

# Pacific Journal of Mathematics

**CARLESON MEASURES FOR FUNCTIONS ORTHOGONAL TO  
INVARIANT SUBSPACES**

WILLIAM S. COHN

## CARLESON MEASURES FOR FUNCTIONS ORTHOGONAL TO INVARIANT SUBSPACES

BILL COHN

Let  $D = \{z: |z| < 1\}$  be the unit disk. Suppose  $\varphi$  is an inner function with singular support  $K$  and let  $M^\perp = H^2 \ominus \varphi H^2$  where  $H^2$  is the usual class of functions holomorphic on  $D$ . If  $\mu$  is a positive measure on  $\bar{D}$ , the closed disk, which assigns zero mass to  $K$ , then call  $\mu$  a Carleson measure for  $M^\perp$  if for a  $c > 0$ ,

$$\int |f|^2 d\mu \leq c \|f\|_2^2$$

for all  $f \in M^\perp$ . (Here and elsewhere,  $\|f\|_2$  denotes the  $H^2$  norm of an  $H^2$  function.) In this paper the Carleson measures for  $M^\perp$  are characterized for all inner functions  $\varphi$  such that for some  $\varepsilon$ ,  $0 < \varepsilon < 1$ , the set  $\{z: |\varphi(z)| < \varepsilon\}$  is connected.

If  $\mu$  is a positive measure on  $D$ , then recall that  $\mu$  is a Carleson measure if there is a positive constant  $c$  such that

$$\mu(R(I)) \leq c |I|,$$

where  $I$  is an arc on the unit circle with center  $e^{i\theta_0}$  and length  $|I|$ , and  $R(I)$  is the "curvilinear rectangle"  $\{re^{i\theta}: 1 - |I|/2\pi \leq r < 1 \text{ and } |\theta - \theta_0| \leq 1/2|I|\}$ .

In [2], Carleson proved that there is a constant  $c > 0$  such that

$$\int |f(z)|^2 d\mu(z) \leq c \|f\|_2^2$$

for all  $f \in H^2$ , if and only if  $\mu$  is a Carleson measure.

Clearly, any Carleson measure is a Carleson measure for  $M^\perp$ . Functions in  $M^\perp$ , however, can be better behaved than typical  $H^2$  functions. Thus one is led to suspect that there are more Carleson measures for  $M^\perp$  than just the Carleson measures alone. This in fact turns out to be the case.

For the sake of simplicity, we state an abridged version of our main result.

**THEOREM.** *Suppose  $\varphi$  is inner and  $\{z: |\varphi(z)| < \varepsilon\}$  is connected for some  $\varepsilon$ ,  $0 < \varepsilon < 1$ . Let  $\mu$  be a measure which assigns zero mass off  $T \setminus K$ , where  $T$  is the unit circle. Then  $\mu$  is a Carleson measure for  $M^\perp$  if and only if, for some constant  $c > 0$ ,*

$$\int_T \frac{1 - |\xi|^2}{|1 - \bar{\xi}e^{i\theta}|^2} d\nu \leq \frac{c}{1 - |\varphi(\xi)|^2}$$

for all  $\xi \in D$ .

Now, it is easy to see that a measure  $\mu$  on the unit circle  $T$  has the property that

$$\int |f|^2 d\mu \leq c \|f\|_2^2$$

for all  $f \in H^2$  if and only if  $d\mu = b d\theta$  where  $b$  is a bounded function. In this case,

$$\int_T \frac{1 - |\xi|^2}{|1 - \bar{\xi}e^{i\theta}|^2} d\mu \leq c$$

for all  $\xi \in D$ . Thus one can see how the situation changes when dealing with  $M^\perp$  instead of  $H^2$ .

This paper is divided into four sections. The assumption that  $\{z: |\varphi(z)| < \varepsilon\}$  is connected implies that  $\varphi$  is a covering map onto the annulus  $\{w: \varepsilon < |w| < 1/\varepsilon\}$ . This is proven in §1. In §2 the covering map hypothesis is used to characterize those Carleson measures restricted to certain subsets of  $\{z: \varepsilon < |\varphi(z)| < 1\}$ . A corollary of this characterization is that for  $\varepsilon < \delta < 1$ , arc length on  $\{z: |\varphi(z)| = \delta\}$  is a Carleson measure. In §3 we prove a theorem about  $M^\perp$  functions which is the key to our main results. Essentially, we show that  $M^\perp$  functions belong to a Hardy space of functions defined on a larger domain than the disk. Section 4 contains some examples and applications.

The measures we consider are always positive measures, even if we do not specifically say so. The constant “ $c$ ” which appears in various theorems changes each time it is used in a different context. If  $F$  and  $E$  are sets,  $F \setminus E$  denotes their set theoretic difference. The symbol  $\bar{F}$  denotes the closure of  $F$ , and  $\partial F$  denotes the topological boundary of  $F$ .

