A NOTE ON $\varepsilon$-SUBGRADIENTS AND MAXIMAL MONOTONICITY

JONATHAN BORWEIN
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It is our desire in this note to provide certain formulae relating subgradients, directional derivatives and ε-subgradients of proper lower semi-continuous convex functions defined on a Banach space.

Our aim is to provide these formulae, which somewhat extend those in [5], [6], [7], as a direct and hopefully straightforward consequence of Ekeland's non convex-version [3], of the Bishop-Phelps-Bronsted-Rockafellar Theorem [1], [2], [3], [4].

As a by-product we obtain somewhat more self contained proofs of the maximality of the subgradient as a monotone relation and of some related results.

1. Preliminaries. Throughout $X$ is a real Hausdorff locally convex space (l.c.s) with topological dual $X^*$. A function $f: X \to [-\infty, \infty]$ is said to be convex if its epigraph, $\text{Epi} f = \{(x, r) \mid f(x) \leq r\}$ is a convex subset of $X \times R$. Also $f$ is lower semi-continuous (l.s.c.) if $\text{Epi} f$ is closed. We will restrict our attention to proper convex functions. These are the functions which are somewhere finite and never $-\infty$. The domain of $f$, $\text{dom} f$, is the set of points in $X$ for which $f(x)$ is finite.

With each convex function we associate its (one-sided) directional derivative at $x$ in $\text{dom} f$ given by

$$f'(x; h) = \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$ (1)

Then $f'(x; \cdot)$ is well defined as a (possibly improper) convex positively homogeneous function. We also define, for each $\varepsilon \geq 0$, the $\varepsilon$-subgradient set for $f$ at $x$ by

$$\partial_\varepsilon f(x) = \{x^* \in X^* \mid x^*(h) + f(x) \leq f(x + h) + \varepsilon, \forall h \in X\}.$$ (2)

When $\varepsilon = 0$, we suppress $\varepsilon$ and the object is the ordinary subgradient. We now may also write

$$\partial f(x) = \{x^* \in X^* \mid x^*(h) \leq f'(x; h), \forall h \in X\}.$$ (3)

For amplification about these concepts the reader is referred to [3], [4], [7].

2. The main result. We begin with a subsidiary proposition
PROPOSITION 1. Let $f$ be a lower semi-continuous proper convex function defined on a locally convex space $X$. For any $x$ in the domain of $f$ one has the following formula:

\[ f'(x; h) = \inf_{t>0} \sup_{x^* \in \partial_t f(x)} \{ x^*(h) \mid x^* \in \partial_t f(x) \}. \]

Proof. Let $\epsilon > 0$ and let $x^* \in \partial_t f(x)$. Then (2) shows that for $t > 0$

\[ x^*(h) \leq \frac{f(x + th) - f(x) + \epsilon}{t}. \]

We let $t = \sqrt{\epsilon}$ and derive

\[ x^*(h) \leq \frac{f(x + \sqrt{\epsilon}h) - f(x) + \sqrt{\epsilon}}{\sqrt{\epsilon}}. \]

Then (5) and (1) combine to show that

\[ f'(x; h) \geq \limsup_{t \to 0} \{ x^*(h) \mid x^* \in \partial_t f(x) \}. \]

Conversely, let $d$ be any real number less than $f'(x; h)$, and let $\epsilon > 0$ be given. For $0 \leq t \leq 1$ one has

\[ f(x + th) \geq f(x) + td. \]

Thus the line segment

\[ L = \{(x, f(x) - \epsilon) + t(h, d) \mid 0 \leq t \leq 1\} \]

can be strictly separated from the closed convex set $\text{Epi} f$, [4]. Simple and standard calculation shows that any separating functional $(x^*, -r^*)$ in $X^* \times \mathbb{R}$ satisfies $r^* > 0$ and that

\[ (\frac{x^*}{r^*})(h) \geq d - \epsilon; \quad \frac{x^*}{r^*} \in \partial_t f(x). \]

The nature of $d$ and (9) show that

\[ \sup_{x^* \in \partial_t f(x)} \{ x^*(h) \mid x^* \in \partial_t f(x) \} \geq f'(x; h) - \epsilon. \]

It is clear from (6) and (10) that (4) holds.

