

# Pacific Journal of Mathematics

**POINTWISE ERGODIC THEOREMS IN L.C.A. GROUPS**

JULIUS RUBIN BLUM AND J. I. REICH

## POINTWISE ERGODIC THEOREMS IN l.c.a. GROUPS

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Let  $G$  be a l.c.a. group and  $\{T_g\}$  be a representation of  $G$  such that each  $T_g$  is a measure-preserving transformation on some probability space  $(\Omega, \mathcal{F}, P)$ . Let  $\{\mu_n\}$  be a sequence of probability measures on  $G$ . We are interested in the a.e. convergence or summability of  $\int_G f(T_g w) d\mu_n(g)$ , for  $f \in L_1(\Omega)$ . Some examples and counterexamples are given, and some partial results are obtained.

1. Let  $G$  be a locally compact abelian group (l.c.a.), and let  $\hat{G}$  be its dual group.  $\hat{G}$  consists of all continuous homomorphisms of  $G$  of absolute value one.  $\hat{G}$  is again l.c.a. Denote by  $\hat{G}_d$  the l.c.a. group obtained from  $\hat{G}$  by endowing it with the discrete topology, and by  $\bar{G}$  the dual of  $\hat{G}_d$ .  $\bar{G}$  is a compact group known as the Bohr compactification of  $G$ , and  $G$  is a dense subset of  $\bar{G}$ . If  $m$  is normalized Haar measure on  $\bar{G}$ , then  $m(G) = 0$ . Note that  $G$  and  $\bar{G}$  have the same characters, namely the elements of  $\hat{G}$ . Now if  $\mu$  is a finite measure on the Borel sets of  $G$ , we may without loss of generality consider it to be a measure on  $\bar{G}$ , for if  $B$  is a Borel subset of  $\bar{G}$  we can define  $\mu(B) = \mu(B \cap G)$ . If  $\{\mu_n, n = 1, 2, \dots\}$  is a sequence of probability measures on  $G$ , we shall call it an ergodic sequence if  $\mu_n$ , considered as a sequence of measures on  $\bar{G}$ , converges weakly to  $m$ , the Haar measure on  $\bar{G}$ . The reason for this terminology is that it was shown in Blum and Eisenberg, [2], that if  $U = \{U_g\}$  is a strongly continuous unitary representation of  $G$  on some Hilbert space  $H$ , and if we consider the sequence  $\int_G U_g f d\mu_n(g)$ , which is defined weakly for each  $f \in H$ , then if  $\{\mu_n\}$  is an ergodic sequence we have a strong limit  $\int_G U_g f d\mu_n(g) = Pf$  for every  $f \in H$ , where  $P$  is the orthogonal projection on the closed linear subspace of  $H$  consisting of those elements of  $H$  invariant under each  $U_g$ . Moreover if this version of the mean ergodic theorem is to hold for every strongly continuous unitary representation of  $G$ , then it is necessary that  $\{\mu_n\}$  be ergodic.

In this paper we shall be concerned with pointwise ergodic theorems.

Let  $(\Omega, \mathcal{F}, P)$  be a probability space and let  $\{T_g\}$  be a group of measure-preserving transformations of  $\Omega$  into itself such that the corresponding unitary operators  $U_g$  on  $L_2(\Omega)$  are a strongly continuous representation of  $G$ . We show by a simple example that the pointwise ergodic theorem does not hold for every ergodic sequence

$\{\mu_n\}$  on  $G$ . We then show that for certain ergodic sequences the pointwise ergodic theorem does hold for a set which is dense in  $L_1(\Omega)$ , but not necessarily for all of  $L_1(\Omega)$ . Finally we exhibit certain ergodic sequences for which the pointwise ergodic theorem does hold.

2. Let  $G = Z$  and for each positive integer  $n$  define  $\mu_n$  by putting mass  $1/[V\sqrt{n}]$  on the integers  $n+1, \dots, n+[V\sqrt{n}]$ , where  $[x]$  is the longest integer not exceeding  $x$ . Now let  $\hat{\mu}_n(\alpha)$  be the Fourier transform of  $\mu_n$  for  $0 \leq \alpha < 2\pi$ . Then  $\hat{\mu}_n(\alpha) = (1/[V\sqrt{n}]) \sum_{j=n+1}^{n+[V\sqrt{n}]} e^{ij\alpha}$  and  $\hat{\mu}_n(0) \rightarrow 1$  while  $\hat{\mu}_n(\alpha) \rightarrow 0$  for  $0 < \alpha < 2\pi$ . But if  $m$  is Haar measure on  $Z$  then  $\hat{m}(\alpha) = \begin{cases} 1, & \alpha = 0 \\ 0, & 0 < \alpha < 2\pi \end{cases}$ . Thus  $\{\mu_n\}$  is an ergodic sequence. Now let  $(\Omega, \mathcal{F}, P)$  be a probability space, and let  $T$  be an invertible ergodic measure-preserving transformation of  $\Omega$  onto itself. It was shown by Akcoglu and Del Junco, [1], that there exists a set  $A \in \mathcal{F}$  with  $0 < P(A) < 1/2$ , and a set  $B \in \mathcal{F}$  with  $P(B) > 1/2$  such that for  $w \in B$  we have

