POINTWISE ERGODIC THEOREMS IN L.C.A. GROUPS

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Let $G$ be a l.c.a. group and $\{T_\sigma\}$ be a representation of $G$ such that each $T_\sigma$ is a measure-preserving transformation on some probability space $(\Omega, \mathcal{F}, P)$. Let $\{\mu_n\}$ be a sequence of probability measures on $G$. We are interested in the a.e. convergence or summability of $\int_\sigma f(T_\sigma w) d\mu_n(\sigma)$, for $f \in L_1(\Omega)$.

Some examples and counterexamples are given, and some partial results are obtained.

1. Let $G$ be a locally compact abelian group (l.c.a.), and let $\hat{G}$ be its dual group. $\hat{G}$ consists of all continuous homomorphisms of $G$ of absolute value one. $\hat{G}$ is again l.c.a. Denote by $\hat{G}_d$ the l.c.a. group obtained from $\hat{G}$ by endowing it with the discrete topology, and by $\hat{G}$ the dual of $\hat{G}_d$. $\hat{G}$ is a compact group known as the Bohr compactification of $G$, and $G$ is a dense subset of $\hat{G}$. If $m$ is normalized Haar measure on $\hat{G}$, then $m(G) = 0$. Note that $G$ and $\hat{G}$ have the same characters, namely the elements of $\hat{G}$. Now if $\mu$ is a finite measure on the Borel sets of $G$, we may without loss of generality consider it to be a measure on $\hat{G}$, for if $B$ is a Borel subset of $\hat{G}$ we can define $\mu(B) = \mu(B \cap G)$. If $\{\mu_n, n = 1, 2, \ldots\}$ is a sequence of probability measures on $G$, we shall call it an ergodic sequence if $\mu_n$, considered as a sequence of measures on $\hat{G}$, converges weakly to $m$, the Haar measure on $\hat{G}$. The reason for this terminology is that it was shown in Blum and Eisenberg, [2], that if $U = \{U_\sigma\}$ is a strongly continuous unitary representation of $G$ on some Hilbert space $H$, and if we consider the sequence $\int_\sigma f d\mu_n(\sigma)$, which is defined weakly for each $f \in H$, then if $\{\mu_n\}$ is an ergodic sequence we have a strong limit $\int_\sigma f d\mu_n(\sigma) = Pf$ for every $f \in H$, where $P$ is the orthogonal projection on the closed linear subspace of $H$ consisting of those elements of $H$ invariant under each $U_\sigma$. Moreover if this version of the mean ergodic theorem is to hold for every strongly continuous unitary representation of $G$, then it is necessary that $\{\mu_n\}$ be ergodic.

In this paper we shall be concerned with pointwise ergodic theorems.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{T_\sigma\}$ be a group of measure-preserving transformations of $\Omega$ into itself such that the corresponding unitary operators $U_\sigma$ on $L_2(\Omega)$ are a strongly continuous representation of $G$. We show by a simple example that the pointwise ergodic theorem does not hold for every ergodic sequence
We then show that for certain ergodic sequences the pointwise ergodic theorem does hold for a set which is dense in $L^1(\Omega)$, but not necessarily for all of $L^1(\Omega)$. Finally we exhibit certain ergodic sequences for which the pointwise ergodic theorem does hold.

2. Let $G = \mathbb{Z}$ and for each positive integer $n$ define $\mu_n$ by putting mass $1/\lfloor 1/\sqrt{n} \rfloor$ on the integers $n + 1, \ldots, n + \lfloor 1/\sqrt{n} \rfloor$, where $[x]$ is the longest integer not exceeding $x$. Now let $\hat{\mu}_n(\alpha)$ be the Fourier transform of $\mu_n$ for $0 \leq \alpha < 2\pi$. Then $\hat{\mu}_n(\alpha) = (1/\lfloor 1/\sqrt{n} \rfloor) \sum_{j=n+1}^{n+\lfloor 1/\sqrt{n} \rfloor} e^{ij\alpha}$ and $\hat{\mu}_n(0) \to 1$ while $\hat{\mu}_n(\alpha) \to 0$ for $0 < \alpha < 2\pi$. But if $m$ is Haar measure on $\mathbb{Z}$ then $\hat{m}(\alpha) = \begin{cases} 1, & \alpha = 0 \\ 0, & 0 < \alpha < 2\pi \end{cases}$. Thus $\{\mu_n\}$ is an ergodic sequence. Now let $(\Omega, \mathcal{F}, P)$ be a probability space, and let $T$ be an invertible ergodic measure-preserving transformation of $\Omega$ onto itself. It was shown by Akcoglu and Del Junco, [1], that there exists a set $A \in \mathcal{F}$ with $0 < P(A) < 1/2$, and a set $B \in \mathcal{F}$ with $P(B) > 1/2$ such that for $w \in B$ we have

