POINTWISE ERGODIC THEOREMS IN L.C.A. GROUPS

Julius Rubin Blum and J. I. Reich
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J. R. BLUM AND J. I. REICH

Let $G$ be a l.c.a. group and $(T_g)$ be a representation of $G$ such that each $T_g$ is a measure-preserving transformation on some probability space $(\Omega, \mathcal{F}, P)$. Let $(\mu_n)$ be a sequence of probability measures on $G$. We are interested in the a.e. convergence or summability of $\int_G f(T_gw) d\mu_n(g)$, for $f \in L_1(\Omega)$.

Some examples and counterexamples are given, and some partial results are obtained.

1. Let $G$ be a locally compact abelian group (l.c.a.), and let $\hat{G}$ be its dual group. $\hat{G}$ consists of all continuous homomorphisms of $G$ of absolute value one. $\hat{G}$ is again l.c.a. Denote by $\hat{\hat{G}}$ the l.c.a. group obtained from $\hat{G}$ by endowing it with the discrete topology, and by $G$ the dual of $\hat{\hat{G}}$. $G$ is a compact group known as the Bohr compactification of $G$, and $\hat{G}$ is a dense subset of $G$. If $m$ is normalized Haar measure on $G$, then $m(G) = 0$. Note that $G$ and $\hat{G}$ have the same characters, namely the elements of $G$. Now if $\mu$ is a finite measure on the Borel sets of $G$, we may without loss of generality consider it to be a measure on $\hat{G}$, for if $B$ is a Borel subset of $\hat{G}$ we can define $\mu(B) = \mu(B \cap G)$. If $(\mu_n, n = 1, 2, \cdots)$ is a sequence of probability measures on $G$, we shall call it an ergodic sequence if $\mu_nf$ considered as a sequence of measures on $\hat{G}$, converges weakly to $\mu$, the Haar measure on $\hat{G}$. The reason for this terminology is that it was shown in Blum and Eisenberg, [2], that if $U = \{U_g\}$ is a strongly continuous unitary representation of $G$ on some Hilbert space $H$, and if we consider the sequence $\int_G U_g f d\mu_n(g)$, which is defined weakly for each $f \in H$, then if $(\mu_n)$ is an ergodic sequence we have a strong limit $\int_G U_g f d\mu_n(g) = Pf$ for every $f \in H$, where $P$ is the orthogonal projection on the closed linear subspace of $H$ consisting of those elements of $H$ invariant under each $U_g$. Moreover if this version of the mean ergodic theorem is to hold for every strongly continuous unitary representation of $G$, then it is necessary that $(\mu_n)$ be ergodic.

In this paper we shall be concerned with pointwise ergodic theorems.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(T_g)$ be a group of measure-preserving transformations of $\Omega$ into itself such that the corresponding unitary operators $U_g$ on $L_2(\Omega)$ are a strongly continuous representation of $G$. We show by a simple example that the pointwise ergodic theorem does not hold for every ergodic sequence
{μₙ} on G. We then show that for certain ergodic sequences the pointwise ergodic theorem does hold for a set which is dense in \( L_2(Ω) \), but not necessarily for all of \( L_1(Ω) \). Finally we exhibit certain ergodic sequences for which the pointwise ergodic theorem does hold.

2. Let \( G = \mathbb{Z} \) and for each positive integer \( n \) define \( μₙ \) by putting mass \( 1/\lceil \sqrt{n} \rceil \) on the integers \( n + 1, \ldots, n + \lceil \sqrt{n} \rceil \), where \( [x] \) is the longest integer not exceeding \( x \). Now let \( \hat{μₙ}(α) \) be the Fourier transform of \( μₙ \) for \( 0 ≤ α < 2π \). Then \( \hat{μₙ}(α) = (1/\lceil \sqrt{n} \rceil) \sum_{j=n+1}^{n+\lceil \sqrt{n} \rceil} e^{ijα} \) and \( \hat{μₙ}(0) → 1 \) while \( \hat{μₙ}(α) → 0 \) for \( 0 < α < 2π \). But if \( m \) is Haar measure on \( \mathbb{Z} \) then \( m(α) = 1/2π \). Thus \( \{μₙ\} \) is an ergodic sequence. Now let \( (Ω, \mathcal{F}, P) \) be a probability space, and let \( T \) be an invertible ergodic measure-preserving transformation of \( Ω \) onto itself. It was shown by Akcoglu and Del Junco, [1], that there exists a set \( A ∈ \mathcal{F} \) with \( 0 < P(A) < 1/2 \). and a set \( B ∈ \mathcal{F} \) with \( P(B) > 1/2 \) such that for \( w ∈ B \) we have