This paper has benefited greatly from correspondence and conversation with P. R. Ahern and C. Belna. It is a pleasure to thank them both.

1. Recall that any inner function  $\varphi$  has the form

$$\varphi(z) = cB(z) \exp\left(-\int_T \frac{\eta + z}{\eta - z} d\sigma(\eta)\right),$$

where  $|c| = 1$ ,  $B$  is a Blaschke product, and  $\sigma$  is a positive measure on  $T$  which is singular with respect to arc length measure. Let

$K$  be the closure of the union of the zero set of  $\varphi$  and the support of  $\sigma$ ;  $K$  is called the *singular support* of  $\varphi$ .

For any complex number  $z, z \neq 0$ , define  $z^* = 1/\bar{z}$ . Let  $0^*$  be the point at  $\infty$  on  $S^2$ , the Riemann sphere. Then  $z^*$  is the reflection of  $z$  through the unit circle. If  $E \subseteq S^2$ , let  $E^* = \{z^*: z \in E\}$  be the reflected set. The equation

$$\varphi(z)\overline{\varphi(z^*)} = 1$$

defines an extension of  $\varphi$  which is holomorphic on  $S^2 \setminus K^*$ . Thus, for  $t > 0$ , the sets

$$D_t = \{z: \varphi \text{ is holomorphic at } z \text{ and } |\varphi(z)| < t\}$$

form a collection of open sets such that

$$D_t \subseteq D_s \text{ for } t \leq s$$

and

$$\bigcup_{t>0} D_t = S^2 \setminus K^* .$$

If  $0 < \varepsilon < 1$ , let  $A_{\varepsilon, 1/\varepsilon}$  be the annulus

$$A_{\varepsilon, 1/\varepsilon} = \{w: \varepsilon < |w| < 1/\varepsilon\} .$$

Define

$$R_\varepsilon = \{z: \varphi \text{ is holomorphic at } z \text{ and } \varepsilon < |\varphi(z)| < 1/\varepsilon\} .$$

Then  $R_\varepsilon = \bigcup_{n=1}^\infty \Omega_n$ , where the  $\Omega_n$  are the distinct connected components of  $R_\varepsilon$ . (The union may be finite.) We will be interested in the situation that  $D_\varepsilon$  is connected for some  $\varepsilon, 0 < \varepsilon < 1$ .

**THEOREM 1.1.** *Suppose for some  $\varepsilon, 0 < \varepsilon < 1$ ,  $D_\varepsilon$  is connected. If  $T \cap K \neq \emptyset$  then  $R_\varepsilon = \bigcup_{n=1}^\infty \Omega_n$  where:*

- (i) *each  $\Omega_n$  is a simply connected set which is symmetric with respect to  $T$ ;*
- (ii) *the map  $\varphi: \Omega_n \rightarrow A_{\varepsilon, 1/\varepsilon}$  is a covering map.*

*Proof.* Fix  $n$ . Let  $z_0 \in \partial\Omega_n$ : we may suppose  $z_0 \notin T$ . Consider the case where  $|z_0| < 1$ .

Let  $\Gamma_\varepsilon$  be the set  $\{z: |z| < 1, |\varphi(z)| = \varepsilon\}$ . Thus  $z_0 \in \Gamma_\varepsilon$ . Observe that  $\varphi'$  never vanishes on  $\Gamma_\varepsilon$ , since  $D_\varepsilon$  is connected. Let  $\gamma$  be the component of  $\Gamma_\varepsilon$  which contains  $z_0$ . Since  $D_\varepsilon$  is connected and  $T \cap K \neq \emptyset$ ,  $\bar{\gamma}$  is not contained in  $D$ . Thus  $\gamma$  is a simple arc whose closure intersects  $T \cap K$ . It is well known that either  $\bar{\gamma} \cap T$  consists of one point or two points.

In the first case,  $\bar{\gamma}$  is a Jordan curve, and  $D_\varepsilon$  must consist entirely of the region which  $\bar{\gamma}$  bounds. Thus  $R_\varepsilon = \Omega_n, \partial\Omega_n = \bar{\gamma} \cup \bar{\gamma}^*$ ,

and (i) holds.

If  $\bar{\gamma}$  contains two points, then  $\gamma$  divides  $D$  into two components, one of which must contain  $D_\varepsilon$ . The other component is entirely contained in  $\Omega_n$ . It follows  $\bar{\gamma} \cup \bar{\gamma}^*$  is a Jordan curve, and  $\Omega_n$  is the simply connected region which  $\bar{\gamma} \cup \bar{\gamma}^*$  bounds. Thus (i) is true.

