

Pacific Journal of Mathematics

THE TWO-OBSTACLE PROBLEM FOR THE BIHARMONIC OPERATOR

LUIS A. CAFFARELLI, AVNER FRIEDMAN AND ALESSANDRO TORELLI

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In this work we consider a two-obstacle problem for the plate, namely, the problem of finding a minimizer u of

$$\int_{\Omega} |\Delta v|^2 dx, \text{ subject to } (v - h) \in H_0^2(\Omega), \quad \phi \leq v \leq \psi$$

where Ω is a bounded domain in R^n ; $n = 2, 3$. We prove that $u \in C^{1,1}$ and that, in general, $u \notin C^2$.

1. **The main results.** Let Ω be a bounded domain in R^n ($n = 2, 3$) with $C^{2+\alpha}$ boundary $\partial\Omega$, where $0 < \alpha < 1$. Let $h(x)$ be a function in $C^{2+\alpha}(\bar{\Omega})$, and let $\phi(x), \psi(x)$ be functions in $C^4(\bar{\Omega})$ satisfying

$$(1.1) \quad \begin{aligned} \phi &\leq \psi && \text{in } \Omega, \\ \phi &< h < \psi && \text{on } \partial\Omega. \end{aligned}$$

Then the set

$$K = \{v; (v - h) \in H_0^2(\Omega), \phi \leq v \leq \psi \text{ a.e.}\}$$

is nonempty.

Consider the variational inequality: find u such that

$$(1.2) \quad \min_{v \in K} \int_{\Omega} |\Delta v|^2 dx = \int_{\Omega} |\Delta u|^2 dx, \quad u \in K.$$

By standard results [4] [5] this problem has a unique solution. We shall prove:

THEOREM 1.1. *u belongs to $C^{1,1}(\Omega)$.*

That means that $\nabla^2 u \in L^\infty(\Omega)$.

We shall also show that, in general,

$$(1.3) \quad u \notin C^2 \text{ locally.}$$

For the corresponding variational inequality (for Δ^2) with one obstacle only (i.e., $u \geq \phi$ instead of $\phi \leq u < \psi$) it was proved by Caffarelli and Friedman [1] that, for $n \geq 2$, $u \in C^{1,1}$ locally and, for $n = 2$, $u \in C^2$ locally.

Notice that if in Theorem 1.1 $\phi < \psi$ in a subdomain Ω_0 of Ω , then the coincidence sets $\{u = \phi\}$, $\{u = \psi\}$ are disjoint in Ω_0 (since u

is continuous). Thus (1.3) can only hold (at least for $n = 2$) in a neighborhood of a point x^0 for which $\phi(x^0) = \psi(x^0)$.

In §2 we shall prove that $\Delta u \in L^\infty(\Omega)$ and in §3 we shall complete the proof of Theorem 1.1. An example for which (1.3) holds is given in §4.

2. Δu is bounded. Set

$$\begin{aligned} \phi_\varepsilon &= \phi - \varepsilon, \quad \varepsilon > 0, \\ K_\varepsilon &= \text{the set } K \text{ with } \phi \text{ replaced by } \phi_\varepsilon. \end{aligned}$$

Denote by u_ε the solution of the variational inequality (1.2) with K replaced by K_ε . Clearly,

$$\int_\Omega |\Delta u_\varepsilon|^2 dx \leq C, \quad C \text{ independent of } \varepsilon.$$

Since $n \leq 3$ we can apply Sobolev's inequality to deduce that

$$(2.1) \quad \begin{aligned} u_\varepsilon &\text{ is uniformly continuous in } x, \text{ with modulus} \\ &\text{of continuity independent of } \varepsilon. \end{aligned}$$