\[
\frac{1}{\lceil \sqrt{n} \rceil} \sum_{j=n+1}^{n+\lceil \sqrt{n} \rceil} \mathbb{1}_A(T^{-j}w) = 1
\]

infinitely often, where \( \mathbb{1}_A \) is the indicator of \( A \). In fact by a slight modification of their argument and by taking \( \lim \sup \) one can make \( P(A) \) arbitrary small and \( P(B) = 1 \). In any case it is clear that the individual ergodic theorem does not hold for this ergodic sequence \( \{μₙ\} \).

3. Now suppose \( \{μₙ\} \) is a sequence of probability measures on \( G \), each of which is absolutely continuous with respect to the Haar measure on \( G \). Denote by \( φₙ \) its density with respect to Haar measure, i.e., \( μₙ(A) = \int_A φₙ(g)dg \), for each Borel subset \( A \) of \( G \), where \( dg \) is Haar measure on \( G \). For \( γ ∈ \hat{G} \) we shall write \( \hat{φₙ}(γ) \) for the Fourier transform of \( μₙ \), i.e., \( \hat{φₙ}(γ) = \int_0<γ, γ'>φₙ(g)dg \). Here \( <g, γ> \) is the usual notation for the character \( γ \) evaluated at \( g \).

Then we have

**Theorem 1.** Suppose for each compact subset \( K \) of \( \hat{G} \) with \( 0 \not∈ K \), and every sequence \( \{γₙ\} \) with \( γₙ ∈ K \) for all \( n \) we have \( \sum_{n=1}^{∞} |\hat{φₙ}(γₙ)|² < ∞ \). Then there exists a set \( D \), dense in \( L_2(Ω) \), and hence in \( L_1(Ω) \), such that for \( f ∈ D \) we have \( \lim_{n→∞} \int_Ω f(Tₙw)dμₙ(g) = Pf \) a.e.

Here \( P \) is as above the orthogonal projection of \( f \) onto the closed subspace of \( L_2(Ω) \) of elements invariant under each \( Uₙ \). Before proving the theorem we exhibit a class of examples for which the
hypotheses of the theorem are satisfied. Let $\mu$ be a probability measure on $G$ which is absolutely continuous with respect to Haar measure on $G$, and suppose $|\hat{\mu}(\gamma)| < 1$ for $\gamma \neq 0$. Then if we let $\mu_n$ be the $n$-fold convolution of $\mu$ with itself it is easily verified that the hypotheses of the theorem are satisfied. For example, if $G = \mathbb{Z}$ and $\mu$ puts mass $p$ on 0 and mass $1 - p$ on 1, where $0 < p < 1$, then $\hat{\mu}(\gamma) = p + (1 - p)e^{i\gamma}$ for $0 < \gamma < 2\pi$ so that $|\hat{\mu}(\gamma)| < 1$. In this case $\mu_n$ is the binomial distribution with parameters $n$ and $p$.

Proof of the theorem. Let $E(\cdot)$ be the spectral measure associated with $\{U_\gamma\}$. Note that $E(\{0\})f = Pf$ for all $f \in L_2(\Omega)$. Denote by $\mathcal{E}$ the closed subspace of $L_2(\Omega)$ spanned by the eigenfunctions of $\{U_\gamma\}$.

Let $f \in \mathcal{E}$. Then $(E(d\gamma)f, f)$ is a continuous, regular Borel measure on $\hat{G}$ and for given $\varepsilon > 0$ we can find a compact set $K \subset \hat{G}$ with $0 \notin K$ such that

$$\|E(K)f - f\|_2 < \varepsilon.$$ 

Since $PE(K)f = 0$ we shall show that $\lim_{n \to \infty} \int_{\hat{G}} E(K)f(T_\gamma w)d\mu_n(g) = 0$ a.e. Henceforth we shall write this integral as $\int_{\hat{G}} E(K)f(T_\gamma w)\varphi_n(g)dg$. Let $\delta > 0$ and define $A_{n,\delta} = \{w \in K | E(K)f(T_\gamma w)\varphi_n(g)dg| < \delta\}$. Then

$$P(A_{n,\delta}) \leq \frac{1}{\delta^2} \left\| \int_{\hat{G}} E(K)f(T_\gamma w)\varphi_n(g)dg \right\|_2^2$$

