KNOT GROUPS IN $S^4$ WITH NONTRIVIAL HOMOLOGY

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In this paper we exhibit smooth 2-manifolds $F^2$ in the 4-sphere $S^4$ having the property that the second homology of the group $\pi_1(S^4 - F^2)$ is nontrivial. In particular, we obtain tori for which $H_2(\pi_1) \cong \mathbb{Z}$ and, by forming connected sums, surfaces of genus $n$ for which $H_2(\pi_1)$ is the direct sum of $n$ copies of $\mathbb{Z}$. Corollaries include: (1) There are knotted surfaces in $S^4$ that cannot be constructed by forming connected sums of unknotted surfaces and knotted 2-spheres. (2) The class of groups that occur as knot groups of surfaces in $S^4$ is not contained in the class of high dimensional knot groups of $S^n$ in $S^{n+2}$.

If $F$ is a compact manifold ($\partial F = \emptyset$) in the $n$-sphere $S^n (n \geq 4)$ then, using Alexander duality and the fact that $H_n(\pi_1(S^n - F))$ is a homomorphic image of $H_n(S^n - F)$, it is easy to show that $H_2(\pi_1(S^n - F))$ is no larger than $H_n(F)$. In the case where $F$ is a 2-sphere in $S^4$, this is Kervaire's proof [6] that $H_2(\pi_1(S^4 - F)) = 0$. Since the property of vanishing second homology is so important in characterizing knot groups of spheres in spheres [6], it is interesting to ask [7, Problem 4.29] [14, Conjecture 4.13] whether it is shared by other manifolds $F$ in $S^4$. The answer we obtain is "sometimes".

For example, if $F^2$ is a closed, orientable 2-manifold embedded in $S^4$ in a standard way (i.e., contained in the equatorial 3-sphere), then $\pi_1(S^4 - F^2) \cong \mathbb{Z}$, which has trivial second homology. If we form the connected sum (analogous to composing knots $S^1 \# S^1$) of such a surface $F^2$ with a knotted 2-sphere $S^2$, then the group of the knotted surface $F^2 \# S^2$ in $S^4$ is just $\pi_1(S^4 - S^2)$; as noted above, this has trivial homology.

On the other hand, in §2, we shall exhibit smooth tori (of genus 1) $F^2$ in $S^4$ such that $H_2(\pi_1(S^4 - F^2)) \cong \mathbb{Z}$. Such a torus cannot be expressed as the connected sum of an unknotted torus and a knotted 2-sphere. Furthermore, $\pi_1(S^4 - F^2)$ cannot occur [6] as the knot group of some $S^n \subset S^{n+2}$. By spinning, we can generate knotted embeddings of the $n$-torus $S^1 \times \cdots \times S^1$ in $S^{n+2}$ having the same "unusual" knot groups.

In §3, we establish a connected-sum lemma, $H_2(\pi_1(S^4 - F^2 \# F^2)) \cong H_2(\pi_1(S^4 - F^2)) \oplus H_2(\pi_1(S^4 - F^2))$. By composing the tori found in §2, we can therefore construct surfaces of any genus $n$, for which

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1 A preliminary report on this paper appeared as [1].
the second homology of the knot group is $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ ($n$ summands). Thus, using the upperbound $H^i(F)$ mentioned above, we conclude that the groups that occur as knot groups of surfaces of genus $n$ in $S^4$ are a proper subset of the groups that arise from surfaces of genus $2n + 1$.

It seems plausible that the number $2n + 1$ (last sentence above) can be pushed closer to $n$. For surfaces of genus 1, we have been unable to find knot groups with second homology larger than $\mathbb{Z}_2$, and we are left with the question: Are there tori in $S^4$ whose knot groups have second homology equal to (even close to) the theoretical upperbound $\mathbb{Z} \oplus \mathbb{Z}$? In this connection, it may be noted that the example given in [12] of a homomorphic image, $G$, of a knot group $(S^1 \times S^3)$ with $H_2(G) \neq 0$ actually has $H_2(G) \cong \mathbb{Z}_2$; the groups $G$ one obtains by killing the longitude of a knot with Property $R$ [11] have $H_2(G) \cong \mathbb{Z}$ [4].