$$\frac{1}{[V\sqrt{n}]} \sum_{j=n+1}^{n+[V\sqrt{n}]} \chi_A(T^{-j}w) = 1$$

infinitely often, where  $\chi_A$  is the indicator of  $A$ . In fact by a slight modification of their argument and by taking lim sup one can make  $P(A)$  arbitrary small and  $P(B) = 1$ . In any case it is clear that the individual ergodic theorem does not hold for this ergodic sequence  $\{\mu_n\}$ .

3. Now suppose  $\{\mu_n\}$  is a sequence of probability measures on  $G$ , each of which is absolutely continuous with respect to the Haar measure on  $G$ . Denote by  $\varphi_n$  its density with respect to Haar measure, i.e.,  $\mu_n(A) = \int_A \varphi_n(g) dg$ , for each Borel subset  $A$  of  $G$ , where  $dg$  is Haar measure on  $G$ . For  $\gamma \in \hat{G}$  we shall write  $\hat{\varphi}_n(\gamma)$  for the Fourier transform of  $\mu_n$ , i.e.,  $\hat{\varphi}_n(\gamma) = \int_G \langle g, \gamma \rangle \varphi_n(g) dg$ . Here  $\langle g, \gamma \rangle$  is the usual notation for the character  $\gamma$  evaluated at  $g$ .

Then we have

**THEOREM 1.** *Suppose for each compact subset  $K$  of  $\hat{G}$  with  $0 \notin K$ , and every sequence  $\{\gamma_n\}$  with  $\gamma_n \in K$  for all  $n$  we have  $\sum_{n=1}^{\infty} |\hat{\varphi}_n(\gamma_n)|^2 < \infty$ . Then there exists a set  $D$ , dense in  $L_2(\Omega)$ , and hence in  $L_1(\Omega)$ , such that for  $f \in D$  we have  $\lim_{n \rightarrow \infty} \int_G f(T_g w) d\mu_n(g) = Pf$  a.e.*

Here  $P$  is as above the orthogonal projection of  $f$  onto the closed subspace of  $L_2(\Omega)$  of elements invariant under each  $U_g$ . Before proving the theorem we exhibit a class of examples for which the

hypotheses of the theorem are satisfied. Let  $\mu$  be a probability measure on  $G$  which is absolutely continuous with respect to Haar measure on  $G$ , and suppose  $|\hat{\mu}(\gamma)| < 1$  for  $\gamma \neq 0$ . Then if we let  $\mu_n$  be the  $n$ -fold convolution of  $\mu$  with itself it is easily verified that the hypotheses of the theorem are satisfied. For example, if  $G = \mathbb{Z}$  and  $\mu$  puts mass  $p$  on 0 and mass  $1 - p$  on 1, where  $0 < p < 1$ , then  $\hat{\mu}(\gamma) = p + (1 - p)e^{i\gamma}$  for  $0 < \gamma < 2\pi$  so that  $|\hat{\mu}(\gamma)| < 1$ . In this case  $\mu_n$  is the binomial distribution with parameters  $n$  and  $p$ .

*Proof of the theorem.* Let  $E(\cdot)$  be the spectral measure associated with  $\{U_g\}$ . Note that  $E(\{0\})f = Pf$  for all  $f \in L_2(\Omega)$ . Denote by  $\mathcal{E}$  the closed subspace of  $L_2(\Omega)$  spanned by the eigenfunctions of  $\{U_g\}$ .

