

Pacific Journal of Mathematics

**A NOTE ON ε -SUBGRADIENTS AND MAXIMAL
MONOTONICITY**

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A NOTE ON ε -SUBGRADIENTS AND MAXIMAL MONOTONICITY

J. M. BORWEIN

It is our desire in this note to provide certain formulae relating subgradients, directional derivatives and ε -subgradients of proper lower semi-continuous convex functions defined on a Banach space.

Our aim is to provide these formulae, which somewhat extend those in [5], [6], [7], as a direct and hopefully straightforward consequence of Ekeland's non convex-version [3], of the Bishop-Phelps-Bronsted-Rockafellar Theorem [1], [2], [3], [4].

As a by-product we obtain somewhat more self contained proofs of the maximality of the subgradient as a monotone relation and of some related results.

1. **Preliminaries.** Throughout X is a real Hausdorff *locally convex space* (l.c.s) with topological dual X^* . A function $f: X \rightarrow [-\infty, \infty]$ is said to be *convex* if its *epigraph*, $\text{Epi } f = \{(x, r) \mid f(x) \leq r\}$ is a convex subset of $X \times R$. Also f is *lower semi-continuous* (l.s.c.) if $\text{Epi } f$ is closed. We will restrict our attention to *proper convex functions*. These are the functions which are somewhere finite and never $-\infty$. The *domain* of f , $\text{dom } f$, is the set of points in X for which $f(x)$ is finite.

With each convex function we associate its (one-sided) *directional derivative* at x in $\text{dom } f$ given by

$$(1) \quad f'(x; h) = \lim_{t \downarrow 0} \frac{f(x + th) - f(x)}{t}.$$

Then $f'(x; \cdot)$ is well defined as a (possibly improper) convex positively homogeneous function. We also define, for each $\varepsilon \geq 0$, the ε -*subgradient* set for f at x by

$$(2) \quad \partial_\varepsilon f(x) = \{x^* \in X^* \mid x^*(h) + f(x) \leq f(x + h) + \varepsilon, \forall h \in X\}.$$

When $\varepsilon = 0$, we suppress ε and the object is the ordinary subgradient. We now may also write

$$(3) \quad \partial f(x) = \{x^* \in X^* \mid x^*(h) \leq f'(x; h), \forall h \in X\}.$$

For amplification about these concepts the reader is referred to [3], [4], [7].

2. **The main result.** We begin with a subsidiary proposition

which may be found in [5] with a different proof.

PROPOSITION 1. *Let f be a lower semi-continuous proper convex function defined on a locally convex space X . For any x in the domain of f one has the following formula:*

$$(4) \quad f'(x; h) = \inf_{\varepsilon \downarrow 0} \sup \{x_\varepsilon^*(h) \mid x_\varepsilon^* \in \partial_\varepsilon f(x)\} .$$

Proof. Let $\varepsilon > 0$ and let $x_\varepsilon^* \in \partial_\varepsilon f(x)$. Then (2) shows that for $t > 0$

$$x_\varepsilon^*(h) \leq \frac{f(x + th) - f(x) + \varepsilon}{t} .$$

We let $t = \sqrt{\varepsilon}$ and derive

$$(5) \quad x_\varepsilon^*(h) \leq \frac{f(x + \sqrt{\varepsilon}h) - f(x)}{\sqrt{\varepsilon}} + \sqrt{\varepsilon} .$$

Then (5) and (1) combine to show that

$$(6) \quad f'(x; h) \geq \limsup_{\varepsilon \downarrow 0} \{x_\varepsilon^*(h) \mid x_\varepsilon^* \in \partial_\varepsilon f(x)\} .$$