If the original point  $z_0$  lies outside of  $\bar{D}$ , then  $|\varphi(z_0)| = 1/\varepsilon$ . By considering  $z_0^*$ , for which  $|\varphi(z_0^*)| = \varepsilon$ , and repeating the arguments above, we complete the proof of property (i).

To prove (ii), let  $\psi: D \rightarrow \Omega_n$  be a conformal map of the unit disk onto  $\Omega_n$ . Since  $\partial\Omega_n$  is a Jordan curve,  $\psi$  extends to a homeomorphism of  $T$ . By symmetry we may assume that

$$(a) \quad \psi(\{w: |w| = 1, \operatorname{Im} w > 0\}) = \partial\Omega_n \cap D$$

$$(b) \quad \psi(\{w: |w| = 1, \operatorname{Im} w < 0\}) = \partial\Omega_n \cap D^*.$$

Let  $g(w) = \varphi(\psi(w))$ . Then  $\varepsilon < |g| < 1/\varepsilon$ , and therefore  $g$  is an outer function. Furthermore,

$$|g(\xi)| = \varepsilon$$

for  $\xi \in T \cap \{\operatorname{Im} \xi > 0\}$ , and

$$|g(\xi)| = 1/\varepsilon$$

for  $\xi \in T \cap \{\operatorname{Im} \xi < 0\}$ . This proves that  $g: D \rightarrow A_{\varepsilon, 1/\varepsilon}$  is a universal cover. Since  $\psi$  is conformal, the theorem is proved.

As a corollary of the proof of Theorem 1.1, we make the following observation.

**COROLLARY 1.1.** *If  $D_\varepsilon$  is connected, then  $D_{1/\varepsilon}$  is simply connected.*

*Proof.* We first show that  $D_{1/\varepsilon}$  is connected. Clearly,  $D \subseteq D_{1/\varepsilon}$ . Let  $z \in D_{1/\varepsilon}$ . Then  $z \in R_\varepsilon$ , and hence,  $z \in \Omega_n$ , for some  $n$ . By the proof of Theorem 1.1,  $\Omega_n \cap D \neq \emptyset$ . Thus  $D_{1/\varepsilon}$  is connected.

To show that  $D_{1/\varepsilon}$  is simply connected, it suffices to show that  $S^2 \setminus D_{1/\varepsilon}$  is connected. But the map  $z \rightarrow z^*$  defines a homeomorphism of  $S^2 \setminus D_{1/\varepsilon}$  and  $\bar{D}_\varepsilon$ . Since  $D_\varepsilon$  is connected, so is  $\bar{D}_\varepsilon$ . This finishes the proof.

It may occur that  $K \subseteq D$ . In this case,  $\varphi$  is a finite Blaschke product and we have the following result.

**THEOREM 1.2.** *Suppose  $\varphi$  is a Blaschke product with  $n$  zeros, counted according to multiplicity. If  $D_\varepsilon$  is connected then the map  $\varphi: R_\varepsilon \rightarrow A_{\varepsilon, 1/\varepsilon}$  is an  $n:1$  covering map. Furthermore,  $D_{1/\varepsilon}$  is simply connected.*

*Proof.* Since  $\varphi$  is a finite Blaschke product, we need only show

$R_\varepsilon$  contains no point where  $\varphi'$  vanishes. But if  $\varphi'(z) = 0$  for some  $z$  such that  $\varepsilon \leq |\varphi(z)| \leq 1/\varepsilon$ , it follows that  $D_\varepsilon$  has at least two components. This is a contradiction. The rest of the proof is elementary and is omitted.

We finish this section with an observation which will prove useful later.

**COROLLARY 1.2.** *If  $D_\varepsilon$  is connected and  $\varepsilon < \delta < 1$ , then  $D_\delta$  is connected.*

*Proof.* By Theorem VIII. 31 in [8] any component of  $D_\delta$  is simply connected and if  $\psi$  is a conformal mapping of the unit disk onto one such component, then  $s = 1/\delta \varphi(\psi)$  is an inner function. Since  $|s|$  takes values less than  $\varepsilon$ ,  $D_\varepsilon$  intersects every component of  $D_\delta$ . Thus  $D_\delta$  is connected.

We immediately get the next result.

**COROLLARY 1.3.** *If  $D_\varepsilon$  is connected and  $\varepsilon < \delta < 1$ , then  $D_{1/\delta}$  is simply connected.*

2. Suppose  $s \in H^\infty$ , and  $\|s\|_\infty \leq 1$ . Let  $0 < \varepsilon < 1$  and set  $A_{\varepsilon,1} = \{w: \varepsilon < |w| < 1\}$ . Suppose further that  $s: s^{-1}(A_{\varepsilon,1}) \rightarrow A_{\varepsilon,1}$  is a covering map. The main result of this section is a characterization of Carleson measures which take all their mass on certain subsets of  $s^{-1}(A_{\varepsilon,1})$ .