It follows that the coincidence sets

$$I_\varepsilon^+ = \{u_\varepsilon = \psi\}, \quad I_\varepsilon^- = \{u_\varepsilon = \phi\},$$

are closed disjoint sets. Furthermore, by (1.1), (2.1),

$$(2.2) \quad d(I_\varepsilon^\pm, \partial\Omega) \geq \delta > 0, \quad \delta \text{ independent of } \varepsilon,$$

where

$$d(A, B) = \text{dist.}(A, B).$$

We now claim that

$$(2.3) \quad u_\varepsilon \longrightarrow u \text{ uniformly in } \Omega, \quad \text{as } \varepsilon \longrightarrow 0.$$

Indeed for any sequence $\varepsilon_m \rightarrow 0$ there is a subsequence $\varepsilon_{m'} \rightarrow 0$ such that

$$u_{\varepsilon_{m'}} \longrightarrow \bar{u} \quad \text{weakly in } H^2(\Omega).$$

The variational inequality for $u_{\varepsilon_{m'}}$ can be written in the form (Minty's lemma)

$$\int_\Omega \Delta v \cdot \Delta(v - u_{\varepsilon_{m'}}) \geq 0 \quad \text{for every } v \in K_{\varepsilon_{m'}}.$$

Taking $m' \rightarrow \infty$ we get

$$\int_{\Omega} \Delta v \cdot \Delta(v - u) \geq 0 \quad \text{for every } v \in K,$$

so that u is the solution u of (1.2); this completes the proof of (2.3).

Since I_{ε}^+ , I_{ε}^- are disjoint closed sets, there is a version of Δu which is subharmonic and upper semicontinuous in $\Omega \setminus I_{\varepsilon}^+$ and superharmonic and lower semicontinuous in $\Omega \setminus I_{\varepsilon}^-$; this is proved exactly as in [1].

Set

$$\Omega_r = \{x \in \Omega; d(x, \partial\Omega) > r\}, \quad r > 0.$$

Let ζ be a $C_0^\infty(\Omega)$ function such that

$$\begin{aligned} \zeta &= 1 \quad \text{in } \Omega_{\delta/2}, \quad \zeta = 0 \quad \text{in } \Omega \setminus \Omega_{\delta/4}, \\ 0 &\leq \zeta \leq 1 \quad \text{elsewhere; } \quad \delta \text{ as in (2.2)}. \end{aligned}$$

We can represent Δu_{ε} as in [1; (3.8)] in the form

$$(2.4) \quad \Delta u_{\varepsilon}(x) = - \int_{\Omega_{\delta}} V(x, y) d\mu(y) + \gamma(x)$$

where $|\gamma(x)|$ is a bounded function in $\Omega_{\delta/2}$, with an upper bound independent of ε , $d\mu = \Delta^2 u_{\varepsilon}$ and V is Green's function for $-\Delta$, for a ball containing $\bar{\Omega}$; here we have used the fact (which follows from (2.2)) that $\Delta^2 u_{\varepsilon} = 0$ in $\Omega \setminus \Omega_{\delta}$ and, consequently, the first two derivatives of u_{ε} are bounded in $\Omega_{\delta/2}$ by a constant independent of ε .

Notice that μ is a signed measure; it can be written as a difference $\mu_1 - \mu_2$ of two positive measures, where μ_1 is $\Delta^2 u_{\varepsilon}$ supported on I_{ε}^- and μ_2 is $\Delta^2 u_{\varepsilon}$ -supported on I_{ε}^+ .

Introduce the notation:

$$\begin{aligned} B(y, \rho) &= \{x; |x - y| < \rho\}, \quad B(\rho) = B(0, \rho), \\ S_{\rho}(y) &= \partial B(y, \rho), \quad S_{\rho} = \partial B(\rho), \\ |S_{\rho}| &= \text{surface area of } S_{\rho}. \end{aligned}$$

We reason as in [1]. Let $x_0 \in I_{\varepsilon}^-$. Then

$$\begin{aligned} u_{\varepsilon}(x_0) &= \frac{1}{|S_{\delta}|} \int_{S_{\delta}(x_0)} u_{\varepsilon} - \int_{B_{\delta}(x_0)} G \Delta u_{\varepsilon}, \\ \phi_{\varepsilon}(x_0) &= \frac{1}{|S_{\delta}|} \int_{S_{\delta}(x_0)} \phi_{\varepsilon} - \int_{B_{\delta}(x_0)} G \Delta \phi_{\varepsilon}. \end{aligned}$$