Let  $f \in \mathcal{E}^\perp$ . Then  $(E(d\gamma)f, f)$  is a continuous, regular Borel measure on  $\hat{G}$  and for given  $\varepsilon > 0$  we can find a compact set  $K \subset \hat{G}$  with  $0 \notin K$  such that

$$\|E(K)f - f\|_2 < \varepsilon.$$

Since  $PE(K)f = 0$  we shall show that  $\lim_{n \rightarrow \infty} \int_G E(K)f(T_g w) d\mu_n(g) = 0$  a.e. Henceforth we shall write this integral as  $\int_G E(K)f(T_g w) \varphi_n(g) dg$ . Let  $\delta > 0$  and define  $A_{n,\delta} = \left\{ w \left| \left| \int_G E(K)f(T_g w) \varphi_n(g) dg \right| > \delta \right. \right\}$ . Then

$$\begin{aligned} P\{A_{n,\delta}\} &\leq \frac{1}{\delta^2} \left\| \int_G E(K)f(T_g w) \varphi_n(g) dg \right\|_2^2 \\ &= \frac{1}{\delta^2} \int_G \int_{\hat{G}} \langle g, \gamma \rangle E d(\gamma) (E(K)f) \varphi_n(g) dg \Big|_2^2 \\ &= \frac{1}{\delta^2} \int_{\hat{G}} |\hat{\varphi}_n(\gamma)|^2 (E d(\gamma) E(K)f, E(K)f) \\ &= \frac{1}{\delta^2} \int_K |\hat{\varphi}_n(\gamma)|^2 (E(d\gamma)f, f) \leq \frac{1}{\delta^2} \max_{\gamma \in K} |\hat{\varphi}_n(\gamma)|^2 \|f\|_2^2. \end{aligned}$$

But  $\hat{\varphi}_n(\gamma)$  is continuous and therefore there exists  $\gamma_n \in K$  such that  $|\hat{\varphi}_n(\gamma_n)|^2 = \max_{\gamma \in K} |\hat{\varphi}_n(\gamma)|^2$ . Hence by the hypotheses we have  $\sum_{n=1}^\infty P(A_{n,\delta}) < \infty$ . It follows from the Borel-Cantelli lemma that  $P(A_\delta) = 1$  where  $A_\delta = \{w | w \text{ is in at most finitely many } A_{n,\delta}\}$  and similarly if  $A = \bigcap_{k=1}^\infty A_{1/k}$ , then  $P\{A\} = 1$ . But for  $w \in A$  we have  $\lim_{n \rightarrow \infty} \int_G E(K)f(T_g w) \varphi_n(g) dg = 0 = PE(K)f$ . Thus we approximate each  $f \in \mathcal{E}^\perp$  by a function for which the theorem holds.

If  $f \in \mathcal{E}$  and  $\varepsilon > 0$  there exist finitely many eigenfunctions, say  $f_{r_1}, \dots, f_{r_M}$  with  $\gamma_j \neq 0, j = 1, \dots, M$  such that

$$\left\| E(\{0\})f + \sum_{j=1}^M f_{r_j} - f \right\|_2 < \varepsilon.$$

Then

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_G \left[ E(\{0\})f + \sum_{j=1}^M f_{r_j}(T_g w) \right] \varphi_n(g) dg \\ = Pf + \sum_{j=1}^M \lim_{n \rightarrow \infty} \int_G \langle g, \gamma_j \rangle f_{r_j}(w) \varphi_n(g) dg \\ = Pf + \sum_{j=1}^M f_{r_j}(w) \lim_{n \rightarrow \infty} \hat{\varphi}_n(\gamma_j) = P(f) \end{aligned}$$

since clearly  $\lim_{n \rightarrow \infty} |\hat{\varphi}_n(\gamma_j)| = 0, j = 1, \dots, M$ .

This concludes the proof of the theorem.

4. As was mentioned earlier the theorem does not hold in general for all of  $L_1(\Omega)$ , or indeed for all bounded functions. As an example, let  $G = Z$  and let  $\mu_n$  be  $B(n, p)$ , the binomial distribution on  $0, 1, \dots, n$  with  $0 < p < 1$ . Let  $(c_j, j = 0, 1, 2, \dots)$  be a sequence of 0's and 1's. Then it was shown by Diaconis and Stein, [4], that  $\lim_{n \rightarrow \infty} \sum_{j=0}^n c_j \binom{n}{j} p^j (1-p)^{n-j} = L$  if and only if for every  $\varepsilon > 0$  we have  $\lim_{n \rightarrow \infty} 1/\varepsilon \sqrt{n} \sum_{j=n+1}^{n+\lceil \varepsilon \sqrt{n} \rceil} c_j = L$ . Now if  $T$  is invertible and ergodic we saw that we can choose a set  $A$  with  $0 < P(A) < 1/2$  and a set  $B$  with  $P(B) = 1$  such that for  $w \in B$  we have

$$\limsup_n \frac{1}{\lfloor \sqrt{n} \rfloor} \sum_{j=n+1}^{n+\lceil \sqrt{n} \rceil} \chi_A(T^{-j}w) = 1.$$

By choosing  $c_j = \chi_A(T^{-j}w)$  it follows from the Diaconis-Stein result with  $\varepsilon = 1$  that the individual ergodic theorem fails to hold for this sequence  $\{\mu_n\}$ . As was mentioned earlier, this sequence does satisfy the hypotheses of the theorem.

