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## **THE TWO-OBSTACLE PROBLEM FOR THE BIHARMONIC OPERATOR**

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## THE TWO-OBSTACLE PROBLEM FOR THE BIHARMONIC OPERATOR

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**In this work we consider a two-obstacle problem for the plate, namely, the problem of finding a minimizer  $u$  of**

$$\int_{\Omega} |\Delta v|^2 dx, \text{ subject to } (v - h) \in H_0^2(\Omega), \quad \phi \leq v \leq \psi$$

**where  $\Omega$  is a bounded domain in  $R^n$ ;  $n = 2, 3$ . We prove that  $u \in C^{1,1}$  and that, in general,  $u \notin C^2$ .**

**1. The main results.** Let  $\Omega$  be a bounded domain in  $R^n$  ( $n = 2, 3$ ) with  $C^{2+\alpha}$  boundary  $\partial\Omega$ , where  $0 < \alpha < 1$ . Let  $h(x)$  be a function in  $C^{2+\alpha}(\bar{\Omega})$ , and let  $\phi(x), \psi(x)$  be functions in  $C^4(\bar{\Omega})$  satisfying

$$(1.1) \quad \begin{aligned} \phi &\leq \psi && \text{in } \Omega, \\ \phi &< h < \psi && \text{on } \partial\Omega. \end{aligned}$$

Then the set

$$K = \{v; (v - h) \in H_0^2(\Omega), \phi \leq v \leq \psi \text{ a.e.}\}$$

is nonempty.

Consider the variational inequality: find  $u$  such that

$$(1.2) \quad \min_{v \in K} \int_{\Omega} |\Delta v|^2 dx = \int_{\Omega} |\Delta u|^2 dx, \quad u \in K.$$

By standard results [4] [5] this problem has a unique solution. We shall prove:

**THEOREM 1.1.**  *$u$  belongs to  $C^{1,1}(\Omega)$ .*

That means that  $\nabla^2 u \in L^\infty(\Omega)$ .

We shall also show that, in general,

$$(1.3) \quad u \notin C^2 \text{ locally.}$$

For the corresponding variational inequality (for  $\Delta^2$ ) with one obstacle only (i.e.,  $u \geq \phi$  instead of  $\phi \leq u < \psi$ ) it was proved by Caffarelli and Friedman [1] that, for  $n \geq 2$ ,  $u \in C^{1,1}$  locally and, for  $n = 2$ ,  $u \in C^2$  locally.

Notice that if in Theorem 1.1  $\phi < \psi$  in a subdomain  $\Omega_0$  of  $\Omega$ , then the coincidence sets  $\{u = \phi\}, \{u = \psi\}$  are disjoint in  $\Omega_0$  (since  $u$

is continuous). Thus (1.3) can only hold (at least for  $n = 2$ ) in a neighborhood of a point  $x^0$  for which  $\phi(x^0) = \psi(x^0)$ .

In §2 we shall prove that  $\Delta u \in L^\infty(\Omega)$  and in §3 we shall complete the proof of Theorem 1.1. An example for which (1.3) holds is given in §4.

2.  $\Delta u$  is bounded. Set

$$\begin{aligned} \phi_\varepsilon &= \phi - \varepsilon, \quad \varepsilon > 0, \\ K_\varepsilon &= \text{the set } K \text{ with } \phi \text{ replaced by } \phi_\varepsilon. \end{aligned}$$

Denote by  $u_\varepsilon$  the solution of the variational inequality (1.2) with  $K$  replaced by  $K_\varepsilon$ . Clearly,

$$\int_{\Omega} |\Delta u_\varepsilon|^2 dx \leq C, \quad C \text{ independent of } \varepsilon.$$

Since  $n \leq 3$  we can apply Sobolev's inequality to deduce that

$$(2.1) \quad \begin{aligned} u_\varepsilon &\text{ is uniformly continuous in } x, \text{ with modulus} \\ &\text{of continuity independent of } \varepsilon. \end{aligned}$$