$$= \frac{1}{\delta^2} \int_{\hat{G}} \int_{\hat{G}} \langle g, \gamma \rangle E\hat{d}(\gamma)(E(K)f)\varphi_n(g)dg \right\|_2^2$$

$$= \frac{1}{\delta^2} \left\| \hat{\varphi}_n(\gamma) \right\|^2(E(d\gamma)f, E(K)f)$$

$$= \frac{1}{\delta^2} \int_{\hat{G}} \left\| \hat{\varphi}_n(\gamma) \right\|^2(E(d\gamma)f, f) \leq \frac{1}{\delta^2} \max_{\gamma \in K} \left\| \hat{\varphi}_n(\gamma) \right\|^2 \|f\|_2^2.$$

But $\hat{\varphi}_n(\gamma)$ is continuous and therefore there exists $\gamma_n \in K$ such that $|\hat{\varphi}_n(\gamma_n)|^2 = \max_{\gamma \in K} |\hat{\varphi}_n(\gamma)|^2$. Hence by the hypotheses we have $\sum_{n=1}^\infty P(A_{n,\delta}) < \infty$. It follows from the Borel-Cantelli lemma that $P(A_\delta) = 1$ where $A_\delta = \{w \in K | w \text{ is in at most finitely many } A_{n,\delta}\}$ and similarly if $A = \bigcap_{n=1}^\infty A_{1/n}$, then $P(A) = 1$. But for $w \in A$ we have $\lim_{n \to \infty} \int_{\hat{G}} E(K)f(T_\gamma w)\varphi_n(g)dg = 0 = PE(K)f$. Thus we approximate each $f \in \mathcal{E}$ by a function for which the theorem holds.

If $f \in \mathcal{E}$ and $\varepsilon > 0$ there exist finitely many eigenfunctions, say $f_{\gamma_1}, \ldots, f_{\gamma_M}$ with $\gamma_j \neq 0$, $j = 1, \ldots, M$ such that

$$\|E(0)f + \sum_{j=1}^M f_{\gamma_j} - f\|_2 < \varepsilon.$$
Then
\[
\lim_{n \to \infty} \int_{\Omega} \left[ E(\{0\})f + \sum_{j=1}^{M} f_{\gamma_j}(T_g w) \right] \varphi_n(g)dg
\]
\[
= Pf + \sum_{j=1}^{M} \lim_{n \to \infty} \int_{\Omega} \langle g, \gamma_j \rangle f_{\gamma_j}(w) \varphi_n(g)dg
\]
\[
= Pf + \sum_{j=1}^{M} \lim_{n \to \infty} \varphi_n(\gamma_j) = P(f)
\]
since clearly \( \lim_{n \to \infty} |\varphi_n(\gamma_j)| = 0, \ j = 1, \cdots, M. \)

This concludes the proof of the theorem.

4. As was mentioned earlier the theorem does not hold in general for all of \( L_1(\Omega) \), or indeed for all bounded functions. As an example, let \( G = \mathbb{Z} \) and let \( \mu_n \) be \( B(n, p) \), the binomial distribution on \( 0, 1, \cdots, n \) with \( 0 < p < 1 \). Let \( \{c_j, j = 0, 1, 2, \cdots\} \) be a sequence of 0's and 1's. Then it was shown by Diaconis and Stein, \([4]\), that

\[\sum_{j=0}^{n} c_j \frac{n^j}{j!} p^j(1 - p)^{n-j} = L \text{ if and only if for every } \varepsilon > 0 \text{ we have } \lim_{n \to \infty} \frac{1}{\varepsilon} \sum_{j=0}^{n} c_j = L. \]

Now if \( T \) is invertible and ergodic we saw that we can choose a set \( A \) with \( 0 < P(A) < 1/2 \) and a set \( B \) with \( P(B) = 1 \) such that for \( w \in B \) we have

\[
\limsup_{n \to \infty} \frac{1}{\sqrt{n}} \sum_{j=n+1}^{n+\lfloor \sqrt{n} \rfloor} \chi_A(T^{-j}w) = 1.
\]

By choosing \( c_j = \chi_A(T^{-j}w) \) it follows from the Diaconis-Stein result with \( \varepsilon = 1 \) that the individual ergodic theorem fails to hold for this sequence \( \{\mu_n\} \). As was mentioned earlier, this sequence does satisfy the hypotheses of the theorem.