If $f$ is actually continuous at $x$ then $\partial_t f(x)$ is weak-star compact [4], and (4) reduces to the standard formula

\[ f'(x; h) = \sup_{x^* \in \partial f(x)} \{ x^*(h) \mid x^* \in \partial f(x) \}. \]
Even in finite dimensions (11) can fail at a point of discontinuity, while in Fréchet space it is possible that \( \partial f \) is empty [4], [5]. In Banach space Rockafellar [5], [6], has given formulae replacing (11), in terms of approximations by subgradients at nearby points. Taylor [8] has given an alternative stronger formula. All these results follow from some form of the Bishop-Phelps [1] or Bronsted-Rockafellar [2] theorems. We now proceed to derive a strong version of Taylor's formula which uses Ekeland's variational form of the previously mentioned theorems [3].

**Theorem 1.** Let \( f \) be a proper convex lower semi-continuous function defined on a Banach space \((X, \| \cdot \|)\). Suppose that \( \varepsilon > 0 \) and \( t \geq 0 \) are given. Suppose that

\[
x^*_\varepsilon \in \partial f(x_0) .
\]

Then one may find points \( x_i \) and \( x^*_i \) such that

\[
x^*_i \in \partial f(x_i) ,
\]

and such that

\[
\| x_i - x_0 \| \leq \sqrt{\varepsilon} ,
\]

\[
| f(x_i) - f(x_0) | \leq \sqrt{\varepsilon} \left( \sqrt{\varepsilon} + \frac{1}{t} \right) ,
\]

\[
\| x^*_i - x^*_\varepsilon \| \leq \sqrt{\varepsilon} \left( 1 + t \| x^*_\varepsilon \| \right) ,
\]

\[
| x^*_i(h) - x^*_\varepsilon(h) | \leq \sqrt{\varepsilon} \left( \| h \| + t \| x^*_\varepsilon(h) \| \right) ,
\]

\[
x^*_i \in \partial_x f(x_0) .
\]

**Proof.** We renorm \( X \) using the equivalent norm given by

\[
\| x \|_t = \| x \| + t | x^* \varepsilon(x) | .
\]

We set \( g(x) = f(x) - x^* \varepsilon(x) \) and observe that \( g \) is l.s.c. and that

\[
g(x_0) \leq \varepsilon + \inf_{x} g(x) .
\]

We now apply Ekeland's theorem [3, p. 29] to \( g \) and \( \| \cdot \|_t \). We are promised the existence of \( x_i \) in \( X \) such that, for \( x \neq x_i \),

\[
g(x) + \sqrt{\varepsilon} \| x - x_i \|_t > g(x_i)
\]

and

\[
g(x_i) + \sqrt{\varepsilon} \| x_0 - x_i \|_t \leq g(x_0) .
\]
Now (21) can be read as saying that
\begin{equation}
0 \in \partial(g + \sqrt{\varepsilon}h)(x_t); \quad h(x) = \|x - x_s\|_t .
\end{equation}

Since \( h \) is continuous, and since
\begin{equation}
\partial h(x_t) = \{x^* + \alpha x^*_s \mid \|x^*\| \leq 1, \ |\alpha| \leq t\} ,
\end{equation}
we may write, using the subgradient sum formula [4],
\begin{equation}
0 \in \partial f(x_t) - x^*_s + \sqrt{\varepsilon} \partial h(x_t) .
\end{equation}
Hence there is some point \( x^*_t \) in \( \partial f(x_t) \) of the form
\begin{equation}
x^*_t = \sqrt{\varepsilon} x^* + (1 - \sqrt{\varepsilon} \alpha(t))x^*_s
\end{equation}
with \( |\alpha(t)| \leq t \) and \( \|x^*\| \leq 1 \). Thus (16) holds. Since (20) holds (22) shows that
\begin{equation}
\sqrt{\varepsilon} \|x_0 - x_s\| + \sqrt{\varepsilon} t |x^*_0(x_0 - x_s)| \leq \varepsilon .
\end{equation}
In particular (14) holds and
\begin{equation}
|x^*_0(x_0 - x_s)| \leq \sqrt{\varepsilon}/t .
\end{equation}
Combined use of (20) and (22) shows that
\begin{equation}
|f(x_t) - f(x_0)| \leq |x^*_0(x_t - x_0)| + \varepsilon .
\end{equation}
Now (15) follows from (28) and (29). Also (26) shows that
\begin{equation}
\|x^*_t(h) - x^*_s(h)\| \leq \sqrt{\varepsilon} (\|h\| + |\alpha(t)| x^*_s(h))
\leq \sqrt{\varepsilon} (\|h\| + t |x^*_s(h)|) .
\end{equation}
Finally, since \( x^*_t \in \partial f(x_t) \),
\begin{equation}
x^*_s(x - x_0) \leq x^*_s(x - x_t) + x^*_s(x_t - x_0)
\leq f(x) - f(x_0) + [f(x_0) - f(x_t) + x^*_s(x_t - x_0)]
+ (x^*_t - x^*_0)(x_t - x_0) .
\end{equation}
Since (12) holds, \( f(x_0) - f(x_t) + x^*_s(x_t - x_0) \leq \varepsilon \), and since (26) holds,
\begin{equation}
|(x^*_t - x^*_s)(x_t - x_0)| \leq \sqrt{\varepsilon} (\|x_t - x_0\| + t |x^*_s(x_t - x_0)|) \leq \varepsilon ,
\end{equation}
on using (27). Then (31) establishes (18). Observe that, with the convention that \( 1/0 = \infty \), the arguments are preserved when \( t = 0 \). Let us also observe that back substitution of (29) into (27) produces a strengthening of (15) to
\begin{equation}
(15)' \quad \|x_s - x_0\| + t |f(x_t) - f(x_0)| \leq \sqrt{\varepsilon} + t\varepsilon ,
\end{equation}
which is slightly less convenient for application.