$$\frac{1}{\lfloor 1/\sqrt{n} \rfloor} \sum_{j=n+1}^{n+\lfloor 1/\sqrt{n} \rfloor} \chi_A(T^{-j}w) = 1$$

infinitely often, where $\chi_A$ is the indicator of $A$. In fact by a slight modification of their argument and by taking lim sup one can make $P(A)$ arbitrary small and $P(B) = 1$. In any case it is clear that the individual ergodic theorem does not hold for this ergodic sequence $\{\mu_n\}$.

3. Now suppose $\{\mu_n\}$ is a sequence of probability measures on $G$, each of which is absolutely continuous with respect to the Haar measure on $G$. Denote by $\varphi_n$ its density with respect to Haar measure, i.e., $\mu_n(A) = \int_A \varphi_n(g)dg$, for each Borel subset $A$ of $G$, where $dg$ is Haar measure on $G$. For $\gamma \in \hat{G}$ we shall write $\hat{\varphi}_n(\gamma)$ for the Fourier transform of $\mu_n$, i.e., $\hat{\varphi}_n(\gamma) = \int \langle \gamma, \gamma \rangle \varphi_n(g)dg$. Here $\langle g, \gamma \rangle$ is the usual notation for the character $\gamma$ evaluated at $g$.

Then we have

**Theorem 1.** Suppose for each compact subset $K$ of $\hat{G}$ with $0 \in K$, and every sequence $\{\gamma_n\}$ with $\gamma_n \in K$ for all $n$ we have $\sum_{n=1}^{\infty} |\hat{\varphi}_n(\gamma_n)|^2 < \infty$. Then there exists a set $D$, dense in $L_1(\Omega)$, and hence in $L_2(\Omega)$, such that for $f \in D$ we have $\lim_{n \to \infty} \int_{\Omega} f(T_gw)d\mu_n(g) = Pf$ a.e.

Here $P$ is as above the orthogonal projection of $f$ onto the closed subspace of $L_1(\Omega)$ of elements invariant under each $U_g$. Before proving the theorem we exhibit a class of examples for which the
hypotheses of the theorem are satisfied. Let $\mu$ be a probability measure on $G$ which is absolutely continuous with respect to Haar measure on $G$, and suppose $|\hat{\mu}(\gamma)| < 1$ for $\gamma \neq 0$. Then if we let $\mu_n$ be the $n$-fold convolution of $\mu$ with itself it is easily verified that the hypotheses of the theorem are satisfied. For example, if $G = \mathbb{Z}$ and $\mu$ puts mass $p$ on 0 and mass $1-p$ on 1, where $0 < p < 1$, then $\hat{\mu}(\gamma) = p + (1 - p)e^{i\gamma}$ for $0 < \gamma < 2\pi$ so that $|\hat{\mu}(\gamma)| < 1$. In this case $\mu_n$ is the binomial distribution with parameters $n$ and $p$.

**Proof of the theorem.** Let $E(\cdot)$ be the spectral measure associated with $\{U_\beta\}$. Note that $E([0])f = Pf$ for all $f \in L_2(\Omega)$. Denote by $\mathcal{E}$ the closed subspace of $L_2(\Omega)$ spanned by the eigenfunctions of $\{U_\beta\}$.

Let $f \in \mathcal{E}^\perp$. Then $(E(d\gamma)f, f)$ is a continuous, regular Borel measure on $\hat{G}$ and for given $\varepsilon > 0$ we can find a compact set $K \subset \hat{G}$ with $0 \notin K$ such that

$$\|E(K)f - f\|_2 < \varepsilon.$$ 

Since $PE(K)f = 0$ we shall show that $\lim_{n \to \infty} \int_{\Omega} E(K)f(T_gw)d\mu_n(g) = 0$ a.e. Henceforth we shall write this integral as $\int_{\Omega} E(K)f(T_gw)\varphi_n(g)dg$. Let $\delta > 0$ and define $A_{n,\delta} = \{w \mid \left| \int_{\Omega} E(K)f(T_gw)\varphi_n(g)dg \right| > \delta \}$. Then