Conversely, let d be any real number less than $f'(x; h)$, and let $\varepsilon > 0$ be given. For $0 \leq t \leq 1$ one has

$$(7) \quad f(x + th) \geq f(x) + td .$$

Thus the line segment

$$(8) \quad L = \{(x, f(x) - \varepsilon) + t(h, d) \mid 0 \leq t \leq 1\}$$

can be strictly separated from the closed convex set $\text{Epi } f$, [4]. Simple and standard calculation shows that any separating functional $(x^*, -r^*)$ in $X^* \times R$ satisfies $r^* > 0$ and that

$$(9) \quad \left(\frac{x^*}{r^*}\right)(h) \geq d - \varepsilon; \frac{x^*}{r^*} \in \partial_\varepsilon f(x) .$$

The nature of d and (9) show that

$$(10) \quad \sup \{x^*(h) \mid x^* \in \partial_\varepsilon f(x)\} \geq f'(x; h) - \varepsilon .$$

It is clear from (6) and (10) that (4) holds. □

If f is actually continuous at x then $\partial_\varepsilon f(x)$ is weak-star compact [4], and (4) reduces to the standard formula

$$(11) \quad f'(x; h) = \sup \{x^*(h) \mid x^* \in \partial f(x)\} .$$

Even in finite dimensions (11) can fail at a point of discontinuity, while in Fréchet space it is possible that ∂f is empty [4], [5]. In Banach space Rockafellar [5], [6], has given formulae replacing (11), in terms of approximations by subgradients at nearby points. Taylor [8] has given an alternative stronger formula. All these results follow from some form of the Bishop-Phelps [1] or Bronsted-Rockafellar [2] theorems. We now proceed to derive a strong version of Taylor's formula which uses Ekeland's variational form of the previously mentioned theorems [3].

THEOREM 1. *Let f be a proper convex lower semi-continuous function defined on a Banach space $(X, \|\cdot\|)$. Suppose that $\varepsilon > 0$ and $t \geq 0$ are given. Suppose that*

$$(12) \quad x_0^* \in \partial_\varepsilon f(x_0).$$

Then one may find points x_ε and x_ε^ such that*

$$(13) \quad x_\varepsilon^* \in \partial f(x_\varepsilon),$$

and such that

$$(14) \quad \|x_\varepsilon - x_0\| \leq \sqrt{\varepsilon},$$

$$(15) \quad |f(x_\varepsilon) - f(x_0)| \leq \sqrt{\varepsilon} \left(\sqrt{\varepsilon} + \frac{1}{t} \right),$$

$$(16) \quad \|x_\varepsilon^* - x_0^*\| \leq \sqrt{\varepsilon} (1 + t \|x_0^*\|),$$

$$(17) \quad |x_\varepsilon^*(h) - x_0^*(h)| \leq \sqrt{\varepsilon} (\|h\| + t |x_0^*(h)|),$$

$$(18) \quad x_\varepsilon^* \in \partial_{2\varepsilon} f(x_0).$$

Proof. We renorm X using the equivalent norm given by

$$(19) \quad \|x\|_t = \|x\| + t |x_0^*(x)|.$$

We set $g(x) = f(x) - x_0^*(x)$ and observe that g is l.s.c. and that

$$(20) \quad g(x_0) \leq \varepsilon + \inf_x g(x).$$

We now apply Ekeland's theorem [3, p. 29] to g and $\|\cdot\|_t$. We are promised the existence of x_ε in X such that, for $x \neq x_\varepsilon$,

$$(21) \quad g(x) + \sqrt{\varepsilon} \|x - x_\varepsilon\|_t > g(x_\varepsilon)$$

and

$$(22) \quad g(x_\varepsilon) + \sqrt{\varepsilon} \|x_0 - x_\varepsilon\|_t \leq g(x_0).$$

Now (21) can be read as saying that

$$(23) \quad 0 \in \partial(g + \sqrt{\varepsilon h})(x_\varepsilon); \quad h(x) = \|x - x_\varepsilon\|_t.$$

Since h is continuous, and since

$$(24) \quad \partial h(x_\varepsilon) = \{x^* + \alpha x_0^* \mid \|x^*\| \leq 1, |\alpha| \leq t\},$$

we may write, using the subgradient sum formula [4],

$$(25) \quad 0 \in \partial f(x_\varepsilon) - x_0^* + \sqrt{\varepsilon} \partial h(x_\varepsilon).$$