Remarks. (1) Our purposes in producing this proof with a parameter \( t \) are three-fold: (a) it leads to a unified development of the Bronsted-Rockafellar theorem \( (t = 0) \) and the improvement of the Taylor result \( (t = 1) \) and allows one to see the differences in the relative approximations in, for example, (15) and (16); (b) since one wishes to approximate in direction \( x^*_\varepsilon \) it is intuitively plausible that \( \| \cdot \|_t \) is the appropriate norm to use; (c) for all the details the proof is really very straightforward and essentially reduces to “apply Ekeland’s theorem to \( g \) and \( \| \cdot \|_t \).” Notice that (17), which is critical to the next result, is considerably more useful than (16) in relating \( x^*_\varepsilon(h) \) and \( x^* (h) \) as \( \varepsilon \) varies. This is because while \( \| x^*_\varepsilon \| \) typically will grow unboundedly as \( \varepsilon \) shrinks, \( |x^*_\varepsilon(h)| \) can generally be given a uniform bound independent of \( \varepsilon \).

Theorem 2. Let \( f \) be a proper convex lower semi-continuous convex function on a Banach space \((X, \| \cdot \|)\). Then, for any \( x_0 \) in the domain of \( f \) and any \( h \) in \( X \),

\[
\inf_{x^*_\varepsilon} \sup_{\varepsilon \in (0, 1)} \{ x^*_\varepsilon(h) | x^*_\varepsilon \in S_\varepsilon(x_0) \}
\]

for \( \varepsilon \in (0, 1) \),

(32)

where

\[
x^*_\varepsilon \in S_\varepsilon(x_0) \iff \begin{cases} (i) & x^*_\varepsilon \in \partial f(x_\varepsilon), \\ (ii) & \| x \varepsilon - x_0 \| \leq \varepsilon , \\ (iii) & \| f(x_\varepsilon) - f(x_0) \| \leq \varepsilon , \\ (iv) & x^*_\varepsilon \in \partial f(x_0). \end{cases}
\]

Proof. Since \( S_\varepsilon(h) \subset \partial f(x_0) \), Proposition 1 shows that it suffices to establish that the right hand side of (32) is no smaller than the left hand side. Suppose first that \( f'(x_0; h) = d < \infty \). Set \( 1 > \delta > 0 \) and pick \( x^*_\varepsilon \), using Proposition 1(9), so that \( x^*_\varepsilon(h) \geq d - \delta \) and \( x^*_\varepsilon \in \partial f(x_0) \). Let us apply Theorem 1 to this \( x^*_\varepsilon \) with \( t = 1 \), \( \delta = \varepsilon \). Then we obtain points \( x^*_\varepsilon \) and \( x \) with \( x^*_\varepsilon \in \partial f(x_\varepsilon) \) which on relabeling satisfy (33) with \( \varepsilon = 2\sqrt{\delta} \). Also (17) shows that

\[
x^*_\varepsilon(h) \geq x^*_\varepsilon(h) - \sqrt{\delta} (\| h \| + |x^*_\varepsilon(h)|).
\]

For sufficiently small \( \delta \), \( x^*_\varepsilon(h) \leq d + 1 \), as follows from (5). Thus

\[
x^*_\varepsilon(h) \geq d - \delta - \sqrt{\delta} (\| h \| - |d| - 1).
\]

Since the right hand side of this expression tends to \( d \) as \( \delta \) tends to zero, (32) is established in this case. Suppose now that \( f'(x_0; h) = \)}
Proposition 1 shows that we can pick \( x^* \in \partial f(x_0) \) and \( x^*(h) \geq 1/\delta \). As before (33) holds with \( \varepsilon = 2\sqrt{\delta} \). In this case (34) implies that

\[
(36) \quad x^*(h) \geq \left(1 - \sqrt{\delta}\right) \frac{1}{\delta} - \|h\|,
\]

and now the right hand side has supremum infinity. Again (32) is established.