It follows that the coincidence sets

$$I_\varepsilon^+ = \{u_\varepsilon = \psi\}, \quad I_\varepsilon^- = \{u_\varepsilon = \phi\},$$

are closed disjoint sets. Furthermore, by (1.1), (2.1),

$$(2.2) \quad d(I_\varepsilon^\pm, \partial\Omega) \geq \delta > 0, \quad \delta \text{ independent of } \varepsilon,$$

where

$$d(A, B) = \text{dist.}(A, B).$$

We now claim that

$$(2.3) \quad u_\varepsilon \longrightarrow u \text{ uniformly in } \Omega, \quad \text{as } \varepsilon \longrightarrow 0.$$

Indeed for any sequence  $\varepsilon_m \rightarrow 0$  there is a subsequence  $\varepsilon_{m'} \rightarrow 0$  such that

$$u_{\varepsilon_{m'}} \longrightarrow \bar{u} \quad \text{weakly in } H^2(\Omega).$$

The variational inequality for  $u_{\varepsilon_{m'}}$  can be written in the form (Minty's lemma)

$$\int_{\Omega} \Delta v \cdot \Delta(v - u_{\varepsilon_{m'}}) \geq 0 \quad \text{for every } v \in K_{\varepsilon_{m'}}.$$

Taking  $m' \rightarrow \infty$  we get

$$\int_{\Omega} \Delta v \cdot \Delta(v - u) \geq 0 \quad \text{for every } v \in K,$$

so that  $u$  is the solution  $u$  of (1.2); this completes the proof of (2.3).

Since  $I_{\varepsilon}^+, I_{\varepsilon}^-$  are disjoint closed sets, there is a version of  $\Delta u$  which is subharmonic and upper semicontinuous in  $\Omega \setminus I_{\varepsilon}^+$  and superharmonic and lower semicontinuous in  $\Omega \setminus I_{\varepsilon}^-$ ; this is proved exactly as in [1].

Set

$$\Omega_r = \{x \in \Omega; d(x, \partial\Omega) > r\}, \quad r > 0.$$

Let  $\zeta$  be a  $C_0^\infty(\Omega)$  function such that

$$\begin{aligned} \zeta &= 1 \quad \text{in } \Omega_{\delta/2}, & \zeta &= 0 \quad \text{in } \Omega \setminus \Omega_{\delta/4}, \\ 0 &\leq \zeta \leq 1 \quad \text{elsewhere;} & \delta &\text{ as in (2.2).} \end{aligned}$$

We can represent  $\Delta u_\varepsilon$  as in [1; (3.8)] in the form

$$(2.4) \quad \Delta u_\varepsilon(x) = - \int_{\Omega_\delta} V(x, y) d\mu(y) + \gamma(x)$$

where  $|\gamma(x)|$  is a bounded function in  $\Omega_{\delta/2}$ , with an upper bound independent of  $\varepsilon$ ,  $d\mu = \Delta^2 u_\varepsilon$  and  $V$  is Green's function for  $-\Delta$ , for a ball containing  $\bar{\Omega}$ ; here we have used the fact (which follows from (2.2)) that  $\Delta^2 u_\varepsilon = 0$  in  $\Omega \setminus \Omega_\delta$  and, consequently, the first two derivatives of  $u_\varepsilon$  are bounded in  $\Omega_{\delta/2}$  by a constant independent of  $\varepsilon$ .

Notice that  $\mu$  is a signed measure; it can be written as a difference  $\mu_1 - \mu_2$  of two positive measures, where  $\mu_1$  is  $\Delta^2 u_\varepsilon$  supported on  $I_{\varepsilon}^-$  and  $\mu_2$  is  $\Delta^2 u_\varepsilon$ -supported on  $I_{\varepsilon}^+$ .