Hence there is some point x_ε^* in $\partial f(x_\varepsilon)$ of the form

$$(26) \quad x_\varepsilon^* = \sqrt{\varepsilon} x^* + (1 - \sqrt{\varepsilon} \alpha(t)) x_0^*$$

with $|\alpha(t)| \leq t$ and $\|x^*\| \leq 1$. Thus (16) holds. Since (20) holds (22) shows that

$$(27) \quad \sqrt{\varepsilon} \|x_0 - x_\varepsilon\| + \sqrt{\varepsilon} t |x_0^*(x_0 - x_\varepsilon)| \leq \varepsilon.$$

In particular (14) holds and

$$(28) \quad |x_0^*(x_0 - x_\varepsilon)| \leq \sqrt{\varepsilon}/t.$$

Combined use of (20) and (22) shows that

$$(29) \quad |f(x_\varepsilon) - f(x_0)| \leq |x_0^*(x_\varepsilon - x_0)| + \varepsilon.$$

Now (15) follows from (28) and (29). Also (26) shows that

$$(30) \quad \begin{aligned} \|x_\varepsilon^*(h) - x_0^*(h)\| &\leq \sqrt{\varepsilon} (\|h\| + |\alpha(t)| |x_0^*(h)|) \\ &\leq \sqrt{\varepsilon} (\|h\| + t |x_0^*(h)|). \end{aligned}$$

Finally, since $x_\varepsilon^* \in \partial f(x_\varepsilon)$,

$$(31) \quad \begin{aligned} x_\varepsilon^*(x - x_0) &\leq x_\varepsilon^*(x - x_\varepsilon) + x_\varepsilon^*(x_\varepsilon - x_0) \\ &\leq f(x) - f(x_0) + [f(x_0) - f(x_\varepsilon) + x_0^*(x_\varepsilon - x_0)] \\ &\quad + (x_\varepsilon^* - x_0^*)(x_\varepsilon - x_0). \end{aligned}$$

Since (12) holds, $f(x_0) - f(x_\varepsilon) + x_0^*(x_\varepsilon - x_0) \leq \varepsilon$, and since (26) holds,

$$|(x_\varepsilon^* - x_0^*)(x_\varepsilon - x_0)| \leq \sqrt{\varepsilon} (\|x_\varepsilon - x_0\| + t |x_0^*(x_\varepsilon - x_0)|) \leq \varepsilon,$$

on using (27). Then (31) establishes (18). Observe that, with the convention that $1/0 = \infty$, the arguments are preserved when $t = 0$. Let us also observe that back substitution of (29) into (27) produces a strengthening of (15) to

$$(15)' \quad \|x_\varepsilon - x_0\| + t |f(x_\varepsilon) - f(x_0)| \leq \sqrt{\varepsilon} + t\varepsilon,$$

which is slightly less convenient for application. □

REMARKS. (1) Our purposes in producing this proof with a parameter t are three-fold: (a) it leads to a unified development of the Bronsted-Rockafellar theorem ($t = 0$) and the improvement of the Taylor result ($t = 1$) and allows one to see the differences in the relative approximations in, for example, (15) and (16); (b) since one wishes to approximate in direction x_0^* it is intuitively plausible that $\|\cdot\|_\epsilon$ is the appropriate norm to use; (c) for all the details the proof is really very straightforward and essentially reduces to "apply Ekeland's theorem to g and $\|\cdot\|_\epsilon$ ". Notice that (17), which is critical to the next result, is considerably more useful than (16) in relating $x_0^*(h)$ and $x_\epsilon^*(h)$ as ϵ varies. This is because while $\|x_\epsilon^*\|$ typically will grow unboundedly as ϵ shrinks, $|x_\epsilon^*(h)|$ can generally be given a uniform bound independent of ϵ .