1. Preliminaries. The spaces and subspaces we discuss are smooth or polyhedral. All homology groups are taken with integer coefficients. If $G$ is a group and $x, y \in G$, then $[x, y]$ denotes $x^{-1}y^{-1}xy$; if $A, B \subseteq G$ then $[A, B]$ denotes the smallest normal subgroup of $G$ containing $\{[a, b]: a \in A, b \in B\}$.

There are several (equivalent) definitions of the second homology of a group.

**DEFINITION 1.1.** If $X$ is a connected $CW$-complex with $\pi_1(X) \cong G$ and $\pi_n(X) = 0$ ($n \geq 2$) then for each $p$, $H_p(G)$ is defined to be $H_p(X)$.

**DEFINITION 1.2.** If $Y$ is connected $CW$-complex with $\pi_1(Y) \cong G$, and $\Sigma_2(Y)$ denotes the subgroup of $H_2(Y)$ generated by all singular 2-cycles representable by maps of a 2-sphere into $Y$, then $H_2(G) = H_2(Y)/\Sigma_2(Y)$. (Informally, $H_2(G) = H_2(Y)/\pi_2(Y)$.)

**DEFINITION 1.3.** If $F$ is a free group, $\theta: F \rightarrow G$ an epimorphism, and $R = \ker \theta$, then $H_2(G) = R \cap [F, F]/[F, R]$.

The equivalence of 1.1 and 1.2 is clear, once one shows that 1.1 is unambiguous, since a space $X$ (as in 1.1) can be built from $Y$ (as in 1.2) by adjoining cells of dimension $\geq 3$. The equivalence of 1.2 and 1.3 is shown in [5]. For computing $H_2(G)$, it may be convenient to view $G$ as a quotient of some group $A$ that (is not free but still) has trivial second homology. The following lemma of J. Stallings [13] provides the necessary instructions.

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2 See concluding Remark.
Lemma 1.4. If $A$ is a group and $N$ is a normal subgroup of $A$ then there is a (natural) exact sequence
$$\begin{align*}
H_2(A) &\longrightarrow H_2(A/N) \longrightarrow N/[A, N] \longrightarrow H_1(A) \longrightarrow H_1(A/N) \longrightarrow 0.
\end{align*}$$

Lemma 1.4.1. If $A$ is a group with $H_2(A) = 0$, $N$ is a normal subgroup of $A$ such that $N \subseteq [A, A]$, and $G = A/N$, then $H_2(G) \cong N/[A, N]$.

Proof. This is just a special case of Lemma 1.4.

Lemma 1.5. Suppose a group $G$ has a presentation of the form $\langle a, b; b = w^{-1}aw \rangle$, where $w$ is some word in $a$ and $b$. Then $H_3(G) = 0$.

Proof. Let $Y$ be a 2-complex formed by attaching one disk to a wedge of two circles, such that $\pi_2(Y) \cong G$. By counting cells, we see the Euler characteristic of $Y$ is 0. Since $\beta_0(Y) = \beta_1(Y) = 1$, we conclude $\beta_2(Y) = 0$ and so, since $Y$ is 2-dimensional, $H_2(Y) = 0$. According to Definition 1.2, $H_2(G) = 0$.

Lemma 1.6. Suppose a group $G$ has a presentation of the form $\langle a, b; b = w^{-1}aw, [b, y] = 1 \rangle$, for some words $w, y$ in $a$ and $b$. Then $H_2(G)$ is isomorphic to the cyclic subgroup generated by $[b, y]$ in the group $C = \langle a, b; b = w^{-1}aw, [a, [b, y]] = 1 \rangle$.

Proof. Let $A = \langle a, b; b = w^{-1}aw \rangle$ and let $N$ be the normal subgroup of $A$ generated by $[b, y]$. By Lemma 1.5, $H_2(A) = 0$. By Lemma 1.4.1, we then have $H_2(G) \cong N/[A, N]$. The subgroup $[A, N]$ is the kernel of the obvious map of $A$ onto $C$, so $H_2(G)$ is isomorphic to the image of $N$ under this map; this image is precisely the cyclic subgroup of $C$ generated by $[b, y]$.

2. Examples of tori in $S^4$. Our first example is illustrated in Figure 1, in the form of successive cross-sections (as in § 6 of [3]). We originally obtained this torus $T$ by the methods of [16], so $T$ is a symmetric ribbon surface. We can, at this point, either compute $\pi_2(S^4 - T)$ from Figure 1 as in [3], or start with a suitable presentation of the group and invoke [16]. In either case, we have the following.