$$P(A_{n,\delta}) \leq \frac{1}{\delta^2} \left\| \int_{\Omega} E(K)f(T_gw)\varphi_n(g)dg \right\|_2^2$$

$$= \frac{1}{\delta^2} \int_{\delta} \left\langle g, \gamma \right\rangle Ed(\gamma)(E(K)f)\varphi_n(g)dg \right\|_2^2$$

$$= \frac{1}{\delta^2} \int_{\delta} |\hat{\varphi}_n(\gamma)|^2(E(\delta\gamma)E(K)f, E(K)f)$$

$$= \frac{1}{\delta^2} \int_{\delta} |\hat{\varphi}_n(\gamma)|^2(E(\delta\gamma)f, f) \leq \frac{1}{\delta^2} \max_{\gamma \in K} |\hat{\varphi}_n(\gamma)|^2 \| f \|_2^2.$$ 

But $\hat{\varphi}_n(\gamma)$ is continuous and therefore there exists $\gamma_n \in K$ such that $|\hat{\varphi}_n(\gamma_n)|^2 = \max_{\gamma \in K} |\hat{\varphi}_n(\gamma)|^2$. Hence by the hypotheses we have $\sum_{n=1}^{\infty} P(A_n, \delta) < \infty$. It follows from the Borel-Cantelli lemma that $P(A_\delta) = 1$ where $A_\delta = \{w \mid w$ is in at most finitely many $A_{n,\delta}\}$ and similarly if $A = \bigcap_{n=1}^{\infty} A_{1/n}$, then $P(A) = 1$. But for $w \in A$ we have $\lim_{n \to \infty} \int_{\Omega} E(K)f(T_gw)\varphi_n(g)dg = 0 = PE(K)f$. Thus we approximate each $f \in \mathcal{E}^\perp$ by a function for which the theorem holds.

If $f \in \mathcal{E}$ and $\varepsilon > 0$ there exist finitely many eigenfunctions, say $f_{i,1}, \ldots, f_{i,M}$ with $\gamma_j \neq 0$, $j = 1, \ldots, M$ such that

$$\left\| E([0])f + \sum_{j=1}^{M} f_{i,j} - f \right\|_2 < \varepsilon.$$
Then
\[
\lim_{n \to \infty} \int_{\Omega} \left[ E(\{0\})f + \sum_{j=1}^{M} f_{\gamma_j}(T_{\gamma_j}w) \right] \nu_n(g) \, dg = Pf + \sum_{j=1}^{M} \lim_{n \to \infty} \int_{\Omega} \langle g, \gamma_j \rangle f_{\gamma_j}(w) \nu_n(g) \, dg
\]

since clearly \( \lim_{n \to \infty} \left| \hat{\nu}_n(\gamma_j) \right| = 0, \ j = 1, \ldots, M. \)

This concludes the proof of the theorem.

4. As was mentioned earlier the theorem does not hold in general for all of \( L_4(\Omega) \), or indeed for all bounded functions. As an example, let \( G = \mathbb{Z} \) and let \( \mu_n \) be \( B(n, p) \), the binomial distribution on \( 0, 1, \ldots, n \) with \( 0 < p < 1 \). Let \( (c_j, j = 0, 1, 2, \ldots) \) be a sequence of 0's and 1's. Then it was shown by Diaconis and Stein, [4], that
\[
\lim_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{j=0}^{n} c_j \left( \frac{n}{j} \right) p^j (1 - p)^{n-j} = L \text{ if and only if for every } \varepsilon > 0 \text{ we have } \lim_{n \to \infty} \sqrt{n} \sum_{j=0}^{n} c_j = L.
\]

Now if \( T \) is invertible and ergodic we saw that we can choose a set \( A \) with \( 0 < P(A) < 1/2 \) and a set \( B \) with \( P(B) = 1 \) such that for \( w \in B \) we have
\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{j=0}^{n} c_j = 1.
\]

By choosing \( c_j = \chi_{A}(T^{-j}w) \) it follows from the Diaconis-Stein result with \( \varepsilon = 1 \) that the individual ergodic theorem fails to hold for this sequence \( \{\mu_n\} \). As was mentioned earlier, this sequence does satisfy the hypotheses of the theorem.