THEOREM 2. *Let f be a proper convex lower semi-continuous convex function on a Banach space $(X, \|\cdot\|)$. Then, for any x_0 in the domain of f and any h in X ,*

$$(32) \quad f'(x_0; h) = \inf_{\epsilon > 0} \sup \{x_\epsilon^*(h) \mid x_\epsilon^* \in S_\epsilon(x_0)\}$$

where

$$(33) \quad x_\epsilon^* \in S_\epsilon(x_0) \iff \begin{cases} \text{(i)} & x_\epsilon^* \in \partial f(x_\epsilon), \\ \text{(ii)} & \|x_\epsilon - x_0\| \leq \epsilon, \\ \text{(iii)} & |f(x_\epsilon) - f(x_0)| \leq \epsilon, \\ \text{(iv)} & x_\epsilon^* \in \partial_\epsilon f(x_0). \end{cases}$$

Proof. Since $S_\epsilon(h) \subset \partial_\epsilon f(x_0)$, Proposition 1 shows that it suffices to establish that the right hand side of (32) is no smaller than the left hand side. Suppose first that $f'(x_0; h) = d < \infty$. Set $1 > \delta > 0$ and pick x_0^* , using Proposition 1(9), so that $x_0^*(h) \geq d - \delta$ and $x_0^* \in \partial_\delta f(x_0)$. Let us apply Theorem 1 to this x_0^* with $t = 1$, $\delta = \epsilon$. Then we obtain points x_δ^* and x_δ with $x_\delta^* \in \partial f(x_\delta)$ which on relabeling satisfy (33) with $\epsilon = 2\sqrt{\delta}$. Also (17) shows that

$$(34) \quad x_\delta^*(h) \geq x_0^*(h) - \sqrt{\delta}(\|h\| + |x_0^*(h)|).$$

For sufficiently small δ , $x_0^*(h) \geq d + 1$, as follows from (5). Thus

$$(35) \quad x_\delta^*(h) \geq d - \delta - \sqrt{\delta}(\|h\| - |d| - 1).$$

Since the right hand side of this expression tends to d as δ tends to zero, (32) is established in this case. Suppose now that $f'(x_0; h) =$

∞ . Proposition 1 shows that we can pick $x_0^* \in \partial_\delta f(x_0)$ and $x_0^*(h) \geq 1/\delta$. As before (33) holds with $\varepsilon = 2\sqrt{\delta}$. In this case (34) implies that

$$(36) \quad x_0^*(h) \geq \left(1 - \sqrt{\delta}\right) \frac{1}{\delta} - \|h\|,$$

and now the right hand side has supremum infinity. Again (32) is established. \square

The approximation in (32) is very strong as we may actually pick subgradients at points which are nearer and nearer x_0 and have converging function values. Observe that application of Theorem 1 with $t = 0$ leads to Theorem 2 except for (33) (iii).

One may recover Taylor's formula [8] on replacing (33) (iii) and (iv) by

$$(37) \quad |x_\varepsilon^*(x_\varepsilon - x_0)| \leq \varepsilon$$

and observing that (37) follows from (33) (i), (ii), (iv) since

$$|x_\varepsilon^*(x_\varepsilon - x_0)| \leq |f(x_\varepsilon) - f(x_0)| + \delta$$

if $x_\varepsilon^* \in \partial_\delta f(x_0) \cap \partial f(x_\varepsilon)$. Thus Taylor's approximating subset is a bigger set than ours. Since (37), (33) (i) and (ii) still force $x_\varepsilon^* \in \partial_{2\varepsilon} f(x_0)$ for small ε , (32) still holds. Indeed, except for scale constants our Theorem 2 and Taylor's Corollary 1 are interderivable.