Introduce the notation:

$$\begin{aligned} B(y, \rho) &= \{x; |x - y| < \rho\}, & B(\rho) &= B(0, \rho), \\ S_\rho(y) &= \partial B(y, \rho), & S_\rho &= \partial B(\rho), \\ |S_\rho| &= \text{surface area of } S_\rho. \end{aligned}$$

We reason as in [1]. Let  $x_0 \in I_{\varepsilon}^-$ . Then

$$\begin{aligned} u_\varepsilon(x_0) &= \frac{1}{|S_\delta|} \int_{S_\delta(x_0)} u_\varepsilon - \int_{B_\delta(x_0)} G \Delta u_\varepsilon, \\ \phi_\varepsilon(x_0) &= \frac{1}{|S_\delta|} \int_{S_\delta(x_0)} \phi_\varepsilon - \int_{B_\delta(x_0)} G \Delta \phi_\varepsilon. \end{aligned}$$

Here  $G$  denotes

$$C\left(\frac{1}{r} - \frac{1}{\delta}\right) \quad \text{in } R^3,$$

$$C \log \frac{r}{\delta} \quad \text{in } R^2$$

for some constant  $C > 0$ . Since

$$u_\varepsilon(x_0) = \phi_\varepsilon(x_0)$$

$$\int_{S_\delta(x_0)} u_\varepsilon \geq \int_{S_\delta(x_0)} \phi_\varepsilon$$

and

$$\frac{1}{|S_\delta|} \int_{S_\delta(x_0)} \Delta u_\varepsilon$$

is a monotone function of  $\delta$ , for  $\delta \rightarrow 0$ , we get

$$(2.5) \quad \Delta u_\varepsilon(x_0) \geq \Delta \phi_\varepsilon(x_0) \quad \text{if } x_0 \in \text{supp } \mu_1 .$$

Similarly

$$(2.6) \quad \Delta u_\varepsilon \leq \Delta \psi_\varepsilon \quad \text{on } \text{supp } \mu_2 .$$

The function

$$(2.7) \quad \hat{V}(x) = \int_{\Omega_\delta} V(x, y) d\mu(y)$$

satisfies, by (2.4)–(2.6),

$$\begin{aligned} \hat{V}(x) &\leq C && \text{on } \text{supp } \mu_1 , \\ \hat{V}(x) &\geq -C && \text{on } \text{supp } \mu_2 \end{aligned}$$

where  $C$  is a constant independent of  $\varepsilon$ . As in the proofs of Theorems 1.6, 1.10 of [3], we then have

$$\limsup_{d(x, \text{supp } \mu_1) \rightarrow 0} \hat{V}(x) \leq C, \quad \limsup_{d(x, \text{supp } \mu_2) \rightarrow 0} \hat{V}(x) \geq -C .$$

Hence, by the maximum principle,

$$|\hat{V}(x)| \leq C \quad \text{in } \Omega_\delta$$

and (2.4) gives

$$|\Delta u_\varepsilon| \leq C \quad \text{in } \Omega_{\delta/2}$$

with another  $C$ . Taking  $\varepsilon \rightarrow 0$  and recalling (2.3), we obtain:

LEMMA 2.1.  $\Delta u$  is in  $L^\infty(\Omega)$ .

3.  $u \in C^{1,1}$ . Let

$$w \in H^2(\Omega), \quad \Delta w \in L^\infty(\Omega), \quad w \geq 0 ,$$

and set

$$J = \{x \in \Omega; w(x) = 0\}, \quad \|\Delta w\|_{L^\infty(\Omega)} \leq M_0.$$

LEMMA 3.1. *There exists a constant  $M$  depending only on  $M_0$  such that if  $x_0 \in J$  then*

$$(3.1) \quad |w(x)| \leq M|x - x_0|^2, \quad |\nabla w(x)| \leq M|x - x_0| \quad \text{if } x \in B(x_0, \rho/2)$$

where  $\rho = d(x_0, \partial\Omega)$ .

*Proof.* Take for simplicity  $x_0 = 0$  and consider the function

$$w_\rho(x) = \frac{1}{\rho^2}w(\rho x) \quad \text{in } B(1).$$

Then

$$w_\rho(0) = 0, \quad |\Delta w_\rho(x)| = |(\Delta w)(\rho x)| \leq M_0.$$

Consider the function

$$\lambda(x) = -\int_{B(1)} V(x-y)\Delta w_\rho(y)dy \quad \text{in } B(1)$$

when  $V$  is Green's function for  $-\Delta$  in  $B(1)$ . Then

$$\Delta\lambda = \Delta w_\rho$$

and

$$(3.2) \quad \|\lambda\|_{L^\infty(B(1))} \leq C_1, \quad |\nabla\lambda|_{L^\infty(B(1))} \leq C_1$$

where the  $C_i$  will be used to denote constants depending only on  $M_0$ .