Here G denotes

$$C\left(\frac{1}{r} - \frac{1}{\delta}\right) \quad \text{in } R^3,$$

$$C \log \frac{r}{\delta} \quad \text{in } R^2$$

for some constant $C > 0$. Since

$$u_\varepsilon(x_0) = \phi_\varepsilon(x_0)$$

$$\int_{S_\delta(x_0)} u_\varepsilon \geq \int_{S_\delta(x_0)} \phi_\varepsilon$$

and

$$\frac{1}{|S_\delta|} \int_{S_\delta(x_0)} \Delta u_\varepsilon$$

is a monotone function of δ , for $\delta \rightarrow 0$, we get

$$(2.5) \quad \Delta u_\varepsilon(x_0) \geq \Delta \phi_\varepsilon(x_0) \quad \text{if } x_0 \in \text{supp } \mu_1 .$$

Similarly

$$(2.6) \quad \Delta u_\varepsilon \leq \Delta \psi_\varepsilon \quad \text{on } \text{supp } \mu_2 .$$

The function

$$(2.7) \quad \hat{V}(x) = \int_{\Omega_\delta} V(x, y) d\mu(y)$$

satisfies, by (2.4)-(2.6),

$$\hat{V}(x) \leq C \quad \text{on } \text{supp } \mu_1 ,$$

$$\hat{V}(x) \geq -C \quad \text{on } \text{supp } \mu_2$$

where C is a constant independent of ε . As in the proofs of Theorems 1.6, 1.10 of [3], we then have

$$\limsup_{d(x, \text{supp } \mu_1) \rightarrow 0} \hat{V}(x) \leq C , \quad \limsup_{d(x, \text{supp } \mu_2) \rightarrow 0} \hat{V}(x) \geq -C .$$

Hence, by the maximum principle,

$$|\hat{V}(x)| \leq C \quad \text{in } \Omega_\delta$$

and (2.4) gives

$$|\Delta u_\varepsilon| \leq C \quad \text{in } \Omega_{\delta/2}$$

with another C . Taking $\varepsilon \rightarrow 0$ and recalling (2.3), we obtain:

LEMMA 2.1. Δu is in $L^\infty(\Omega)$.

3. $u \in C^{1,1}$. Let

$$w \in H^2(\Omega) , \quad \Delta w \in L^\infty(\Omega) , \quad w \geq 0 ,$$

and set

$$J = \{x \in \Omega; w(x) = 0\}, \quad \|\Delta w\|_{L^\infty(\Omega)} \leq M_0.$$

LEMMA 3.1. *There exists a constant M depending only on M_0 such that if $x_0 \in J$ then*

$$(3.1) \quad |w(x)| \leq M|x - x_0|^2, \quad |\nabla w(x)| \leq M|x - x_0| \quad \text{if } x \in B(x_0, \rho/2)$$

where $\rho = d(x_0, \partial\Omega)$.

Proof. Take for simplicity $x_0 = 0$ and consider the function

$$w_\rho(x) = \frac{1}{\rho^2}w(\rho x) \quad \text{in } B(1).$$

Then

$$w_\rho(0) = 0, \quad |\Delta w_\rho(x)| = |(\Delta w)(\rho x)| \leq M_0.$$

Consider the function

$$\lambda(x) = -\int_{B(1)} V(x-y)\Delta w_\rho(y)dy \quad \text{in } B(1)$$

when V is Green's function for $-\Delta$ in $B(1)$. Then

$$\Delta \lambda = \Delta w_\rho$$

and

$$(3.2) \quad \|\lambda\|_{L^\infty(B(1))} \leq C_1, \quad |\nabla \lambda|_{L^\infty(B(1))} \leq C_1$$

where the C_i will be used to denote constants depending only on M_0 .