It is of some interest to point out that in the case when \( \mu_n \) is the \( n \)-fold convolution of a measure \( \mu \) on \( G \), we do have a version of the individual ergodic theorem as follows:

**Theorem 2.** Let \( \mu \) be a probability measure on \( G \), and let \( \{T_{\gamma_j}\} \) be a measure-preserving representation of \( G \) on some probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \). For each \( n \) let \( \mu_n \) be the \( n \)-fold convolution of \( \mu \) with itself. Let \( f \in L_1(\Omega) \). Then \( \lim_{n \to \infty} 1/N \sum_{j=1}^{n} f(T_{\gamma_j}w) d\mu_n(g) \) exists a.e. Let \( f \in L_1(\Omega) \). Then \( \lim_{N \to \infty} 1/N \sum_{j=1}^{N} f(T_{\gamma_j}w) d\mu_n(g) \) exists a.e.

The proof follows at once from the Dunford-Schwartz ergodic theorem. (See e.g., Garsia, [5].) Define the operator \( S \) on \( L_1(\Omega) \) by \( \langle Sf \rangle(w) = \int_{\Omega} f(T_{\gamma}w) d\mu(g) \). Then clearly \( \|S\|_1 \leq 1 \) and \( \|S\|_\infty \leq 1 \). It is easily verified that \( \langle S^n f \rangle(w) = \int_{\Omega} f(T_{\gamma}w) d\mu_n(g) \), and the theorem follows from the Dunford-Schwartz theorem.
The limit in Theorem 2 clearly depends on the nature of \( \mu \). If \( \mu \) is absolutely continuous with respect to Haar measure on \( G \), and if its density \( \varphi \) has a Fourier transform \( \hat{\varphi}(\gamma) \) such that \( \hat{\varphi}(\gamma) \neq 1 \) for \( \gamma \neq 0 \), then it is not difficult to show that for \( f \in \mathcal{L}_2(\Omega) \) the limit is again \( Pf \).

5. In the case when \( G = \mathbb{Z} \) or \( G = \mathbb{R} \), it is of interest to ask what summability methods other than Cesaro averaging yield the individual ergodic theorem. Some results along this line may be obtained from a paper by Davydov and Ibragimov, [3]. We shall give one of their theorems for the real line \( \mathbb{R} \). Let \( \mu \) be a probability measure on \( \mathbb{R} \) and for each \( n \) let \( \mu_n \) be the \( n \)-fold convolution of \( \mu \) with itself. Let \( f \) be a measurable, real-valued, bounded function on \( \mathbb{R} \). Then we have the

**Theorem (Davydov-Ibragimov).** Suppose \( \mu \) belongs to the domain of attraction of a symmetric stable law and suppose for some \( n \) the distribution \( \mu_n \) has an absolutely continuous component. Then

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) d\mu_n(x) = L \quad \text{if and only if} \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) dx = L.
\]

A similar result holds when \( G = \mathbb{Z} \), i.e., when \( \mu \) is a lattice distribution.

The way we can apply this is as follows. Let \( (\Omega, \mathcal{F}, P) \) be a probability space, and let \( \{T_t, -\infty < t < \infty\} \) be a measurable, measure-preserving flow on \( \Omega \) which is a representation of \( \mathbb{R} \), i.e., \( T_0 = I \) and \( T_t T_s = T_{t+s} \). Let \( g \in \mathcal{L}_1(\Omega) \). Then it follows from the individual ergodic theorem that \( \lim_{T \to \infty} 1/2T \int_{-T}^{T} g(T_t w) dt \) exists a.e. and equals \( Pg \) when \( g \in \mathcal{L}_1(\Omega) \). Now let \( w \) be in this set of probability one. Without loss of generality we may assume that \( g(T_t w) \) is bounded in \( t \). Define \( f(t) = g(T_t w) \). Then it follows from the theorem above that \( \lim_{n \to \infty} \int_{-\infty}^{\infty} g(T_t w) d\mu_n(t) \) exists and equals \( \lim_{T \to \infty} 1/2T \int_{-T}^{T} g(T_t w) dt \). For example, if \( \mu \) is the normal distribution with mean zero and variance \( \sigma^2 > 0 \), then \( \mu_n \) is the normal distribution with mean zero and variance \( n\sigma^2 \), and we have that

\[
\lim_{n \to \infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{n\sigma^2}} \int_{-\infty}^{\infty} g(T_t w)e^{-\frac{(1/2n\sigma^2)t^2}{2\sigma^2}} dt
\]

exists a.e. for \( g \in \mathcal{L}_1(\Omega) \). In the case when \( G = \mathbb{Z} \), the Davydov-Ibragimov theorem only requires that \( \mu \) belong to the domain of
attraction of a symmetric stable law. For example, if $\mu$ is any distribution on the integers with mean zero and positive second moment the theorem applies.

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