Proposition 2.1. If $T$ is the torus in Figure 1 then the group $G = \pi_2(S^4 - T)$ has a presentation $\langle a, b; b = a^{-1}b'a^{-2}a, b = [b^{-1}, a^{-1}b]^{-1}b[b^{-1}, a^{-1}b] \rangle$. 

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A torus with $H_2(G) \cong \mathbb{Z}_2$

Figure 1

**Theorem 2.2.** If $G$ is the group in 2.1 then $H_2(G) \cong \mathbb{Z}_2$.

**Proof.** Let $\lambda$ denote $[ba^{-1}, a^{-1}b]$, $w$ denote $b^{-1}a^{-1}b'ab^{-2}a$, $A = \langle a, b; w = 1 \rangle$ and $C = \langle a, b; w = [a, [b, \lambda]] = [b, [b, \lambda]] = 1 \rangle$. By Lemma 1.6, $H_2(G)$ is isomorphic to the cyclic subgroup of $C$ generated by $[b, \lambda]$.

First note that in $A$, hence in $C$, $b^{-1}ab = \lambda^{-1}$. (To see that $b^{-1}ab = 1$ in $A$, first cyclically reduce $b^{-1}ab$; then replace a subword, $a^{-1}b'ab^{-2}a$, of this with "$b$"; then note that the word so obtained is a cyclic permutation of $w^{-1}$.) Thus $[b, \lambda] = \lambda^5$ and $[b, [b, \lambda]] = \lambda^4$ in $A$. 
In $C$, since $[b, [b, \lambda]] = 1$, we have $\lambda^4 = 1$, i.e., $[\delta, \lambda]^2 = 1$. We thus have $H_2(G) \cong 0$ or $\mathbb{Z}$; to establish the latter, we need to show $\lambda$ (i.e., $[\delta, \lambda]$) is not trivial in $C$. Since $\lambda \in [C, C]$, we can compute the order of $\lambda$ in $C$ by computing its order in $[C, C]$.

Claim 2.3. $[C, C]$ has a presentation $\langle B_o, B_{-1}; [B_o, [B_o, B_{-1}]] = [B_o, [B_o, B_{-1}]]^4 = 1 \rangle$, where $\lambda^8 = [B_o, B_{-1}]^4$.

Proof of 2.3. To establish 2.3, we can use the Reidemeister-Schreier process [9, §2.3], with coset representatives $[a^n]_{n \in \mathbb{Z}}$ and rewriting function $\rho(b) = \rho(a) = a$, applied to the presentation $C \cong \langle a, b; w = [a, \lambda]^2 = \lambda^4 = 1 \rangle$. The presentation initially obtained will have infinitely many generators $B_n(a^n b^{-n} a^{-n}, n \in \mathbb{Z})$, but almost all the generators and relations can be eliminated, leaving 2.3. Alternatively, we can argue as follows.

Let $D = \langle u, v; [u, [u, v]]^2 = [v, [u, v]] = [u, v]^4 = 1 \rangle$. The function $\theta(u) = v, \theta(v) = vu$ sends $[u, v]$ to $[u, v]^1$ and therefore defines an automorphism of $D$. Extend $D$ to a group $\bar{D} = \langle D, b; b^{-1}g = \theta(g), \forall g \in D \rangle$. We then have $D = [\bar{D}, \bar{D}]$, and $\bar{D} \cong \langle u, v, b; b^{-1}ub = v, b^{-1}vb = vu, [u, v] = [u, [u, v]] = [v, [u, v]]^4 = 1 \rangle$. Use $v = b^{-1}ub$ to eliminate the generator $v$, introduce a new generator $a = u^{-1}b$, and use $u = ba^{-1}$ to eliminate the generator $u$. Since, as noted earlier, the relation $w = 1$ implies $b^{-1}\lambda b = \lambda^{-1}$, it is easy to show that $\bar{D}$ is exactly $C$. We know $D = [\bar{D}, \bar{D}]$, and if we identify $u$ with $B_o$, $v$ with $B_{-1}$, we obtain 2.3.