The function

$$(3.3) \quad z = w_\rho - \lambda$$

is harmonic in $B(1)$ and

$$|z(0)| = |\lambda(0)| \leq C_1, \quad z \geq -C_1.$$

By Harnack's inequality we obtain

$$|z(x)| \leq C_2 \quad \text{in } B(3/4);$$

therefore

$$|\nabla z(x)| \leq C_3 \quad \text{in } B(1/2).$$

Recalling (3.2), (3.3) are get

$$|w_\rho(x)| \leq M, \quad |\nabla w_\rho(x)| \leq M \quad \text{in } B(1/2)$$

and (3.1) follows.

Set

$$\begin{aligned} I^- &= \{x \in \Omega; u(x) = \phi(x)\}, \\ I^+ &= \{x \in \Omega; u(x) = \psi(x)\}, \\ I &= I^- \cup I^+. \end{aligned}$$

Since $u \in C(\bar{\Omega})$,

$$(3.4) \quad d(I, \partial\Omega) > 0.$$

In view of Lemma 2.1 we can apply Lemma 3.1 to $u - \phi$ and conclude, upon using also (3.4), that

$$(3.5) \quad \begin{aligned} |(u - \phi)(x)| &\leq M(d(x, I^-))^2, \\ |\nabla(u - \phi)(x)| &\leq Md(x, I^-). \end{aligned}$$

Similar estimates hold for $u - \psi$.

LEMMA 3.2. *There exists a positive constant N such that*

$$(3.6) \quad |D^2u(x)| \leq N \quad \text{in } \Omega \setminus I.$$

Proof. Let $x^0 \in \Omega_\delta \setminus I$, $d(x^0, I) < d(I, \partial\Omega)$. Suppose for definiteness that

$$d(x^0, I) = d(x^0, I^-).$$

Consider the function

$$w_d(x) = \frac{1}{d^2}(u - \phi)(d(x - x^0)) \quad (d = d(x^0, I))$$

and take for simplicity $x^0 = 0$. Then, by (3.5),

$$\begin{cases} |w_d(x)| \leq M \\ |\nabla w_d(x)| \leq M \end{cases} \quad \text{in } B(1).$$

Also

$$(3.7) \quad \Delta^2 w_d(x) = \Delta^2 \phi(dx).$$

By elliptic estimates it then follows that

$$(3.8) \quad |D^2 w_d(x)| \leq C \quad \text{in } B(1/2).$$

Thus

$$|D^2(u - \phi)(x)| \leq C \quad \text{in } B\left(x^0, \frac{1}{2}d\right)$$

and consequently,

$$|D^2u(x)| \leq C \quad \text{if } |x - x^0| < \frac{1}{2}d(x^0, I)$$

provided $d(x^0, I) < d(I, \partial\Omega)$. Recalling (3.4), the assertion (3.6) follows.

We can now complete the proof of Theorem 1.1. Let e_1 be the unit vector in the direction of the positive x_1 -axis and $h = h_1 e_1$, h_1 real. Consider the finite difference

$$D_h^2 u(x) = \frac{u(x+h) + u(x-h) - 2u(x)}{2h_1^2}$$

for $x \in \Omega$ and $|h_1|$ small enough.

If $d(x, I) < 4|h_1|$ then we choose a point $x_0 \in I$ with $|x - x_0| = d(x, I)$ and suppose, for definiteness, that $x_0 \in I^-$. Using (3.5) we get

$$\begin{aligned} |D_h^2(u - \phi)(x)| &\leq \frac{1}{h_1^2} \{ |u(x+h) - \phi(x+h)| + |u(x-h) - \phi(x-h)| \\ &\quad + 2|u(x) - \phi(x)| \} \\ &\leq \frac{1}{h_1^2} C h_1^2, \end{aligned}$$