Let  $|z| = (1 + \varepsilon)/2$  and set  $B(z, (1 - \varepsilon)/2)$  equal to the open disk centered at  $z$  with radius  $(1 - \varepsilon)/2$ . Since  $s$  is a covering map, we have

$$s^{-1}\left(B\left(z, \frac{1 - \varepsilon}{2}\right)\right) = \cup C_{n,z},$$

where the  $C_{n,z}$  are pairwise disjoint and  $s: C_{n,z} \rightarrow B(z, (1 - \varepsilon)/2)$  is a homeomorphism. Let  $\mathcal{C}$  be the collection of all such  $C_{n,z}$ , where  $z$  ranges over the circle of radius  $(1 + \varepsilon)/2$ .

We prove the following theorem.

**THEOREM 2.1.** *Let  $F$  be a compact subset of  $A_{\varepsilon,1}$  and let  $\mu$  be a measure on  $D$  which assigns zero mass off  $s^{-1}(F)$ . Then the following conditions are equivalent:*

- (i)  $\mu$  is a Carleson measure.
- (ii) There is a constant  $c > 0$  such that  $\int_{C_{n,z}} |s'(z)| d\mu(z) \leq c$  for all  $C_{n,z} \in \mathcal{C}$ .

*Proof.* Since  $F$  is compact, it is contained in the finite union of noneuclidean disks of the form

$$N(z_0, r) = \left\{ \xi: \left| \frac{\xi - z_0}{1 - \bar{z}_0 \xi} \right| < r \right\}.$$

We may also assume that for each  $N(z_0, r)$ , there is an  $t$ ,  $r < t < 1$ , and a  $z$ ,  $|z| = (1 + \varepsilon)/2$ , such that

$$N(z_0, r) \subseteq N(z_0, t) \subseteq B\left(z, \frac{1 - \varepsilon}{2}\right).$$

Thus we may assume that  $\mu$  assigns zero mass off the set  $s^{-1}(N(z_0, r))$ .

Write  $s^{-1}(N(z_0, r)) = \cup G_n$ , and  $s^{-1}(N(z_0, t)) = \cup R_n$ , where  $G_n \subseteq R_n$  and  $s: R_n \rightarrow N(z_0, t)$  is a homeomorphism. Let  $a_n$  be the point in  $G_n$  for which  $s(a_n) = z_0$ . We make the following observation.

LEMMA 2.1. *The sequence  $\{a_n\}$  is uniformly separated.*

*Proof.* Let  $h(z) = (z_0 - s(z))/(1 - \bar{z}_0 s(z))$ . Then  $\{a_n\}$  is the zero set of  $h$ , and  $|h| \equiv r$  on  $\partial G_n$ . Let  $B_n$  be the Blaschke product with factors  $\bar{a}_k/|a_k| (a_k - z)/(1 - \bar{a}_k z)$ ,  $k \neq n$ . Then  $|B_n|$  never vanishes on  $G_n$ . Furthermore, for  $z \in \partial G_n$ ,

$$|B_n(z)| \geq |h(z)| = r.$$

It follows from the minimum principle that

$$|B_n(a_n)| \geq r,$$

and the lemma is proved.

Define the measure  $\delta_z$  to be point mass at  $z$ . We have the immediate corollary; see [2].

COROLLARY 2.1. *The measure  $\nu = \sum \delta_{a_n} \cdot (1 - |a_n|^2)$  is a Carleson measure.*

Let  $I$  be an arc on the unit circle with center  $e^{i\theta}$  and length  $|I|$ . For  $m > 0$ , define  $mI$  to be the arc with center  $e^{i\theta}$  and length  $m|I|$ . The next lemma enables us to compare  $\mu$  to  $\nu$ .

LEMMA 2.2. *Suppose condition (ii) of Theorem 2.1 is true. Then there are constants  $c_1 > 0$  and  $m > 0$  such that*

- (1)  $\mu(G_n) \leq c_1(1 - |a_n|^2)$  for all  $n$ .
- (2) if  $I$  is an arc on  $T$  and  $R(I) \cap G_n \neq \emptyset$ , then  $G_n \subseteq R(mI)$ .

Accepting Lemma 2.2, for the moment, we show that condition (ii) implies condition (i). For  $I$  an arc on  $T$  we have

$$\begin{aligned} \mu(R(I)) &= \sum_n \mu(R(I) \cap G_n) \\ &\leq \sum \mu(G_n) \\ G_n \cap R(I) &\neq \emptyset . \end{aligned}$$

Since Lemma 2.2 is in force,  $G_n \cap R(I) \neq \emptyset$  implies  $\alpha_n \in R(mI)$ . Thus

$$\begin{aligned} \mu(R(I)) &\leq \sum_{\alpha_n \in R(mI)} \mu(G_n) \\ &\leq \sum_{\alpha_n \in R(mI)} c_1 \cdot (1 - |\alpha_n|^2) \\ &= c_1 \nu(R(mI)) \\ &\leq c_1 \gamma(\nu) \cdot m \cdot |I| , \end{aligned}$$

where  $\gamma(\nu)$  is the Carleson constant for  $\nu$ . Thus  $\mu$  is a Carleson measure.