The function

$$(3.3) \quad z = w_\rho - \lambda$$

is harmonic in  $B(1)$  and

$$|z(0)| = |\lambda(0)| \leq C_1, \quad z \geq -C_1.$$

By Harnack's inequality we obtain

$$|z(x)| \leq C_2 \quad \text{in } B(3/4);$$

therefore

$$|\nabla z(x)| \leq C_3 \quad \text{in } B(1/2).$$

Recalling (3.2), (3.3) are get

$$|w_\rho(x)| \leq M, \quad |\nabla w_\rho(x)| \leq M \quad \text{in } B(1/2)$$

and (3.1) follows.

Set

$$\begin{aligned} I^- &= \{x \in \Omega; u(x) = \phi(x)\}, \\ I^+ &= \{x \in \Omega; u(x) = \psi(x)\}, \\ I &= I^- \cup I^+ . \end{aligned}$$

Since  $u \in C(\bar{\Omega})$ ,

$$(3.4) \quad d(I, \partial\Omega) > 0 .$$

In view of Lemma 2.1 we can apply Lemma 3.1 to  $u - \phi$  and conclude, upon using also (3.4), that

$$(3.5) \quad \begin{aligned} |(u - \phi)(x)| &\leq M(d(x, I^-))^2, \\ |\nabla(u - \phi)(x)| &\leq Md(x, I^-) . \end{aligned}$$

Similar estimates hold for  $u - \psi$ .

LEMMA 3.2. *There exists a positive constant  $N$  such that*

$$(3.6) \quad |D^2u(x)| \leq N \quad \text{in } \Omega \setminus I .$$

*Proof.* Let  $x^0 \in \Omega_\delta \setminus I$ ,  $d(x^0, I) < d(I, \partial\Omega)$ . Suppose for definiteness that

$$d(x^0, I) = d(x^0, I^-) .$$

Consider the function

$$w_d(x) = \frac{1}{d^2}(u - \phi)(d(x - x^0)) \quad (d = d(x^0, I))$$

and take for simplicity  $x^0 = 0$ . Then, by (3.5),

$$\left. \begin{aligned} |w_d(x)| &\leq M \\ |\nabla w_d(x)| &\leq M \end{aligned} \right\} \quad \text{in } B(1) .$$

Also

$$(3.7) \quad \Delta^2 w_d(x) = \Delta^2 \phi(dx) .$$

By elliptic estimates it then follows that

$$(3.8) \quad |D^2 w_d(x)| \leq C \quad \text{in } B(1/2) .$$

Thus

$$|D^2(u - \phi)(x)| \leq C \quad \text{in } B\left(x^0, \frac{1}{2}d\right)$$

and consequently,

$$|D^2u(x)| \leq C \quad \text{if } |x - x^0| < \frac{1}{2}d(x^0, I)$$

provided  $d(x^0, I) < d(I, \partial\Omega)$ . Recalling (3.4), the assertion (3.6) follows.

We can now complete the proof of Theorem 1.1. Let  $e_1$  be the unit vector in the direction of the positive  $x_1$ -axis and  $h = h_1 e_1$ ,  $h_1$  real. Consider the finite difference

$$D_h^2 u(x) = \frac{u(x+h) + u(x-h) - 2u(x)}{2h_1^2}$$

for  $x \in \Omega$  and  $|h_1|$  small enough.

