KNOT GROUPS IN $S^4$ WITH NONTRIVIAL HOMOLOGY

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In this paper we exhibit smooth 2-manifolds $F^2$ in the 4-sphere $S^4$ having the property that the second homology of the group $\pi_1(S^4 - F^2)$ is nontrivial. In particular, we obtain tori for which $H_2(\pi_1) \cong \mathbb{Z}$ and, by forming connected sums, surfaces of genus $n$ for which $H_2(\pi_1)$ is the direct sum of $n$ copies of $\mathbb{Z}$. Corollaries include: (1) There are knotted surfaces in $S^4$ that cannot be constructed by forming connected sums of unknotted surfaces and knotted 2-spheres. (2) The class of groups that occur as knot groups of surfaces in $S^4$ is not contained in the class of high dimensional knot groups of $S^n$ in $S^{n+2}$.

If $F$ is a compact manifold ($\partial F = \phi$) in the $n$-sphere $S^n (n \geq 4)$ then, using Alexander duality and the fact that $H_*(\pi_1(S^n - F))$ is a homomorphic image of $H_*(S^n - F)$, it is easy to show that $H_*(\pi_1(S^n - F))$ is no larger than $H^{n-3}(F)$. In the case where $F$ is a 2-sphere in $S^n$, this is Kervaire's proof [6] that $H_*(\pi_1(S^n - F)) = 0$. Since the property of vanishing second homology is so important in characterizing knot groups of spheres in spheres [6], it is interesting to ask [7, Problem 4.29] [14, Conjecture 4.13] whether it is shared by other manifolds $F$ in $S^n$. The answer we obtain is "sometimes".

For example, if $F^2$ is a closed, orientable 2-manifold embedded in $S^4$ in a standard way (i.e., contained in the equatorial 3-sphere), then $\pi_1(S^4 - F^2) \cong \mathbb{Z}$, which has trivial second homology. If we form the connected sum (analogous to composing knots $S^1 \subset S^3$) of such a surface $F^2$ with a knotted 2-sphere $S^2$, then the group of the knotted surface $F^2 \# S^2$ in $S^4$ is just $\pi_1(S^4 - S^2)$; as noted above, this has trivial homology.

On the other hand, in § 2, we shall exhibit smooth tori (of genus 1) $F^2$ in $S^4$ such that $H_2(\pi_1(S^4 - F^2)) \cong \mathbb{Z}$. Such a torus cannot be expressed as the connected sum of an unknotted torus and a knotted 2-sphere. Furthermore, $\pi_1(S^4 - F^2)$ cannot occur [6] as the knot group of some $S^n \subset S^{n+2}$. By spinning, we can generate knotted embeddings of the $n$-torus $S^1 \times \cdots \times S^1$ in $S^{n+2}$ having the same "unusual" knot groups.

In § 3, we establish a connected-sum lemma, $H_2(\pi_1(S^4 - F_1^2 \# F_2^2)) \cong H_2(\pi_1(S^4 - F_1^2)) \oplus H_2(\pi_1(S^4 - F_2^2))$. By composing the tori found in § 2, we can therefore construct surfaces of any genus $n$, for which

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A preliminary report on this paper appeared as [1].
the second homology of the knot group is $\mathbb{Z}_2 \oplus \cdots \oplus \mathbb{Z}_2$ ($n$ summands). Thus, using the upperbound $H^1(F)$ mentioned above, we conclude that the groups that occur as knot groups of surfaces of genus $n$ in $S^4$ are a proper subset of the groups that arise from surfaces of genus $2n + 1$.

It seems plausible that the number $2n + 1$ (last sentence above) can be pushed closer to $n$. For surfaces of genus 1, we have been unable to find knot groups with second homology larger than $\mathbb{Z}_2$, and we are left with the question: Are there tori in $S^4$ whose knot groups have second homology equal to (even close to) the theoretical upperbound $\mathbb{Z} \oplus \mathbb{Z}$? In this connection, it may be noted that the example given in [12] of a homomorphic image, $G$, of a knot group $(S^1 \subset S^3)$ with $H_2(G) \neq 0$ actually has $H_2(G) \cong \mathbb{Z}_2$; the groups $G$ one obtains by killing the longitude of a knot with Property R [11] have $H_2(G) \cong \mathbb{Z}$ [4].