We now prove Lemma 2.2. Fix  $w$ . Since  $h$  is a 1:1 map of  $R_n$  onto the disk  $\{w: |w| < t\}$  we may choose a branch of  $h^{-1}$  such that

$$g(z) = h^{-1}(z/t)$$

maps the unit disk onto  $R_n$ . By the Schwarz-Pick theorem, if  $z_1, z_2 \in R_n$ ,

$$(*) \quad \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| \leq \left| \frac{h(z_1)/t - h(z_2)/t}{1 - t^{-2} \overline{h(z_1)} h(z_2)} \right| .$$

If  $z_1$  and  $z_2$  are restricted to  $G_n$  we see that

$$(**) \quad \left| \frac{z_1 - z_2}{1 - \bar{z}_1 z_2} \right| \leq c < 1 ,$$

where  $c$  depends only on  $r$  and  $t$ . It is not difficult to see that for an  $m$  depending only on  $c$ , if  $z_1 \in R(I)$  then  $N(z_1, c) \subseteq R(mI)$ . This establishes (2) of Lemma 2.2.

Next, equation (\*\*\*) yields a  $\beta > 0$ , independent of  $n$ , for which

$$\inf_{z_1, z_2 \in G_n} \frac{1 - |z_1|^2}{1 - |z_2|^2} \geq \beta .$$

In particular, if  $z \in G_n$ , then

$$\frac{1}{1 - |z|^2} \geq \frac{\beta}{1 - |\alpha_n|^2} .$$



Now let  $z_1 \rightarrow z_2 = z$  in equation (\*) and use the last inequality to conclude that

$$\begin{aligned} |h'(z)| &\geq \frac{t(1 - (r/t)^2)}{1 - |z|^2} \\ &\geq \frac{t\beta(1 - (r/t)^2)}{1 - |a_n|^2} \end{aligned}$$

for all  $z \in G_n$ . Since

$$h'(z) = \frac{1 - |z_0|^2}{(1 - \bar{z}_0 s(z))^2} s'(z),$$

we see that for some constant  $c_2 > 0$ ,

$$|s'(z)| \geq \frac{c_2}{1 - |a_n|^2}$$

for all  $z \in G_n$ .

Finally, by this last inequality and condition (ii) of Theorem 2.1,

$$\mu(G_n) \cdot \frac{c_2}{1 - |a_n|^2} \leq \int_{G_n} |s'(z)| d\mu \leq c.$$

This proves (1) of Lemma 2.2, and the theorem in one direction.

To show that condition (i) implies condition (ii), observe that  $\mu(C_{k,z}) \leq \mu(G_n)$  for some  $n$ . By the proof of Lemma 2.2, if  $z \in G_n$ , then for a constant  $c_3 > 0$ ,

$$\frac{2}{1 - |z|^2} \leq \frac{c_3}{1 - |a_n|^2}.$$

Furthermore,  $G_n \subseteq N(a_n, c)$ . If  $\mu$  is a Carleson measure then for a constant  $c_4$ , depending on  $c_3$ ,

$$\mu(N(a_n, c)) \leq c_4 \cdot \gamma(\mu)(1 - |a_n|^2),$$

where  $\gamma(\mu)$  is the Carleson constant of  $\mu$ . Thus

$$\begin{aligned} \int_{C_{k,z}} |s'(z)| d\mu &\leq \int_{G_n} |s'(z)| d\mu \leq \int_{G_n} \frac{1 - |s(z)|^2}{1 - |z|^2} d\mu(Z) \\ &\leq \frac{c_3}{1 - |a_n|^2} \cdot \mu(G_n) \leq \frac{c_3}{1 - |a_n|^2} \mu(N(a_n, c)) \leq c_3 \cdot c_4 \cdot \gamma(\mu). \end{aligned}$$

This proves the theorem.

As an application, suppose  $\varphi$  is an inner function and  $D_\varepsilon$  is connected. Let  $\varepsilon < \delta < 1$  and set  $\Gamma = \{z: |\varphi(z)| = \delta\}$ . Let  $\mu$  be arclength measure on  $\Gamma$ . By Theorems 1.1 and 1.2 we may apply Theorem 2.1, with  $\varphi$  in place of  $s$ , to  $\mu$ . Since for any  $C_{n,z} \in \mathcal{C}$ ,

$$\int_{c_{n,z}} |\varphi'(z)| d\mu(z) = \int_{\Gamma \cap c_{n,z}} |\varphi'(z)| |dz| \leq 2\pi,$$

we have the following result.