If  $d(x, I) < 4|h_1|$  then we choose a point  $x_0 \in I$  with  $|x - x_0| = d(x, I)$  and suppose, for definiteness, that  $x_0 \in I^-$ . Using (3.5) we get

$$\begin{aligned} |D_h^2(u - \phi)(x)| &\leq \frac{1}{h_1^2} \{ |u(x+h) - \phi(x+h)| + |u(x-h) - \phi(x-h)| \\ &\quad + 2|u(x) - \phi(x)| \} \\ &\leq \frac{1}{h_1^2} C h_1^2, \end{aligned}$$

so that

$$|D_h^2 u(x)| \leq C + |D_h^2 \phi(x)|.$$

If  $d(x, I) > 4|h_1|$  then

$$|D_h^2 u(x)| = |D_{x_1 x_1} u(\bar{x})|$$

for some  $\bar{x}$  in  $\Omega \setminus I$ , and  $d(\bar{x}, I) < 2d(x, I)$ . Using Lemma 3.2 we obtain

$$|D_h^2 u(x)| \leq M.$$

We have thus proved that for any  $x \in \Omega$

$$|D_h^2 u(x)| \leq C \quad \text{if } |h_1| \text{ is small enough,}$$

where  $C$  is a constant independent of  $x$ ,  $h_1$ . This implies that

$$\frac{\partial^2 u}{\partial x_1^2} \in L^\infty(\Omega).$$

Similarly one can show that each second derivative of  $u$  belongs to  $L^\infty(\Omega)$ .

REMARK 1. The assumption  $\phi, \psi \in C^4(\bar{\Omega})$  was used in order to deduce (3.8) from (3.7). One can actually justify this derivation assuming merely that  $\phi, \psi \in C^{2+\alpha}(\bar{\Omega})$ .

REMARK 2. The assumption  $n = 2, 3$  made in Theorem 1.1 is

used only at one point, namely, in deducing (2.1). The remaining arguments are all valid for any  $n \geq 2$ .

**REMARK 3.** Theorem 1.1 extends, with obvious modifications in the proof, to the case  $n = 1$ .

**4. Counterexample.** We shall show by a counterexample that, in general,  $u$  is not in  $C^2$ , locally.

Take  $\Omega$  the unit ball in  $R^n$ ,  $n \geq 2$ , and

$$\begin{aligned}\phi(x) &= -|x|^2 - |x|^4, \\ \psi(x) &= |x|^2 + |x|^4.\end{aligned}$$

For  $K$  we take

$$K = \left\{ v \in H^2(\Omega); \phi \leq v \leq \psi; v = A, \frac{\partial v}{\partial \nu} = B \text{ on } \partial\Omega \right\}$$

where  $A, B$  are constants satisfying

$$(4.1) \quad |A| < 2$$

and

$$(4.2) \quad 2A \neq B, \quad \text{or } |A| > 1, \quad \text{or } |B| > 2.$$

Notice that

$$\phi = -2 < A < 2 = \psi \quad \text{on } \partial\Omega$$

and that  $K$  is nonempty.

**THEOREM 4.1.** *If (4.1), (4.2) hold then the solution  $u$  is not in  $C^2$ , locally in  $\Omega$ .*

*Proof.* Notice that

$$(4.3) \quad I^+ \cap I^- = \{0\}.$$

It is clear, by symmetrization, that the solution  $u$  must be a function of  $\rho = |x|$ . We shall write

$$u = u(\rho), \quad \phi = \phi(\rho), \quad \psi = \psi(\rho).$$

Since  $u(\rho)$  is in  $H^2$ , it is continuously differentiable for  $0 < \rho < 1$ . In view of (4.3),  $u$  then has the same regularity properties in  $\Omega \setminus \{0\}$  as the solution of the one obstacle problem; i.e., by [2] [6],

$$(4.4) \quad u(\rho) \in C^2(0, 1).$$

We claim that

$$(4.5) \quad \text{int } I^+ = \emptyset .$$

Indeed (cf. [1]) in  $\text{int } I^+$  we have  $\Delta^2 u = \Delta^2 \psi > 0$  and also (since  $u > \phi$  in a neighborhood of  $(\text{int } I^+) \setminus \{0\}$ )  $\Delta^2 u \leq 0$ ; thus (4.5) follows.

Similarly one shows that  $\text{int } I^- = \emptyset$ .