1. Preliminaries. The spaces and subspaces we discuss are smooth or polyhedral. All homology groups are taken with integer coefficients. If $G$ is a group and $x, y \in G$, then $[x, y]$ denotes $x^{-1}yx$; if $A, B \subseteq G$ then $[A, B]$ denotes the smallest normal subgroup of $G$ containing $\{(a, b): a \in A, b \in B\}$.

There are several (equivalent) definitions of the second homology of a group.

**Definition 1.1.** If $X$ is a connected CW-complex with $\pi_1(X) \cong G$ and $\pi_n(X) = 0$ ($n \geq 2$) then for each $p$, $H_p(G)$ is defined to be $H_p(X)$.

**Definition 1.2.** If $Y$ is connected CW-complex with $\pi_1(Y) \cong G$, and $\sum_2(Y)$ denotes the subgroup of $H_2(Y)$ generated by all singular 2-cycles representable by maps of a 2-sphere into $Y$, then $H_2(G) = H_2(Y)/\sum_2(Y)$. (Informally, $H_2(G) = H_2(Y)/\pi_2(Y)$.)

**Definition 1.3.** If $F$ is a free group, $\theta: F \to G$ an epimorphism, and $R = \ker \theta$, then $H_2(G) = R \cap [F, F]/[F, R]$.

The equivalence of 1.1 and 1.2 is clear, once one shows that 1.1 is unambiguous, since a space $X$ (as in 1.1) can be built from $Y$ (as in 1.2) by adjoining cells of dimension $\geq 3$. The equivalence of 1.2 and 1.3 is shown in [5]. For computing $H_2(G)$, it may be convenient to view $G$ as a quotient of some group $A$ that (is not free but still) has trivial second homology. The following lemma of J. Stallings [13] provides the necessary instructions.

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See concluding Remark.
Lemma 1.4. If $A$ is a group and $N$ is a normal subgroup of $A$ then there is a (natural) exact sequence

$$H_3(A) \longrightarrow H_3(A/N) \longrightarrow N/[A, N] \longrightarrow H_1(A) \longrightarrow H_1(A/N) \longrightarrow 0.$$ 

Lemma 1.4.1. If $A$ is a group with $H_2(A) = 0$, $N$ is a normal subgroup of $A$ such that $N \subseteq [A, A]$, and $G = A/N$, then $H_2(G) \cong N/[A, N]$.

Proof. This is just a special case of Lemma 1.4.

Lemma 1.5. Suppose a group $G$ has a presentation of the form $\langle a, b; b = w^{-1}aw \rangle$, where $w$ is some word in $a$ and $b$. Then $H_2(G) = 0$.

Proof. Let $Y$ be a 2-complex formed by attaching one disk to a wedge of two circles, such that $\pi_1(Y) \cong G$. By counting cells, we see the Euler characteristic of $Y$ is 0. Since $\beta_0(Y) = \beta_1(Y) = 1$, we conclude $\beta_2(Y) = 0$ and so, since $Y$ is 2-dimensional, $H_2(Y) = 0$. According to Definition 1.2, $H_2(G) = 0$.

Lemma 1.6. Suppose a group $G$ has a presentation of the form $\langle a, b; b = w^{-1}aw, [b, y] = 1 \rangle$, for some words $w, y$ in $a$ and $b$. Then $H_2(G)$ is isomorphic to the cyclic subgroup generated by $[b, y]$ in the group $C = \langle a, b; b = w^{-1}aw, [a, [b, y]] = 1, [b, [b, y]] = 1 \rangle$.

Proof. Let $A = \langle a, b; b = w^{-1}aw \rangle$ and let $N$ be the normal subgroup of $A$ generated by $[b, y]$. By Lemma 1.5, $H_2(A) = 0$. By Lemma 1.4.1, we then have $H_2(G) \cong N/[A, N]$. The subgroup $[A, N]$ is the kernel of the obvious map of $A$ onto $C$, so $H_2(G)$ is isomorphic to the image of $N$ under this map; this image is precisely the cyclic subgroup of $C$ generated by $[b, y]$.