**COROLLARY 2.1.** *Let  $\varphi$  be inner and  $D_\varepsilon$  connected. Then if  $\varepsilon < \delta < 1$ , arclength on  $\{z: |\varphi(z)| = \delta\}$  is a Carleson measure.*

In §1 we showed that under the hypotheses of Corollary 2.1  $D_{1/\delta}$  was simply connected. In fact, more is true.

**THEOREM 2.2.** *Let  $\varphi$  be inner and  $D_\varepsilon$  connected. Then if  $\varepsilon < \delta < 1$  and  $|\varphi(0)| < \delta$ ,  $\partial D_{1/\delta}$  is a rectifiable Jordan curve.*

*Proof.* We first prove that  $\partial D_{1/\delta}$  is a Jordan curve.

Let  $R_\delta$  be defined as in Theorem 1.1, and write  $R_\delta = \bigcup_{n=1}^\infty \Omega_n$ , where the  $\Omega_n$  are the components of  $R_\delta$ . Let  $\gamma_n = \partial\Omega_n \setminus \bar{D}$ . Then if  $J_n = \Omega_n \cap T$  and  $F = T \setminus \bigcup J_n$ , we see that

$$\partial D_{1/\delta} = F \cup \bigcup_{n=1}^\infty \gamma_n.$$

Let  $\alpha_n: \bar{J}_n \rightarrow \bar{\gamma}_n$  be a homeomorphism which fixes the endpoints of  $\bar{J}_n$ . Define the mapping of  $T$  onto  $\partial\Omega_{1/\delta}$  by the formula

$$\alpha(e^{i\theta}) = \begin{cases} \alpha_n(e^{i\theta}), & \text{if } e^{i\theta} \in \bar{J}_n \\ e^{i\theta}, & \text{if } e^{i\theta} \in F. \end{cases}$$

We must show that  $\alpha$  is continuous. It suffices to do this for  $e^{i\theta} \in F$ . This amounts to showing that if a sequence of arcs  $J_n$  approach  $e^{i\theta}$ , then the associated arcs  $\gamma_n$  must approach  $e^{i\theta}$ . If this fails to be the case then there is a cluster point of the arcs  $\gamma_n$ ,  $z_0$ , such that  $z_0 \neq e^{i\theta}$ . If  $|z_0| < 1$  then it follows that  $|\varphi(z_0)| = \delta$ , and  $z_0 \in \Gamma = \{z: |z| < 1, |\varphi(z)| = \delta\}$ . As in §1,  $\varphi'$  never vanishes on  $\Gamma$ . Thus there is a ball centered around  $z_0$  which  $\Gamma$  divides into two regions; on one of those regions  $|\varphi| > \delta$ , and on the other  $|\varphi| < \delta$ . This contradicts the assertion that  $z_0$  is a cluster point of the arcs  $\gamma_n$ .

If there is no  $z_0$  with  $|z_0| < 1$ , and  $z_0 \neq e^{i\theta}$ , then it is easy to see that

$$\liminf_{r \rightarrow 1} |\varphi(re^{i\theta})| \leq \delta$$

for all  $e^{i\theta}$  on an arc connecting  $z_0$  to  $e^{i\theta}$ . Since  $\varphi$  is inner, this is impossible. Thus  $\alpha$  is continuous at  $e^{i\theta}$ , and  $\partial\Omega_{1/\delta}$  is a Jordan curve.

Turning to the rectifiability, it isn't hard to see that  $\alpha$  has

total variation

$$\|d\alpha\| = \sum_{n=1}^{\infty} |\gamma_n| + |F|,$$

where  $|\gamma_n|$  denotes the length of the arc  $\gamma_n$  and  $|F|$  denotes the measure of  $F$ . Since  $\infty \notin \partial D_{1/\delta}$ , by Corollary 2.1,  $\|d\alpha\| < \infty$ . This proves Theorem 2.2.

REMARK. It follows from the rectifiability of  $\partial D_{1/\delta}$  and Theorems VIII 30 and 31 in [8], that  $|F| = 0$ . Thus arclength measure on  $\partial\Omega_{1/\delta}$  is equivalent to arc length measure on  $\partial\Omega_{1/\delta} \setminus T$ .

3. In this section we characterize Carleson measures for  $(\varphi H^2)^\perp$  in the case that  $D_\varepsilon$  is connected.

