_{p-2}, \xi, \xi']$ where $\deg \alpha = 2$, $\deg \mu = 3$, $\deg \gamma_i = 2i$, $\deg \chi_i = 2i + 2$, $\deg \xi = \deg \xi' = 2p$ with the relations $p\alpha = p\mu = sp^{n-3}\gamma_i = p\chi_i = p^{n-1}\xi = p^2\xi' = 0$, $\alpha^p = 0$, $\alpha\gamma_i = \alpha\chi_i = 0$, $\mu^2 = 0$, $\mu\gamma_i = \mu\chi_i = 0$, $\gamma_i\gamma_j = 0$, and $\chi_i\chi_j = 0$ for all i and j . □*

$H^{\text{even}}(G, \mathbf{Z})$ is generated by $\alpha, \gamma_1, \dots, \gamma_{p-1}, \chi_1, \dots, \chi_{p-2}, \xi, \xi'$. $\alpha = c_1(\hat{\alpha})$ is the first Chern class of the 1-dimensional representation given by $\hat{\alpha}(A) = 1/p$. $\gamma_i = \text{Cor}(\gamma^i)$ for $1 \leq i < p$ and $\chi_i = \text{Cor}(\beta^{i+1})$, $1 \leq i \leq p - 2$. Then by using a special Reimann-Rock formula [12, Theorem 2] we get: $\text{Cor}(\gamma^i) = S_i(i, \hat{\gamma})$, $2 \leq i \leq p - 2$; $\text{Cor}(\gamma^{p-1}) = S_{p-1}(i, \hat{\gamma}) + (p - 1)\alpha^{p-1}$ and $\text{Cor}(\beta^i) = S_i(i, \hat{\beta})$, $2 \leq i \leq p - 2$; $\text{Cor}(\beta^{p-1}) = S_{p-1}(i, \hat{\beta}) + (p - 1)\alpha^{p-1}$ where α is the inflation of the generator of $H^2(\langle \hat{A} \rangle, \mathbf{Z})$ and $\hat{\beta}, \hat{\alpha}$ are two representations given by $\hat{\beta}: B \rightarrow 1/p$, $C \rightarrow 0$ and $\hat{\gamma}, B \rightarrow 0$; $C \rightarrow 1/sp^{n-2}$. The two generators $\xi = \mathcal{N}(\gamma) = c_p(\hat{\gamma})$ and $\xi' = \mathcal{N}(\beta) = c_p(\hat{\beta})$ are given in terms of p th Chern classes [7, Theorem 4]. By [2-Appendix] we have:

THEOREM 10. $H^{\text{even}}(G, \mathbf{Z})$ is generated by Chern classes and

hence G satisfies Atiyah's Conjecture. □

The author is greatly indebted to Dr. C. B. Thomas, who, as his former research supervisor, gave invaluable assistance during the preparation of this work at Mathematics Department, University College London. The author also wishes to thank the referee for several helpful suggestions.

REFERENCES

1. K. Alzubaidy, *Metacyclic P-groups and Chern classes*, Illinois J. Math., **26** (3) (1982), 423-431.
2. M. F. Atiyah, *Characters and Cohomology of finite groups*, I.H.E.S. Publ. Math., **9** (1961), 23-64.
3. W. Burnside, *Theory of groups of finite order*, 2nd edition, Dover Publ. 1955.
4. H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton, N.J., 1956.
5. C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebra*, Inter-Science, New York, 1962.
6. L. Evans, *A generalization of the transfer map in cohomology of groups*, Trans. Amer. Math. Soc., **108** (1963), 54-65.
7. ———, *On the Chern classes of representations of finite groups*, Trans. Amer. Math. Soc., **115** (1965), 180-193.
8. B. Huppert, *Endliche Gruppen I*, Die Grundlehren der Math. Wiss., **134** Springer-Verlag, Berlin, 1968.
9. G. Lewis, *The integral cohomology rings of groups of order p^3* , Trans. Amer. Math. Soc., **132** (1968), 501-529.
10. ———, *Free actions on $S^n \times S^n$* , Trans. Amer. Math. Soc., **132** (1968), 531-540.
11. C. B. Thomas, *Chern classes and metacyclic p-groups*, Mathematika, **18** (1971), 169-200.
12. ———, *Rieman-Roch formulae for group representation*, Mathematika, **20** (1973), 253-262.
13. E. A. Weiss, *Kohomologiering und darstellungsring endlicher Gruppen*, Bonner Math. Schriften, **36** (1969).