LEMMA 4.2. *There holds:*

$$(4.6) \quad 0 \in \overline{I \setminus \{0\}} \quad \text{where } I = I^+ \cup I^- .$$

*Proof.* If the assertion is not true then

$$\Delta^2 u(\rho) = 0 \quad \text{if } 0 < \rho < \delta, \quad \text{for some } \delta > 0 .$$

Thus

$$\left( \frac{d^2}{d\rho^2} + \frac{n-1}{\rho} \frac{d}{d\rho} \right)^2 u(\rho) = 0 .$$

One can now either use a general theorem on removable singularities for solution of  $\Delta^2 w = 0$  or else write  $u$  explicitly (i.e.,

$$u = c_1 + c_2 \rho^2 + c_3 \log \rho + c_4 \rho^2 \log \rho \quad \text{if } n = 2, \text{ etc.})$$

in order to deduce (after making use of the fact that  $\phi \leq u \leq \psi$ ) that  $u(\rho) = c\rho^2$  if  $0 < \rho < \delta$  and  $|c| < 1$ .

By analytic continuation we then get  $u = c\rho^2$  if  $0 < \rho < 1$ . Hence  $B = 2A$  and  $|A| < 1$ . Since, by (4.1),  $|A| < 2$ , we now get a contradiction to (4.2).

LEMMA 4.3. *Suppose*

$$\alpha, \beta \in I^+, \quad 0 < \alpha < \beta < 1, \quad (\alpha, \beta) \subset (0, 1) \setminus I .$$

*Then there exists a  $\bar{\rho} \in [\alpha, \beta]$  such that*

$$\Delta u(\bar{\rho}) = \Delta \psi(\bar{\rho}) .$$

*Proof.* Since  $\psi - u$  takes minimum at  $\alpha, \beta$ , we have (using (4.4))

$$\Delta(\psi - u)(\alpha) \geq 0, \quad \Delta(\psi - u)(\beta) \geq 0 .$$

Hence if the assertion is not true then

$$\Delta(\psi - u)(\rho) > 0 \quad \text{for all } \rho \in [\alpha, \beta] .$$

Recalling that  $(\psi - u)(\alpha) = (\psi - u)(\beta) = 0$ , and applying the maximum

principle, we get  $\psi < u$  in  $(\alpha, \beta)$ , which is impossible.

LEMMA 4.4. *There holds:*

$$(4.7) \quad 0 \in \overline{I^- \setminus \{0\}}, \quad 0 \in \overline{I^+ \setminus \{0\}}.$$

*Proof.* It is enough to prove the first assertion. If this assertion is not true then

$$(4.8) \quad (0, \delta) \cap I^- = \emptyset \quad \text{for some } \delta > 0.$$

By Lemma 4.2 we then have

$$0 \in \overline{I^+ \setminus \{0\}}.$$

Recalling (4.5) we deduce that there exist

$$\alpha_i \in I^+, \quad \beta_i \in I^+ \quad (i = 1, 2)$$

such that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \delta$$

and

$$(\alpha_i, \beta_i) \subset (0, 1) \setminus I.$$

From Lemma 4.3 it follows that there exist  $\rho_i \in [\alpha_i, \beta_i]$  such that

$$(4.9) \quad \Delta(\psi - u)(\rho_i) = 0.$$

Since  $u$  does not touch the lower obstacle in  $0 < \rho < \delta$ , we have

$$\Delta^2 u \leq 0 \quad \text{in } 0 < \rho < \delta$$

and consequently,

$$\Delta^2(\psi - u) > 0 \quad \text{in } (\rho_1, \rho_2).$$

We can therefore apply the maximum principle to conclude that

$$\Delta(\psi - u)(\rho) < 0 \quad \text{in } (\rho_1, \rho_2).$$

But this contradicts the fact that  $\Delta(\psi - u)(\alpha_2) \geq 0$ .

From Lemma 4.4 it follows that there exist sequences  $\rho_m \rightarrow 0$ ,  $\tilde{\rho}_m \rightarrow 0$  such that

$$\begin{aligned} u(\rho) &= \rho^2 + \rho^4 & \text{if } \rho = \rho_m, \\ u(\rho) &= -\rho^2 - \rho^4 & \text{if } \rho = \tilde{\rho}_m. \end{aligned}$$

This implies that  $u \notin C^2$  in any neighborhood of  $\rho = 0$ .

REMARK. In the above example  $u$  touches both the upper obstacle and the lower obstacle (by Lemma 4.4).

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