For  $\xi \in D$ , define the function

$$K_\xi(z) = \frac{1 - \overline{\varphi(\xi)}\varphi(z)}{1 - \overline{\xi}z}.$$

Then  $K_\xi \in M^\perp$  and

$$\|K_\xi\|_2^2 = \frac{1 - |\varphi(\xi)|^2}{1 - |\xi|^2}.$$

See [1], page 194 for the proofs. Let  $\mu$  be a measure on  $\bar{D}$  which assigns zero mass to  $K$ . Let  $\mu_\delta$  be the restriction of  $\mu$  to  $D_\delta$ . Then if  $0 < \delta < 1$ ,

$$\int |K_\xi(z)|^2 \frac{1 - |\xi|^2}{1 - |\varphi(\xi)|^2} d\mu_\delta(z) \geq (1 - \delta)^2 \int \frac{1 - |\xi|^2}{|1 - \overline{\xi}z|^2} d\mu_\delta(z).$$

Suppose  $\mu$  is a Carleson measure for  $M^\perp$ . Then the last inequality yields

$$\frac{c}{(1 - \delta)^2} \geq \int \frac{1 - |\xi|^2}{|1 - \overline{\xi}z|^2} d\mu_\delta,$$

where  $c$  is independent of  $\xi$ . It follows that  $\mu_\delta$  is a Carleson measure for  $D$ . Conversely, if  $\mu_\delta$  is a Carleson measure for  $D$ , then  $\mu_\delta$  is a Carleson measure for  $M^\perp$ . We have proven the following lemma.

LEMMA 3.1. *The following properties are equivalent:*

- (i)  $\mu$  is a Carleson measure for  $M^\perp$ .
- (ii) (a)  $\mu_\delta$  is a Carleson measure for  $D$  and  
(b)  $\mu - \mu_\delta$  is a Carleson measure for  $M^\perp$ .

We turn, therefore, to the problem of characterizing Carleson

measures for  $M^\perp$  which assign zero mass to  $K \cup D_\delta$ .

Assume that  $0 < \varepsilon < \delta < 1$ . Then  $D_{1/\delta}$  is simply connected and we may choose a conformal map  $\sigma: D \rightarrow D_{1/\delta}$ . Let  $\psi = \sigma^{-1}$ .

Suppose  $\mu$  is a measure on  $\bar{D}$  which assigns zero mass to  $K \cup D_\delta$  and set  $\mu_1 = |\psi'| \mu$ . Then if  $E \subseteq D$ , the equation

$$\nu(E) = \mu_1(\sigma(E))$$

defines a measure on  $D$ . We prove the following theorem.

**THEOREM 3.1.** *The following properties are equivalent:*

- (i) *The measure  $\nu$  is a Carleson measure.*
- (ii) *The measure  $\mu$  is a Carleson measure for  $M^\perp$ .*
- (iii) *There is a constant  $c > 0$  such that*

$$\int_{\sigma(C_{n,z})} |\varphi'(z)| d\mu(z) \leq c$$

for all sets  $C_{n,z} \in \mathcal{C}$ , where  $\mathcal{C}$  is the collection defined in Theorem 2.1 with  $s = \delta\varphi(\sigma)$  and  $\varepsilon \cdot \delta$  in place of  $\varepsilon$ .

*Proof.* We show first that (i) implies (ii). If  $f \in M^\perp$ , then it is well known that  $f$  has a holomorphic extension to  $D_{1/\delta}$ . See [4]. We need an explicit expression for  $f(z)$  when  $|z| > 1$ . Since  $f \in M^\perp$ ,

$$\int_T \bar{\varphi} f \cdot \bar{b} d\theta = 0$$

for all  $b \in H^\infty$ . Thus

$$\bar{\varphi} f = e^{-i\theta} \bar{h} \text{ a.e. } [d\theta]$$

where  $h \in H^2$ . For all  $z, |z| \geq 1$  define

$$(1) \quad F(z) = \varphi(z) \frac{1}{z} \bar{h}(1/\bar{z}).$$

Then  $F(e^{i\theta}) = f(e^{i\theta})$  for  $e^{i\theta} \notin K$  and it follows that  $F(z) = f(z)$  for all  $z \notin K \cup K^*$ . Equation (1) will imply that  $f$  is well behaved on  $D_{1/\delta}$ .

To make this precise, let  $T_n$  be the circle of radius  $1 - 1/n$  centered at 0, and set  $C_n = \sigma(T_n)$ . Then  $E^2(D_{1/\delta})$  is the class of analytic functions defined on  $D_{1/\delta}$  which satisfy the condition

$$\lim_{n \rightarrow \infty} \int_{C_n} |f(z)|^2 |dz| < \infty.$$

The space  $E^2$  is closely related to  $H^2(D)$ . In fact  $f \in E^2(D_{1/\delta})$  if and only if

$$f(\sigma(w))\sigma'(w)^{1/2} = g(w)$$

for some  $g \in H^2(D)$ . For a full discussion, see [5], pages 168–169.