2. Examples of tori in $S^4$. Our first example is illustrated in Figure 1, in the form of successive cross-sections (as in § 6 of [3]). We originally obtained this torus $T$ by the methods of [16], so $T$ is a symmetric ribbon surface. We can, at this point, either compute $\pi_1(S^4 - T)$ from Figure 1 as in [3], or start with a suitable presentation of the group and invoke [16]. In either case, we have the following.

Proposition 2.1. If $T$ is the torus in Figure 1 then the group $G = \pi_1(S^4 - T)$ has a presentation

$$\langle a, b; b = a^{-1}b^2ab^{-2}a, b = [ba^{-1}, a^{-1}b]^{-1}b[ba^{-1}, a^{-1}b] \rangle.$$
A torus with $H_3(G) \cong \mathbb{Z}_2$

**Figure 1**

**Theorem 2.2.** If $G$ is the group in 2.1 then $H_3(G) \cong \mathbb{Z}_2$.

**Proof.** Let $\lambda$ denote $[ba^{-1}, a^{-1}b]$, $w$ denote $b^{-1}a^{-1}b^2ab^{-2}a$, $A = \langle a, b; w = 1 \rangle$ and $C = \langle a, b; w = [a, [b, \lambda]] = [b, [b, \lambda]] = 1 \rangle$. By Lemma 1.6, $H_3(G)$ is isomorphic to the cyclic subgroup of $C$ generated by $[b, \lambda]$.

First note that in $A$, hence in $C$, $b^{-1}\lambda b = \lambda^{-1}$. (To see that $b^{-1}\lambda b\lambda = 1$ in $A$, first cyclically reduce $b^{-1}\lambda b\lambda$; then replace a sub-word, $a^{-1}b^2ab^{-2}a$, of this with "$b$"; then note that the word so obtained is a cyclic permutation of $w^{-1}$.) Thus $[b, \lambda] = \lambda^2$ and $[b, [b, \lambda]] = \lambda^4$ in $A$. 
In $C$, since $[6, [\delta, \lambda]] = 1$, we have $\lambda^4 = 1$, i.e., $[\delta, \lambda]^2 = 1$. We thus have $H_2(G) \cong 0$ or $\mathbb{Z}_2$; to establish the latter, we need to show $\lambda^2$ (i.e., $[\delta, \lambda]$) $\neq 1$ in $C$. Since $\lambda \in [C, C]$, we can compute the order of $\lambda$ in $C$ by computing its order in $[C, C]$.

Claim 2.3. $[C, C]$ has a presentation $\langle B_0, B_{-1}; [B_0, [B_0, B_{-1}]] = [B_{-1}, [B_0, B_{-1}]] = [B_0, B_{-1}]^4 = 1 \rangle$, where $\lambda^2 = [B_0, B_{-1}]^6$.

Proof of 2.3. To establish 2.3, we can use the Reidemeister-Schreier process [9, § 2.3], with coset representatives $\{a_n\}_{n \in \mathbb{Z}}$ and rewriting function $q(b) = q(a)$, applied to the presentation $C = \langle a, b; w = [a, \lambda^2] = \lambda^4 = 1 \rangle$. The presentation initially obtained will have infinitely many generators $B_n$ (i.e., $[B_0, B_{-1}]$ $\neq 1$ in $C$). Since $\lambda \in [C, C]$, we can compute the order of $\lambda$ in $C$ by computing its order in $[C, C]$.

We now map $[C, C]$ onto the group $D = \langle \delta, \delta = a^{-1}b^2ab^{-2}a \rangle$ by setting $B_0 = B_0^2 = B_{-1}^2 = 1$. Under this map, $\lambda^2 \rightarrow (B_0B_{-1})^4$. Since the order of $B_0B_{-1}$ in $D$ is exactly 8 [2, §§ 4.3, 4.4], we conclude $\lambda^2 \neq 1$ in $C$. This completes the proof of Theorem 2.2.