Received September 22, 1980 and in revised form September 23, 1981.

GARYOUNIS UNIVERSITY BENGHAZI
LIBYA

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)

University of California
Los Angeles, CA 90024

HUGO ROSSI

University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ARTHUR AGUS

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF AAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies,

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966, Regular subscription rate: \$114.00 a year (6 Vol., 12 issues). Special rate: \$57.00 a year to individual members of supporting institution.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 103, No. 2

April, 1982

Alberto Alesina and Leonede De Michele , A dichotomy for a class of positive definite functions	251
Kahtan Alzubaidy , Rank ₂ p -groups, $p > 3$, and Chern classes	259
James Arney and Edward A. Bender , Random mappings with constraints on coalescence and number of origins	269
Bruce C. Berndt , An arithmetic Poisson formula	295
Julius Rubin Blum and J. I. Reich , Pointwise ergodic theorems in l.c.a. groups	301
Jonathan Borwein , A note on ε -subgradients and maximal monotonicity	307
Andrew Michael Brunner, Edward James Mayland, Jr. and Jonathan Simon , Knot groups in S^4 with nontrivial homology	315
Luis A. Caffarelli, Avner Friedman and Alessandro Torelli , The two-obstacle problem for the biharmonic operator	325
Aleksander Całka , On local isometries of finitely compact metric spaces	337
William S. Cohn , Carleson measures for functions orthogonal to invariant subspaces	347
Roger Fenn and Denis Karmen Sjerve , Duality and cohomology for one-relator groups	365
Gen Hua Shi , On the least number of fixed points for infinite complexes	377
George Golightly , Shadow and inverse-shadow inner products for a class of linear transformations	389
Joachim Georg Hartung , An extension of Sion's minimax theorem with an application to a method for constrained games	401
Vikram Jha and Michael Joseph Kallaher , On the Lorimer-Rahilly and Johnson-Walker translation planes	409
Kenneth Richard Johnson , Unitary analogs of generalized Ramanujan sums	429
Peter Dexter Johnson, Jr. and R. N. Mohapatra , Best possible results in a class of inequalities	433
Dieter Jungnickel and Sharad S. Sane , On extensions of nets	437
Johan Henricus Bernardus Kemperman and Morris Skibinsky , On the characterization of an interesting property of the arcsin distribution	457
Karl Andrew Kosler , On hereditary rings and Noetherian V -rings	467
William A. Lampe , Congruence lattices of algebras of fixed similarity type. II	475
M. N. Mishra, N. N. Nayak and Swadeenananda Pattanayak , Strong result for real zeros of random polynomials	509
Sidney Allen Morris and Peter Robert Nickolas , Locally invariant topologies on free groups	523
Richard Cole Penney , A Fourier transform theorem on nilmanifolds and nil-theta functions	539
Andrei Shkalikov , Estimates of meromorphic functions and summability theorems	569
László Székelyhidi , Note on exponential polynomials	583
William Thomas Watkins , Homeomorphic classification of certain inverse limit spaces with open bonding maps	589
David G. Wright , Countable decompositions of E^n	603
Takayuki Kawada , Correction to: "Sample functions of Pólya processes"	611
Z. A. Chanturia , Errata: "On the absolute convergence of Fourier series of the classes $H^\omega \cap V[v]$ "	611