It is of some interest to point out that in the case when  $\mu_n$  is the  $n$ -fold convolution of a measure  $\mu$  on  $G$ , we do have a version of the individual ergodic theorem as follows:

**THEOREM 2.** *Let  $\mu$  be a probability measure on  $G$ , and let  $\{T_g\}$  be a measure-preserving representation of  $G$  on some probability space  $(\Omega, \mathcal{F}, P)$ . For each  $n$  let  $\mu_n$  be the  $n$ -fold convolution of  $\mu$  with itself. Let  $f \in L_1(\Omega)$ . Then  $\lim_{N \rightarrow \infty} 1/N \sum_{j=1}^N \int_G f(T_g w) d\mu_n(g)$  exists a.e.*

The proof follows at once from the Dunford-Schwartz ergodic theorem. (See e.g., Garsia, [5].) Define the operator  $S$  on  $L_1(\Omega)$  by  $(Sf)(w) = \int_G f(T_g w) d\mu(g)$ . Then clearly  $\|S\|_1 \leq 1$  and  $\|S\|_\infty \leq 1$ . It is easily verified that  $(S^n f)(w) = \int_G f(T_g w) d\mu_n(g)$ , and the theorem follows from the Dunford-Schwartz theorem.

The limit in Theorem 2 clearly depends on the nature of  $\mu$ . If  $\mu$  is absolutely continuous with respect to Haar measure on  $G$ , and if its density  $\varphi$  has a Fourier transform  $\hat{\varphi}(\gamma)$  such that  $\hat{\varphi}(\gamma) \neq 1$  for  $\gamma \neq 0$ , then it is not difficult to show that for  $f \in L_2(\Omega)$  the limit is again  $Pf$ .

5. In the case when  $G = Z$  or  $G = R$ , it is of interest to ask what summability methods other than Cesaro averaging yield the individual ergodic theorem. Some results along this line may be obtained from a paper by Davydov and Ibragimov, [3]. We shall give one of their theorems for the real line  $R$ . Let  $\mu$  be a probability measure on  $R$  and for each  $n$  let  $\mu_n$  be the  $n$ -fold convolution of  $\mu$  with itself. Let  $f$  be a measurable, real-valued, bounded function on  $R$ . Then we have the

**THEOREM (Davydov-Ibragimov).** *Suppose  $\mu$  belongs to the domain of attraction of a symmetric stable law and suppose for some  $n$  the distribution  $\mu_n$  has an absolutely continuous component. Then*

$$\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} f(x) d\mu_n(x) = L \quad \text{if and only if}$$

$$\lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T f(x) dx = L.$$

A similar result holds when  $G = Z$ , i.e., when  $\mu$  is a lattice distribution.

The way we can apply this is as follows. Let  $(\Omega, \mathcal{F}, P)$  be a probability space, and let  $\{T_t, -\infty < t < \infty\}$  be a measurable, measure-preserving flow on  $\Omega$  which is a representation of  $R$ , i.e.,  $T_0 = I$  and  $T_t T_s = T_{t+s}$ . Let  $g \in L_1(\Omega)$ . Then it follows from the individual ergodic theorem that  $\lim_{T \rightarrow \infty} 1/2T \int_{-T}^T g(T_t w) dt$  exists a.e. and equals  $Pg$  when  $g \in L_2(\Omega)$ . Now let  $w$  be in this set of probability one. Without loss of generality we may assume that  $g(T_t w)$  is bounded in  $t$ . Define  $f(t) = g(T_t w)$ . Then it follows from the theorem above that  $\lim_{n \rightarrow \infty} \int_{-\infty}^{\infty} g(T_t w) d\mu_n(t)$  exists and equals  $\lim_{T \rightarrow \infty} 1/2T \int_{-T}^T g(T_t w) dt$ . For example, if  $\mu$  is the normal distribution with mean zero and variance  $\sigma^2 > 0$ , then  $\mu_n$  is the normal distribution with mean zero and variance  $n\sigma^2$ , and we have that

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi} \sqrt{n\sigma^2}} \int_{-\infty}^{\infty} g(T_t w) e^{-(1/2n\sigma^2)t^2} dt$$

exists a.e. for  $g \in L_1(\Omega)$ . In the case when  $G = Z$ , the Davydov-Ibragimov theorem only requires that  $\mu$  belong to the domain of

attraction of a symmetric stable law. For example, if  $\mu$  is any distribution on the integers with mean zero and positive second moment the theorem applies.

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