so that

$$|D_h^2 u(x)| \leq C + |D_h^2 \phi(x)|.$$

If $d(x, I) > 4|h_1|$ then

$$|D_h^2 u(x)| = |D_{x_1 x_1} u(\bar{x})|$$

for some \bar{x} in $\Omega \setminus I$, and $d(\bar{x}, I) < 2d(x, I)$. Using Lemma 3.2 we obtain

$$|D_h^2 u(x)| \leq M.$$

We have thus proved that for any $x \in \Omega$

$$|D_h^2 u(x)| \leq C \quad \text{if } |h_1| \text{ is small enough,}$$

where C is a constant independent of x, h_1 . This implies that

$$\frac{\partial^2 u}{\partial x_1^2} \in L^\infty(\Omega).$$

Similarly one can show that each second derivative of u belongs to $L^\infty(\Omega)$.

REMARK 1. The assumption $\phi, \psi \in C^4(\bar{\Omega})$ was used in order to deduce (3.8) from (3.7). One can actually justify this derivation assuming merely that $\phi, \psi \in C^{2+\alpha}(\bar{\Omega})$.

REMARK 2. The assumption $n = 2, 3$ made in Theorem 1.1 is

used only at one point, namely, in deducing (2.1). The remaining arguments are all valid for any $n \geq 2$.

REMARK 3. Theorem 1.1 extends, with obvious modifications in the proof, to the case $n = 1$.

4. **Counterexample.** We shall show by a counterexample that, in general, u is not in C^2 , locally.

Take Ω the unit ball in R^n , $n \geq 2$, and

$$\begin{aligned}\phi(x) &= -|x|^2 - |x|^4, \\ \psi(x) &= |x|^2 + |x|^4.\end{aligned}$$

For K we take

$$K = \left\{ v \in H^2(\Omega); \phi \leq v \leq \psi; v = A, \frac{\partial v}{\partial \nu} = B \text{ on } \partial\Omega \right\}$$

where A, B are constants satisfying

$$(4.1) \quad |A| < 2$$

and

$$(4.2) \quad 2A \neq B, \quad \text{or } |A| > 1, \quad \text{or } |B| > 2.$$

Notice that

$$\phi = -2 < A < 2 = \psi \quad \text{on } \partial\Omega$$

and that K is nonempty.

THEOREM 4.1. *If (4.1), (4.2) hold then the solution u is not in C^2 , locally in Ω .*

Proof. Notice that

$$(4.3) \quad I^+ \cap I^- = \{0\}.$$

It is clear, by symmetrization, that the solution u must be a function of $\rho = |x|$. We shall write

$$u = u(\rho), \quad \phi = \phi(\rho), \quad \psi = \psi(\rho).$$

Since $u(\rho)$ is in H^2 , it is continuously differentiable for $0 < \rho < 1$. In view of (4.3), u then has the same regularity properties in $\Omega \setminus \{0\}$ as the solution of the one obstacle problem; i.e., by [2] [6],

$$(4.4) \quad u(\rho) \in C^2(0, 1).$$

We claim that

$$(4.5) \quad \text{int } I^+ = \emptyset .$$

Indeed (cf. [1]) in $\text{int } I^+$ we have $\Delta^2 u = \Delta^2 \psi > 0$ and also (since $u > \phi$ in a neighborhood of $(\text{int } I^+) \setminus \{0\}$) $\Delta^2 u \leq 0$; thus (4.5) follows.

Similarly one shows that $\text{int } I^- = \emptyset$.

LEMMA 4.2. *There holds:*

$$(4.6) \quad 0 \in \overline{I \setminus \{0\}} \quad \text{where } I = I^+ \cup I^- .$$

Proof. If the assertion is not true then

$$\Delta^2 u(\rho) = 0 \quad \text{if } 0 < \rho < \delta , \quad \text{for some } \delta > 0 .$$

Thus

$$\left(\frac{d^2}{d\rho^2} + \frac{n-1}{\rho} \frac{d}{d\rho} \right)^2 u(\rho) = 0 .$$

One can now either use a general theorem on removable singularities for solution of $\Delta^2 w = 0$ or else write u explicitly (i.e.,

$$u = c_1 + c_2 \rho^2 + c_3 \log \rho + c_4 \rho^2 \log \rho \quad \text{if } n = 2 , \text{ etc.})$$

in order to deduce (after making use of the fact that $\phi \leq u \leq \psi$) that $u(\rho) = c\rho^2$ if $0 < \rho < \delta$ and $|c| < 1$.