Remark 2.4. It can be shown that the group $A = \langle a, b; b = a^{-1}b^2ab^{-2}a \rangle$, sometimes called the Fibonacci group, is a $\mathbb{Z}_2$-extension of the group $K$ of the “figure-8” knot [8, § V.2]. By erasing the lower band in Figure 1, we can see a symmetric ribbon 2-sphere with knot group $A$. The elements $b^2$ and $\lambda = [ba^{-1}, a^{-1}b]$ are, respectively, the meridian and longitude for $K$. The fact that $K$ admits an outer automorphism $\alpha$ (conjugation by $b$ in $A$) with certain properties (e.g., $\alpha(\lambda) = \lambda^{-1}$) can be used as the basis for an alternate proof that $H_2(G) \cong \mathbb{Z}_2$. This analysis is the motivation for our next examples, and, in fact, the group $G_i$ below is isomorphic to the group $G$ of Theorem 2.2.

We originally built the groups $H_n$ (below) as $\mathbb{Z}_2$-extensions of the knot groups $\mathcal{K}_n$ of the knots $K(n, n)$ shown in Figure 2. By [10, p. 229-230], $\mathcal{K}_n \cong \langle a, b, t; t^{-1}a^nb = a^n, t^{-1}b^nt = a^{-1}b^n \rangle$. The
function \( \theta(t) = t, \theta(b) = t'Vtb' \) defines an automorphism of \( \mathbb{H}_n \) such that \( \theta^g(t) = t' Vgt \) (all \( g \in \mathbb{H}_n \)). Let \( H_n = \langle \mathbb{H}_n, s; s^2 = t, s^{-1}gs = \theta(g) \rangle \) (all \( g \in \mathbb{H}_n \)), and \( \lambda = [s^{-1}b^n s, b^n] \) (=the longitude of \( K(n, n) \)). We can show, using arguments similar to [10, proof of Cor. 4.7] that for \( n \) odd, centralizing \([b, \lambda]\) in \( H_n \) does not kill \([b, \lambda]\). It follows that for \( n \) odd, \( H_2(G_n) \cong \mathbb{Z}_2 \), where \( G_n = H_n/[b, \lambda] \). The proof below is somewhat removed from its knot theoretic origins, but the notation is consistent with the preceding remarks.

**Theorem 2.5.** There exists an infinite family \( \{G_n\} \) of groups such that

1. For each \( n \), there is a smooth torus \( T_n \cong S^1 \times S^1 \subset S^4 \) such that \( \pi_1(S^4 - T_n) \cong G_n \).
2. \( G_m \not\cong G_n \) (\( m \not= n \)).
3. \( H_2(G_n) \cong \mathbb{Z}_2 \) (\( n \) odd).

**Proof.** (Remark: Our proof that \( H_2(G_n) \neq 0 \) requires \( n \) to be odd, though another argument might make the assumption unnecessary.) Let \( G_n = \langle b, s; s^{-2}b^ns^2 = s^{-1}bsb^n, [s, \lambda] = 1 \rangle \), where \( \lambda = [s^{-1}b^ns, b^n] \).

**Claim 2.6.** \( G_n \) has a Wirtinger presentation

\[
\langle x, s; x = (s^{-1}xs^{-1})^n s(s^{-1}xs^{-1})^{-n}, s = \lambda^{-1}s\lambda \rangle
\]

where \( x = b^nsb^{-n} \) (and \( \lambda \) now is expressed as a word in \( x \) and \( s \)).

**Proof of 2.6.** Rewrite the relation \( s^{-2}b^ns^2 = s^{-1}bsb^n \) as \( b = s^{-1}b^ns^2b^{-n}s^{-1} \). Introduce the new generator \( x \) and replace the first relation with \( b = s^{-1}x^2s^{-1} \). Use the latter to eliminate the generator \( b \).

**Claim 2.7.** For each \( n \), \( G_n \) is the group of a smooth torus in \( S^4 \).
Proof of 2.7. This follows from 2.6 and the methods of [16]. Figure 1 illustrates how to weave bands between two unknotted curves, following the instructions of a Wirtinger presentation of a group, to obtain a surface with that knot group.

Claim 2.8. For \( m \neq n \), \( G_m \not\cong G_n \).

Proof of 2.8. These groups are distinguished by their Alexander polynomials \( \Delta(t) = nt^2 + t - n \).