The approximation in (32) is very strong as we may actually pick subgradients at points which are nearer and nearer \( x_0 \) and have converging function values. Observe that application of Theorem 1 with \( t = 0 \) leads to Theorem 2 except for (33) (iii).

One may recover Taylor’s formula [8] on replacing (33) (iii) and (iv) by

\[
(37) \quad |x^*(x_i - x_0)| \leq \varepsilon
\]

and observing that (37) follows from (33) (i), (ii), (iv) since

\[
|x^*(x_i - x_0)| \leq |f(x_i) - f(x_0)| + \delta
\]

if \( x^*_i \in \partial f(x_0) \cap \delta f(x_0) \). Thus Taylor’s approximating subset is a bigger set than ours. Since (37), (33) (i) and (ii) still force \( x^*_i \in \partial f(x_0) \) for small \( \varepsilon \), (32) still holds. Indeed, except for scale constants our Theorem 2 and Taylor’s Corollary 1 are interderivable.

Recall that \( \partial f \) is a monotone relation [3]: if \( x^*_i \in \partial f(x_i) \) \((i = 1, 2)\) then

\[
(38) \quad (x^*_s - x^*_i)(x_s - x_i) \geq 0.
\]

Rockafellar [5] produced a proof that \( \partial f \) is always maximal as a monotone relation. Rockafellar’s proof was irremediably flawed and he subsequently gave a correct proof using conjugate functions in [6]. Taylor [8] then produced an essentially correct proof more in the spirit of [5]. This proof is slightly flawed technically (\( d < \infty \) is assumed). We provide here a derivation of the result from Theorem 2.

**Corollary 1.** If \( f \) is a proper lower semi-continuous convex function on a Banach space \( X \) then \( \partial f \) is maximal as a monotone relation in \( X \times X^* \).

**Proof.** As in [5], [8] we may assume by translation that \( 0 \in \partial f(0) \). A one dimensional argument now produces a point \( x_0 \) in \( \text{dom} f \) with \( f'(x_i; -x_0) > 2\delta > 0 \). Note that it may well be that
\( f''(x_0; -x_0) \) is infinite, contrary to the implicit assumption in [8].

By any account, we have, from Theorem 2, points \( x_ε \) and \( x^*_ε \in \partial f(x_ε) \) with

\[
\begin{align*}
(\text{i}) & \quad x^*_ε(-x_0) > 2δ , \\
(\text{ii}) & \quad x^*_ε \in \partial_x f(x_0) , \\
(\text{iii}) & \quad |f(x_ε) - f(x_0)| ≤ δ .
\end{align*}
\]

Since (ii) holds

\[
x^*_ε(x_δ) ≤ x^*_ε(x_0) + f(x_δ) - f(x_0) + δ \tag{40}
\]

and thus (i) and (iii) combine to show

\[
x^*_ε(x_δ) < -2δ + δ + δ < 0 . \tag{41}
\]

Thus one cannot have \((x^* - 0)(x - 0) > 0\) for each \( x^* \in \partial f(x) \) and so \( \partial f \) is maximal.

**Corollary 2.** If \( f \) is a proper convex lower semi-continuous function on a Banach space \( X \) then

\[
f(x) = \limsup_{x_δ \downarrow x^*} \{ x^*(x) - f^*(x^*) | x^* \in S_δ(x) \} , \tag{42}
\]

where \( f^* \) is the conjugate function

\[
f^*(x^*) = \sup \{ x^*(x) - f(x) | x \in \text{dom } f \} . \tag{43}
\]

**Proof.** For any \( x^* \) in the nonempty set \( S_δ(x) \) one has

\[
x^*(y) - f(y) ≤ x^*(x) - f(x) + ε \tag{44}
\]

or

\[
x^*(x) - f^*(x^*) ≥ f(x) - ε . \tag{45}
\]

Thus the right hand side of (42) dominates \( f(x) \). The opposite inequality follows directly from (43) or Young’s inequality. \( \square \)