Recall that ∂f is a *monotone relation* [3]: if $x_i^* \in \partial f(x_i)$ ($i = 1, 2$) then

$$(38) \quad (x_2^* - x_1^*)(x_2 - x_1) \geq 0.$$

Rockafellar [5] produced a proof that ∂f is always maximal as a monotone relation. Rockafellar's proof was irremediably flawed and he subsequently gave a correct proof using conjugate functions in [6]. Taylor [8] then produced an essentially correct proof more in the spirit of [5]. This proof is slightly flawed technically ($d < \infty$ is assumed). We provide here a derivation of the result from Theorem 2.

COROLLARY 1. *If f is a proper lower semi-continuous convex function on a Banach space X then ∂f is maximal as a monotone relation in $X \times X^*$.*

Proof. As in [5], [8] we may assume by translation that $0 \notin \partial f(0)$. A one dimensional argument now produces a point x_0 in $\text{dom } f$ with $f'(x_0; -x_0) > 2\delta > 0$. Note that it may well be that

$f'(x_0; -x_0)$ is infinite, contrary to the implicit assumption in [8]. By any account, we have, from Theorem 2, points x_δ and $x_\delta^* \in \partial f(x_\delta)$ with

$$(39) \quad \begin{aligned} & \text{(i) } x_\delta^*(-x_0) > 2\delta, \\ & \text{(ii) } x_\delta^* \in \partial_\delta f(x_0), \\ & \text{(iii) } |f(x_\delta) - f(x_0)| \leq \delta. \end{aligned}$$

Since (ii) holds

$$(40) \quad x_\delta^*(x_\delta) \leq x_\delta^*(x_0) + f(x_\delta) - f(x_0) + \delta$$

and thus (i) and (iii) combine to show

$$(41) \quad x_\delta^*(x_\delta) < -2\delta + \delta + \delta < 0.$$

Thus one cannot have $(x^* - 0)(x - 0) > 0$ for each $x^* \in \partial f(x)$ and so ∂f is maximal. □

COROLLARY 2. *If f is a proper convex lower semi-continuous function on a Banach space X then*

$$(42) \quad f(x) = \limsup_{\varepsilon \downarrow 0} \{x^*(x) - f^*(x^*) \mid x^* \in S_\varepsilon(x)\},$$

where f^* is the conjugate function

$$(43) \quad f^*(x^*) = \sup \{x^*(x) - f(x) \mid x \in \text{dom } f\}.$$

Proof. For any x^* in the nonempty set $S_\varepsilon(x)$ one has

$$(44) \quad x^*(y) - f(y) \leq x^*(x) - f(x) + \varepsilon$$

or

$$(45) \quad x^*(x) - f^*(x^*) \geq f(x) - \varepsilon.$$

Thus the right hand side of (42) dominates $f(x)$. The opposite inequality follows directly from (43) or Young's inequality. □

COROLLARY 3. *If f is a proper, lower semi-continuous convex function on a Banach space X the following mean-value theorem holds. For each x_1 and x_2 in $\text{dom } f$ one can find z in (x_1, x_2) and sequences of points $\{z_n\}$ in X and $\{z_n^*\}$ in X^* with*

$$(46) \quad \begin{aligned} & \text{(i) } \|z_n - z\| \leq \frac{1}{n}, \\ & \text{(ii) } |f(z_n) - f(z)| \leq \frac{1}{n}, \\ & \text{(iii) } z_n^* \in \partial f(z_n), \end{aligned}$$

$$(iv) \quad z_n^* \in \partial \frac{1}{n} f(z),$$

and such that

$$(47) \quad \lim_n z_n^*(x_1 - x_2) = f(x_1) - f(x_2).$$

Proof. It is straightforward to show that for some z in (x_1, x_2) one has

$$(48) \quad f'(z; x_1 - x_2) \geq f(x_1) - f(x_2) \geq -f'(z; x_2 - x_1).$$

The result now follows from Theorem 1. □

In the case that f is continuous at z , as observed before $\partial_z f(z)$ is w^* compact, and (47) reduces to the better known

$$(49) \quad f(x_1) - f(x_2) \in \partial f(z)(x_1 - x_2).$$

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