Claim 2.9. For each \( n \), \( H_2(G_n) \cong 0 \) or \( \mathbb{Z}_2 \).

Proof of 2.9. Let \( H_n = \langle b, s; s^{-1}b^ns^2 = s^{-1}bbs^n \rangle \) and let \( \lambda = [s^{-1}b^ns, b^n] \) in \( H_n \). Note that \( s^{-1}\lambda s = [s^{-2}b^ns, s^{-1}b^ns] = (\text{substitute}) [s^{-1}bbs^n, s^{-1}b^ns] = \lambda^{-1} \).

We observe that \( G_n \) is obtained from \( H_n \) by killing \( [s, \lambda] \) and so, by Claim 2.6 and Lemma 1.6, \( H_2(G_n) \) is isomorphic to the cyclic subgroup of \( C_n = H_n/[H_n, [s, \lambda]] \) generated by \( [s, \lambda] \). Since \( [s, \lambda] = \lambda^2 \) in \( H_n \), we have \( [s, [s, \lambda]] = \lambda^4 \). Thus, in \( C_n \), \( [s, \lambda]^2 = \lambda^4 = 1 \), so \( [s, \lambda] \) has order 1 or 2 in \( C_n \).

Claim 2.10. \( H_2(G_n) \cong \mathbb{Z}_2 \) for \( n \) odd.

Proof of 2.10. From the proof of 2.9, we have \( \lambda^4 = 1 \) in \( C_n \) and need to show \( \lambda^2 \neq 1 \). We shall construct a homomorphic image \( D_n \) of \( C_n \) in which \( \lambda^2 \) is central but nontrivial.

Let \( F \) denote the free nilpotent group of class 2 \( \langle u, v; [[X, Y], Z] \rangle \). By a theorem of Gruenberg [9, §6.5], \( F \) is residually a finite 2-group. Thus, since \( [u, v]^2 \neq 1 \) in \( F \), there is, for some integer \( m \), a group \( \hat{F} \) in the variety of groups satisfying the laws \( [[X, Y], Z] = 1 \) and \( X^{2m} = 1 \) that is a homomorphic image of \( F \), and in which \( [u, v] \) has order \( 2^r \) for some \( r \geq 2 \). Since \( \hat{F} \) is nilpotent of class 2, the cyclic subgroup generated by \( [u, v] \) is central, hence normal, and we can pass to a quotient \( F^* \) in which \( [u, v]^4 = 1 \) (but \( [u, v]^2 \neq 1 \)). Since \( F^* \) is nilpotent and generated by \( (u, v) \), any commutator \( [g, h] \) equals some power of \( [u, v] \), so \( [g, h]^4 = 1 \). Thus we may choose \( F^* \) to be the free group of rank 2 in the variety defined by the laws \( X^{2m} = [[X, Y], Z] = [X, Y]^4 = 1 \).

For any integer \( \nu \), the free group \( \langle x, y \rangle \) has an automorphism \( \tau \) given by \( \tau(x) = y, \tau(y) = y^\nu x \). Since \( F^* \) is a reduced free group (i.e., \( \text{(free group)}/(\text{verbal subgroup}) \)), \( \tau \) induces an automorphism \( \tau^* \) of \( F^* \). Let \( D_\nu \) be the extension of \( F^* \), \( D_\nu = \langle u, v, t; t^{-1}ut = v, t^{-1}vt = v^\nu u \rangle \), relations for \( F^*(u, v) \). By eliminating \( v(-t^{-1}ut) \), we obtain \( D_\nu = \langle u, t; t^{-2}ut = t^{-1}u^\nu t \rangle \), relations for \( F^*(u, t^{-1}ut) \). Note that in
$D_\nu, [u, t^{-ut}]$ has order exactly 4. We now restrict $\nu$ so that $\nu n \equiv 1$ modulo $(2^n)$.