We now map $[C, C]$ onto the group $\mathcal{Z}_o = \langle B_o, B_{-1}; B_o^8 = B_{-1}^8 = (B_o B_{-1})^8 = 1 \rangle$ by setting $B_o^8 = B_{-1}^8 = 1$. Under this map, $\lambda^8 \to (B_o B_{-1})^4$. Since the order of $B_o B_{-1}$ in $\mathcal{Z}_o$ is exactly 8 [2, §§4.3, 4.4], we conclude $\lambda^8 \neq 1$ in $C$. This completes the proof of Theorem 2.2.

Remark 2.4. It can be shown that the group $A = \langle a, b; b = a^{-1}b^{-1}a \rangle$, sometimes called the Fibonacci group, is a $\mathbb{Z}_2$-extension of the group $K$ of the "figure-8" knot [8, §V.2]. By erasing the lower band in Figure 1, we can see a symmetric ribbon 2-sphere with knot group $A$. The elements $b^i$ and $\lambda = [ba^{-1}, a^{-1}b]$ are, respectively, the meridian and longitude for $K$. The fact that $K$ admits an outer automorphism $\alpha$ (conjugation by $b$ in $A$) with certain properties (e.g., $\alpha(\lambda) = \lambda^{-1}$) can be used as the basis for an alternate proof that $H_2(G) \cong \mathbb{Z}$. This analysis is the motivation for our next examples, and, in fact, the group $G$ below is isomorphic to the group $G$ of Theorem 2.2.

We originally built the groups $H_n$ (below) as $\mathbb{Z}_2$-extensions of the knot groups $\mathcal{K}_n$ of the knots $K(n, n)$ shown in Figure 2. By [10, p. 229-230], $\mathcal{K}_n \cong \langle a, b, t; t^{-1}a^5 bt = a^5, t^{-1}b^5 t = a^{-1}b^5 \rangle$. The
function $\theta(t) = t, \theta(b) = t^{-1}b^n b^{-n}$ defines an automorphism of $\mathcal{K}$ such that $\theta(g) = t^{-1}gt$ (all $g \in \mathcal{K}$). Let $H_n = \langle \mathcal{K}, s; s^2 = t, s^{-1}gs = \theta(g) \rangle$ (all $g \in \mathcal{K}$), and $\lambda = [s^{-1}b^n s, b^n]$ (= the longitude of $K(n, n)$). We can show, using arguments similar to [10, proof of Cor. 4.7] that for $n$ odd, centralizing $[b, \lambda]$ in $H_n$ does not kill $[b, \lambda]$. It follows that for $n$ odd, $H_2(G_n) = Z_2$, where $G_n = H_n/[b, \lambda]$. The proof below is somewhat removed from its knot theoretic origins, but the notation is consistent with the preceding remarks.

**Theorem 2.5.** There exists an infinite family $\{G_n\}$ of groups such that

(i) For each $n$, there is a smooth torus $T_n \cong S^1 \times S^1 \subseteq S^4$ such that $\pi_1(S^4 - T_n) \cong G_n$.

(ii) $G_m \cong G_n$ ($m \neq n$).

(iii) $H_2(G_n) \cong Z_2$ ($n$ odd).

**Proof.** (Remark: Our proof that $H_2(G_n) \neq 0$ requires $n$ to be odd, though another argument might make the assumption unnecessary.) Let $G_n = \langle b, s; s^{-1}b^n s = s^{-1}bs^n, [s, \lambda] = 1 \rangle$, where $\lambda = [s^{-1}b^n s, b^n]$.

**Claim 2.6.** $G_n$ has a Wirtinger presentation

$\langle x, s; x = (s^{-1}xs^{-1})^n s(s^{-1}xs^{-1})^{-n}, s = \lambda s \lambda \rangle$

where $x = b^s b^{-n}$ (and $\lambda$ now is expressed as a word in $x$ and $s$).

**Proof of 2.6.** Rewrite the relation $s^{-1}b^n s = s^{-1}bs^n$ as $b = s^{-1}b^n s^{-1}$. Introduce the new generator $x$ and replace the first relation with $b = s^{-1}x^n x^{-1}$. Use the latter to eliminate the generator $b$.

**Claim 2.7.** For each $n$, $G_n$ is the group of a smooth torus in $S^4$. 
Proof of 2.7. This follows from 2.6 and the methods of [16]. Figure 1 illustrates how to weave bands between two unknotted curves, following the instructions of a Wirtinger presentation of a group, to obtain a surface with that knot group.