It is of some interest to point out that in the case when \( \mu_n \) is the \( n \)-fold convolution of a measure \( \mu \) on \( G \), we do have a version of the individual ergodic theorem as follows:

**Theorem 2.** Let \( \mu \) be a probability measure on \( G \), and let \( \{T_g\} \) be a measure-preserving representation of \( G \) on some probability space \( (\Omega, \mathcal{F}, P) \). For each \( n \) let \( \mu_n \) be the \( n \)-fold convolution of \( \mu \) with itself. Let \( f \in L_1(\Omega) \). Then \( \lim_{n \to \infty} 1/N \sum_{j=1}^{N} f(T_g w)d\mu_n(g) \) exists a.e.

The proof follows at once from the Dunford-Schwartz ergodic theorem. (See e.g., Garsia, \([5]\).) Define the operator \( S \) on \( L_1(\Omega) \) by

\[
(Sf)(w) = \int_{\Omega} f(T_g w)d\mu(g).
\]

Then clearly \( \|S\|_1 \leq 1 \) and \( \|S\|_\infty \leq 1 \). It is easily verified that \( (S^n f)(w) = \int_{\Omega} f(T_g w)d\mu_n(g) \), and the theorem follows from the Dunford-Schwartz theorem.
The limit in Theorem 2 clearly depends on the nature of \( \mu \). If \( \mu \) is absolutely continuous with respect to Haar measure on \( G \), and if its density \( \varphi \) has a Fourier transform \( \hat{\varphi}(\gamma) \) such that \( \hat{\varphi}(\gamma) \neq 1 \) for \( \gamma \neq 0 \), then it is not difficult to show that for \( f \in L^2(\Omega) \) the limit is again \( Pf \).

5. In the case when \( G = \mathbb{Z} \) or \( G = \mathbb{R} \), it is of interest to ask what summability methods other than Cesaro averaging yield the individual ergodic theorem. Some results along this line may be obtained from a paper by Davydov and Ibragimov, [3]. We shall give one of their theorems for the real line \( \mathbb{R} \). Let \( \mu \) be a probability measure on \( \mathbb{R} \) and for each \( n \) let \( \mu_n \) be the \( n \)-fold convolution of \( \mu \) with itself. Let \( f \) be a measurable, real-valued, bounded function on \( \mathbb{R} \). Then we have the

**Theorem (Davydov-Ibragimov).** Suppose \( \mu \) belongs to the domain of attraction of a symmetric stable law and suppose for some \( n \) the distribution \( \mu_n \) has an absolutely continuous component. Then

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} f(x) d\mu_n(x) = L \quad \text{if and only if} \quad \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) dx = L.
\]

A similar result holds when \( G = \mathbb{Z} \), i.e., when \( \mu \) is a lattice distribution.

The way we can apply this is as follows. Let \( (\Omega, \mathcal{F}, P) \) be a probability space, and let \( \{T_t, -\infty < t < \infty\} \) be a measurable, measure-preserving flow on \( \Omega \) which is a representation of \( \mathbb{R} \), i.e., \( T_0 = I \) and \( T_t T_s = T_{t+s} \). Let \( g \in L_1(\Omega) \). Then it follows from the individual ergodic theorem that \( \lim_{T \to \infty} 1/2T \int_{-T}^{T} g(T_tw) dt \) exists a.e. and equals \( Pg \) when \( g \in L_1(\Omega) \). Now let \( w \) be in this set of probability one. Without loss of generality we may assume that \( g(T_tw) \) is bounded in \( t \). Define \( f(t) = g(T_tw) \). Then it follows from the theorem above that \( \lim_{T \to \infty} \int_{-\infty}^{\infty} g(T_tw) d\mu_n(t) \) exists and equals \( \lim_{T \to \infty} 1/2T \int_{-T}^{T} g(T_tw) dt \).

For example, if \( \mu \) is the normal distribution with mean zero and variance \( \sigma^2 > 0 \), then \( \mu_n \) is the normal distribution with mean zero and variance \( n\sigma^2 \), and we have that

\[
\lim_{n \to \infty} \frac{1}{\sqrt{2\pi n\sigma^2}} \int_{-\infty}^{\infty} g(T_tw) e^{-\left(1/2n\sigma^2\right)t^2} dt
\]

exists a.e. for \( g \in L_1(\Omega) \). In the case when \( G = \mathbb{Z} \), the Davydov-Ibragimov theorem only requires that \( \mu \) belong to the domain of
attraction of a symmetric stable law. For example, if \( \mu \) is any distribution on the integers with mean zero and positive second moment the theorem applies.

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