**Corollary 3.** If \( f \) is a proper, lower semi-continuous convex function on a Banach space \( X \) the following mean-value theorem holds. For each \( x_1 \) and \( x_2 \) in \( \text{dom } f \) one can find \( z \) in \((x_1, x_2)\) and sequences of points \( \{z_n\} \) in \( X \) and \( \{z^*_n\} \) in \( X^* \) with

\[
\begin{align*}
(\text{i}) & \quad \|z_n - z\| ≤ \frac{1}{n} , \\
(\text{ii}) & \quad |f(z_n) - f(z)| ≤ \frac{1}{n} , \\
(\text{iii}) & \quad z^*_n \in \partial f(z_n) ,
\end{align*}
\]

\[
\tag{46}
\]
and such that
\begin{equation}
\lim_{n} z_n^{*} = \frac{1}{n} f(x) ,
\end{equation}
\begin{equation}
\lim_{n} z_n^{*}(x_i - x_j) = f(x_i) - f(x_j) .
\end{equation}

Proof. It is straightforward to show that for some $z$ in $(x_i, x_j)$ one has
\begin{equation}
f'(z; x_i - x_j) \geq f(x_i) - f(x_j) \geq -f'(z; x_j - x_i).
\end{equation}
The result now follows from Theorem 1.

In the case that $f$ is continuous at $z$, as observed before $\partial_s f(z)$ is $w^*$ compact, and (47) reduces to the better known
\begin{equation}
f(x_i) - f(x_j) \in \partial f(x)(x_i - x_j). 
\end{equation}

References

Received December 19, 1980.

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PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
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Alberto Alesina and Leonede De Michele, A dichotomy for a class of positive definite functions ........................................ 251
Kahtan Alzubaidy, Rank\textsubscript{2} $p$-groups, $p > 3$, and Chern classes ............... 259
James Arney and Edward A. Bender, Random mappings with constraints on coalescence and number of origins .................................................. 269
Bruce C. Berndt, An arithmetic Poisson formula ............................................................... 295
Julius Rubin Blum and J. I. Reich, Pointwise ergodic theorems in l.c.a. groups ............ 301
Jonathan Borwein, A note on $\varepsilon$-subgradients and maximal monotonicity ............. 307
Andrew Michael Brunner, Edward James Mayland, Jr. and Jonathan Simon, Knot groups in $S^4$ with nontrivial homology .................................................. 315
Luis A. Caffarelli, Avner Friedman and Alessandro Torelli, The two-obstacle problem for the biharmonic operator .................................................. 325
Aleksander Całka, On local isometries of finitely compact metric spaces ..................... 337
William S. Cohn, Carleson measures for functions orthogonal to invariant subspaces .................................................. 347
Roger Fenn and Denis Karmen Sjerve, Duality and cohomology for one-relator groups .................................................. 365
Gen Hua Shi, On the least number of fixed points for infinite complexes ....................... 377
George Golightly, Shadow and inverse-shadow inner products for a class of linear transformations .................................................. 389
Joachim Georg Hartung, An extension of Sion’s minimax theorem with an application to a method for constrained games .................................................. 401
Vikram Jha and Michael Joseph Kallaher, On the Lorimer-Rahilly and Johnson-Walker translation planes .................................................. 409
Kenneth Richard Johnson, Unitary analogs of generalized Ramanujan sums ............. 429
Peter Dexter Johnson, Jr. and R. N. Mohapatra, Best possible results in a class of inequalities .................................................. 433
Dieter Jungnickel and Sharad S. Sane, On extensions of nets .......................................... 437
Johan Henricus Bernardus Kemperman and Morris Skibinsky, On the characterization of an interesting property of the arcsin distribution .................................................. 457
Karl Andrew Kosler, On hereditary rings and Noetherian $V$-rings ................................ 467
William A. Lampe, Congruence lattices of algebras of fixed similarity type. II ............ 475
M. N. Mishra, N. N. Nayak and Swadeenananda Pattanayak, Strong result for real zeros of random polynomials .................................................. 509
Sidney Allen Morris and Peter Robert Nickolas, Locally invariant topologies on free groups .................................................. 523
Richard Cole Penney, A Fourier transform theorem on nilmanifolds and nil-theta functions .................................................. 539
Andrei Shkalikov, Estimates of meromorphic functions and summability theorems .................................................. 569
László Székelyhidi, Note on exponential polynomials .................................................. 583
William Thomas Watkins, Homeomorphic classification of certain inverse limit spaces with open bonding maps .................................................. 589
David G. Wright, Countable decompositions of $E^n$ .................................................. 603
Takayuki Kawada, Correction to: “Sample functions of Pólya processes” .................. 611
Z. A. Chanturia, Errata: “On the absolute convergence of Fourier series of the classes $H^\omega \cap V[u]”$ .................................................. 611