Claim 2.8. For \( m \neq n \), \( G_m \ncong G_n \).

Proof of 2.8. These groups are distinguished by their Alexander polynomials \( A(t) = nt^2 + t - n \).

Claim 2.9. For each \( n \), \( H_2(G_n) \cong 0 \) or \( \mathbb{Z}_2 \).

Proof of 2.9. Let \( H_n = \langle b, s; s^{-1}bsb = s^{-1}bsb \rangle \) and let \( \lambda = [s^{-1}bsb, s, \lambda] \) in \( H_n \). Note that \( s^{-1}\lambda s = [s^{-1}bsb, s^{-1}bsb] = (\text{substitute}) [s^{-1}bsb, s^{-1}bsb] = \lambda^{-1} \).

We observe that \( G_n \) is obtained from \( H_n \) by killing \([s, \lambda]\) and so, by Claim 2.6 and Lemma 1.6, \( H_2(G_n) \) is isomorphic to the cyclic subgroup of \( C_n = H_n/[H_n, [s, \lambda]] \) generated by \([s, \lambda]\). Since \([s, \lambda] = \lambda^r \) in \( H_n \), we have \([s, [s, \lambda]] = \lambda^4 \). Thus, in \( C_n \), \([s, \lambda]^r = \lambda^4 = 1 \), so \([s, \lambda]\) has order 1 or 2 in \( C_n \).

Claim 2.10. \( H_2(G_n) \cong \mathbb{Z}_2 \) for \( n \) odd.

Proof of 2.10. From the proof of 2.9, we have \( \lambda^4 = 1 \) in \( C_n \) and need to show \( \lambda^2 \neq 1 \). We shall construct a homomorphic image \( D_\nu \) of \( C_n \) in which \( \lambda^2 \) is central but nontrivial.

Let \( F \) denote the free nilpotent group of class 2 \( \langle u, v; [[X, Y], Z] \rangle \). By a theorem of Gruenberg [9, §6.5], \( F \) is residually a finite 2-group. Thus, since \([u, v]^r \neq 1 \) in \( F \), there is, for some integer \( m \), a group \( F^0 \) in the variety of groups satisfying the laws \([[[X, Y], Z] = 1 \) and \( X^{2m} = 1 \) that is a homomorphic image of \( F \), in which \([u, v] \) has order \( 2^r \) for some \( r \geq 2 \). Since \( F^0 \) is nilpotent of class 2, the cyclic subgroup generated by \([u, v] \) is central, hence normal, and we can pass to a quotient \( F^* \) in which \([u, v]^r = 1 \) (but \([u, v]^2 \neq 1 \)). Since \( F^* \) is nilpotent and generated by \( u \) and \( v \), any commutator \([g, h] \) equals some power of \([u, v] \), so \([g, h]^r = 1 \). Thus we may choose \( F^* \) to be the free group of rank 2 in the variety defined by the laws \( X^{2m} = [[X, Y], Z] = [X, Y]^r = 1 \).

For any integer \( r \), the free group \( \langle x, y \rangle \) has an automorphism \( \tau \) given by \( \tau(x) = y \), \( \tau(y) = y^r x \). Since \( F^* \) is a reduced free group (i.e., \( \langle \text{free group}/(\text{verbal subgroup}) \rangle \), \( \tau \) induces an automorphism \( \tau^* \) of \( F^* \). Let \( D_\nu \) be the extension of \( F^* \), \( D_\nu = \langle u, v, t; t^{-1}ut = v, t^{-1}vt = v^r u, \text{relations for } F^* \rangle \). By eliminating \( v(=t^{-1}ut) \), we obtain \( D_\nu = \langle u, t; t^{-2}ut^2 = t^{-2}w'tu, \text{relations for } F^*(u, t^{-1}ut) \rangle \). Note that in
D\nu = [u, t^{-1}ut] has order exactly 4. We now restrict \nu so that \nu n \equiv 1 modulo (2^n).

The group \( C_n = H_n/[H_n, [s, \lambda]] \) has a presentation \( \langle b, s; s^{-1}b^2s = s^{-1}bb^s, [b, \lambda^2] = \lambda = 1 \rangle \). Add the relation \( b^{2m} = 1 \) to obtain a homomorph \( \hat{C}_n \) of \( C_n \). Introduce a new generator \( r = b^n \). By choice of \( \nu \), we then have \( r^\nu = b \); using this to eliminate \( b \), we obtain \( \hat{C}_n \cong \langle r, s; r^{2m} = 1, s^{-rs} = s^{-r}sr, [s, \lambda^2] = \lambda = 1 \rangle \), where \( \lambda = [s^{-rs}, r] \).