By analytic continuation we then get $u = c\rho^2$ if $0 < \rho < 1$. Hence $B = 2A$ and $|A| < 1$. Since, by (4.1), $|A| < 2$, we now get a contradiction to (4.2).

LEMMA 4.3. *Suppose*

$$\alpha, \beta \in I^+ , \quad 0 < \alpha < \beta < 1 , \quad (\alpha, \beta) \subset (0, 1) \setminus I .$$

Then there exists a $\bar{\rho} \in [\alpha, \beta]$ such that

$$\Delta u(\bar{\rho}) = \Delta \psi(\bar{\rho}) .$$

Proof. Since $\psi - u$ takes minimum at α, β , we have (using (4.4))

$$\Delta(\psi - u)(\alpha) \geq 0 , \quad \Delta(\psi - u)(\beta) \geq 0 .$$

Hence if the assertion is not true then

$$\Delta(\psi - u)(\rho) > 0 \quad \text{for all } \rho \in [\alpha, \beta] .$$

Recalling that $(\psi - u)(\alpha) = (\psi - u)(\beta) = 0$, and applying the maximum

principle, we get $\psi < u$ in (α, β) , which is impossible.

LEMMA 4.4. *There holds:*

$$(4.7) \quad 0 \in \overline{I^- \setminus \{0\}}, \quad 0 \in \overline{I^+ \setminus \{0\}}.$$

Proof. It is enough to prove the first assertion. If this assertion is not true then

$$(4.8) \quad (0, \delta) \cap I^- = \emptyset \quad \text{for some } \delta > 0.$$

By Lemma 4.2 we then have

$$0 \in \overline{I^+ \setminus \{0\}}.$$

Recalling (4.5) we deduce that there exist

$$\alpha_i \in I^+, \quad \beta_i \in I^+ \quad (i = 1, 2)$$

such that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \delta$$

and

$$(\alpha_i, \beta_i) \subset (0, 1) \setminus I.$$

From Lemma 4.3 it follows that there exist $\rho_i \in [\alpha_i, \beta_i]$ such that

$$(4.9) \quad \Delta(\psi - u)(\rho_i) = 0.$$

Since u does not touch the lower obstacle in $0 < \rho < \delta$, we have

$$\Delta^2 u \leq 0 \quad \text{in } 0 < \rho < \delta$$

and consequently,

$$\Delta^2(\psi - u) > 0 \quad \text{in } (\rho_1, \rho_2).$$

We can therefore apply the maximum principle to conclude that

$$\Delta(\psi - u)(\rho) < 0 \quad \text{in } (\rho_1, \rho_2).$$

But this contradicts the fact that $\Delta(\psi - u)(\alpha_2) \geq 0$.

From Lemma 4.4 it follows that there exist sequences $\rho_m \rightarrow 0$, $\tilde{\rho}_m \rightarrow 0$ such that

$$\begin{aligned} u(\rho) &= \rho^2 + \rho^4 & \text{if } \rho = \rho_m, \\ u(\rho) &= -\rho^2 - \rho^4 & \text{if } \rho = \tilde{\rho}_m. \end{aligned}$$

This implies that $u \notin C^2$ in any neighborhood of $\rho = 0$.

REMARK. In the above example u touches both the upper obstacle and the lower obstacle (by Lemma 4.4).

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Received September 29, 1980. This paper is partially supported by the National Science Foundation Grants 7406375 A01 and MCS 791 5171 and by C.N.R. of Italy through L.A.N. of Pavia.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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Vol. 103, No. 2

April, 1982

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