The group $C_n = H_n/[H_n, [s, \lambda]]$ has a presentation $\langle b, s; s^{-2}b^n s^2 = s^{-2}b b^n, [b, \lambda^n] = \lambda^4 = 1 \rangle$. Add the relation $b^{2m} = 1$ to obtain a homomorph $\hat{C}_n$ of $C_n$. Introduce a new generator $r = b^n$. By choice of $\nu$, we then have $r^\nu = b$; using this to eliminate $b$, we obtain $\hat{C}_n \cong \langle r, s; r^{2m} = 1, s^{-2}r s^2 = s^{-2}r^{\nu} s r, [r, \lambda^\nu] = \lambda^4 = 1 \rangle$, where $\lambda = [s^{-1}r s, r]$. The mapping $r \to u, s \to t$ defines an epimorphism of $\hat{C}_n$ onto $D_\nu$. Since $\lambda^\nu$ is central and has order exactly 2 in $D_\nu$, this completes the proof of 2.10.

3. Connected sums. As with classical knots, one can compose knotted surfaces $T_0, T_1$ in 4-space (assuming $T_0, T_1$ are separated by a flat 3-plane or 3-sphere) by connecting $T_0$ and $T_1$ with a straight arc $\alpha$ and using $\alpha$ as a guide for an annulus from $T_0$ to $T_1$. We denote the surface so obtained by $T_0 \# T_1$. The group $\pi_1(S^4-T_0 \# T_1)$ is of the form $G_0 \ast_{\mu_0 \ast \mu_1} G_1$, where $G_i = \pi_1(S^4-T_i)$ and $\mu_i$ is a meridian of $T_i$ (in particular, $\mu_i$ generates $G_i/[G_i,G_i]$). The following lemma implies that second homology of groups is additive under this type of composition.

**Lemma 3.1.** Let $G$ and $H$ be groups, $g \in G, h \in H$, and suppose $g$ has infinite order in $G/[G,G]$ and $h$ has infinite order in $H$. Let $\mathcal{G}$ denote $G *_{g = h} H$. Then $H_2(\mathcal{G}) \cong H_2(G) \oplus H_2(H)$.

**Proof.** Let $X_G, X_H$ be connected, aspherical CW-complexes with fundamental groups $G, H$. Adjoin a cylinder $S^1 \times [0,1]$ to the disjoint union of $X_G$ and $X_H$ using attaching maps of $S^1 \times \{0\} \to X_G$, $S^1 \times \{1\} \to X_H$ that trace out $g, h$. The space $W$ so obtained has $\pi_1(W) \cong \mathcal{G}$. Furthermore, since $g$ and $h$ are of infinite order, it follows from [15, Theorem 5] that $W$ is aspherical. According to Definition 1.1, $H_2(\mathcal{G}) \cong H_2(W), H_2(G) \cong H_2(X_G)$, and $H_2(H) \cong H_2(X_H)$. Since, by hypothesis, $\langle g \rangle \to G/[G,G]$ is injective, the Mayer-Vietoris sequence for $(W, X_G \cup S^1 \times [0,1], X_H \cup S^1 \times (0,1])$ states that $H_2(W) \cong H_2(X_G) \oplus H_2(X_H)$.

**Theorem 3.2.** If $T_0, T_1$ are surfaces in $S^4$ with knot groups $G_0, G_1$ respectively, then $H_2(\pi_1(S^4 - T_0 \# T_1)) \cong H_2(G_0) \oplus H_2(G_1)$.

**Corollary 3.3.** The tori exhibited in §2 are not compositions of unknotted tori with knotted 2-spheres.

**Corollary 3.4.** For each $n \geq 1$, there exists a closed orientable
surface of genus \( n \), \( F_n \), in \( S^4 \) such that \( H_2(\pi_1(S^4 - F_n)) \cong \bigoplus_{n} \mathbb{Z} \).

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Remark. We have learned that T. Maeda ("On the groups with Wirtinger presentations", Math. Seminar Notes, Kwansei Gakuin Univ., Sept. 1977) also has obtained an example of a group with nontrivial second homology \( (\mathbb{Z}_2) \) that occurs as \( \pi_1(S^4 - F^2) \) for some surface \( F^2 \). More recently, using methods similar to ours, C. Gordon has obtained tori in \( S^4 \) with \( H_2(G) = \mathbb{Z}_n \) for any desired \( n \geq 0 \). Finally, R. Litherland has found tori realizing all the groups \( \mathbb{Z}_p \oplus \mathbb{Z}_q \) (\( p, q \geq 0 \)).

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