The mapping \( r \to u, s \to t \) defines an epimorphism of \( \hat{C}_n \) onto \( D_\nu \). Since \( \lambda^2 \) is central and has order exactly 2 in \( D_\nu \), this completes the proof of 2.10.

3. Connected sums. As with classical knots, one can compose knotted surfaces \( T_0, T_1 \) in 4-space (assuming \( T_0, T_1 \) are separated by a flat 3-plane or 3-sphere) by connecting \( T_0 \) and \( T_1 \) with a straight arc \( \alpha \) and using \( \alpha \) as a guide for an annulus from \( T_0 \) to \( T_1 \). We denote the surface so obtained by \( T_0 \#_2 T_1 \). The group \( \pi_2(T_0 \#_2 T_1) \) is of the form \( G_0 \# G_1 \) where \( G_i = \pi_1(S^4 - T_i) \) and \( \mu_i \) is a meridian of \( T_i \) (in particular, \( \mu_i \) generates \( \pi_1(G_i) \)). The following lemma implies that second homology of groups is additive under this type of composition.

**Lemma 3.1.** Let \( G \) and \( H \) be groups, \( g \in G, h \in H \), and suppose \( g \) has infinite order in \( G/[G, G] \) and \( h \) has infinite order in \( H \). Let \( \mathscr{G} \) denote \( G \ast_{g=h} H \). Then \( H_2(\mathscr{G}) \cong H_2(G) \oplus H_2(H) \).

**Proof.** Let \( X_G, X_H \) be connected, aspherical CW-complexes with fundamental groups \( G, H \). Adjoin a cylinder \( S^1 \times [0, 1] \) to the disjoint union of \( X_G \) and \( X_H \), using attaching maps of \( S^1 \times \{0\} \to X_G \), \( S^1 \times \{1\} \to X_H \) that trace out \( g, h \). The space \( W \) so obtained has \( \pi_2(W) \cong \mathscr{G} \). Furthermore, since \( g \) and \( h \) are of infinite order, it follows from [15, Theorem 5] that \( W \) is aspherical.

According to Definition 1.1, \( H_2(\mathscr{G}) \cong H_2(W), H_2(G) \cong H_2(X_G), \) and \( H_2(H) \cong H_2(X_H) \).

Since, by hypothesis, \( \langle g \rangle \to G/[G, G] \) is injective, the Mayer-Vietoris sequence for \( (W, X_G \cup S^1 \times [0, 1], X_H \cup S^1 \times (0, 1]) \) states that \( H_2(W) \cong H_2(X_G) \oplus H_2(X_H) \).

**Theorem 3.2.** If \( T_0, T_1 \) are surfaces in \( S^4 \) with knot groups \( G_0, G_1 \) respectively, then \( H_2(\pi_1(S^4 - T_0 \# T_1)) \cong H_2(G_0) \oplus H_2(G_1) \).

**Corollary 3.3.** The tori exhibited in §2 are not compositions of unknotted tori with knotted 2-spheres.

**Corollary 3.4.** For each \( n \geq 1 \), there exists a closed orientable
surface of genus \( n \), \( F_n \), in \( S^4 \) such that \( H_2(\pi_1(S^4 - F_n)) \cong \bigoplus_{n\geq 0} \mathbb{Z}_2 \).

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Remark. We have learned that T. Maeda ("On the groups with Wirtinger presentations", Math. Seminar Notes, Kwansei Gakuin Univ., Sept. 1977) also has obtained an example of a group with nontrivial second homology \( (\mathbb{Z}_2) \) that occurs as \( \pi_1(S^4 - F_n) \) for some surface \( F_n \). More recently, using methods similar to ours, C. Gordon has obtained tori in \( S^4 \) with \( H_2(G) = \mathbb{Z}_n \) for any desired \( n \geq 0 \). Finally, R. Litherland has found tori realizing all the groups \( \mathbb{Z}_p \oplus \mathbb{Z}_q \) \((p, q \geq 0)\).

References

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