Since  $\partial D_{1/\delta}$  is a rectifiable Jordan curve,  $\sigma' \in H^1(D)$ , and the measure on  $T$  given by  $|\sigma'(w)||dw|$  is arclength measure for  $\partial D_{1/\delta}$ .

Since  $g$  and  $\sigma'$  both have radial limits a.e.  $[d\theta]$ , it follows that  $\lim_{r \rightarrow 1} f(\sigma(re^{i\theta}))$  exists a.e.  $[d\theta]$ . Thus we may write

$$\begin{aligned} \|g\|_2 &= \int_T |g(w)|^2 |dw| = \int_T |f(\sigma(w))|^2 |\sigma'(w)| |dw| \\ &= \int_{\partial D_{1/\delta}} |f(z)|^2 |dz|. \end{aligned}$$

Thus  $E^2$  is a Hilbert space with norm defined by the equation

$$\|f\|_{E^2}^2 = \int_{\partial D_{1/\delta}} |f(z)|^2 |dz|,$$

and  $g \rightarrow f$  is an isometry of  $H^2$  onto  $E^2$ . Recall that  $F = \partial D_{1/\delta} \cap T$  is a set of measure 0. Thus

$$\|f\|_{E^2}^2 = \int_{\partial D_{1/\delta} \setminus T} |f(z)|^2 |dz|.$$

These observations and equation (1) are the key to the next lemma.

**LEMMA 3.1.** *If  $f \in M^+$  then the extension of  $f$  to  $D_{1/\delta}$  belongs to  $E^2(D_{1/\delta})$ . Furthermore,*

$$\|f\|_{E^2}^2 \leq c \|f\|_2^2,$$

where  $c$  is independent of  $f$ .

*Proof.* Suppose  $f \in M^+ \cap H^\infty$ . If  $f$  and  $h$  are related by the equation

$$f = \varphi e^{-i\theta} \bar{h} \text{ a.e. } [d\theta],$$

then  $\|h\|_\infty = \|f\|_\infty$ . Thus equation (1) shows that  $f$  is bounded on  $D_{1/\delta}$ . Since  $\sigma' \in H^1$ ,  $f \in E^2$ . We calculate  $\|f\|_{E^2}^2$  using the fact that  $f$  is continuous off the singular support of  $\varphi$ , and the fact that arclength on  $\Gamma = \{z: |\varphi(z)| = \delta\}$  is a Carleson measure. Thus

$$\begin{aligned} \|f\|_{E^2}^2 &= \int_{\partial D_{1/\delta} \setminus T} |f(z)|^2 |dz| = \int_{\partial D_{1/\delta} \setminus T} |\varphi(z)|^2 |h(1/\bar{z})|^2 \frac{|dz|}{|z|^2} \\ &= 1/\delta^2 \int_\Gamma |h(w)|^2 |dw|^2 \leq \frac{\gamma}{\delta^2} \|h\|_2^2 = \frac{\gamma}{\delta^2} \|f\|_2^2, \end{aligned}$$

where  $\gamma$  depends only on the Carleson constant of  $|dw|$  on  $\Gamma$ . This shows that the conclusion of the lemma is valid for  $M^+ \cap H^\infty$ . Since

linear combinations of the functions  $K_\varepsilon$  are dense in  $M^\perp$ ,  $M^\perp \cap H^\infty$  is dense in  $M^\perp$ . A standard argument proves the lemma for all of  $M^\perp$ .

We complete the proof that condition (i) of Theorem 3.1 implies condition (ii). Since  $g \in H^2$  if and only if  $g(w) = f(\sigma(w))\sigma'(w)^{1/2}$  for  $f \in E^2$ , it follows that

$$\int |f(z)|^2 d\mu(z) = \int |g(w)|^2 d\nu(w).$$

If  $\nu$  is a Carleson measure, then from the last equation,

$$\int |f(z)|^2 d\mu \leq \gamma(\nu) \|g\|_2^2 = \gamma(\nu) \|f\|_{E^2}^2 \leq \gamma(\nu)c \cdot \|f\|_2^2.$$

Thus  $d\mu$  is a Carleson measure for  $M^\perp$ .

We next show that condition (iii) implies condition (i). Let  $s(w) = \delta\varphi(\sigma(w))$ . Then by the results of §1,  $s: s^{-1}(A_{\varepsilon\delta,1}) \rightarrow A_{\varepsilon\delta,1}$  is a covering map. Observe that  $\nu$  assigns zero mass off  $\{w: \delta^2 \leq |s(w)| \leq \delta\}$ . By Theorem 2.1  $\nu$  is a Carleson measure if and only if for some  $c > 0$ ,

$$\int_{C_{n,z}} |s'(w)| d\nu \leq c$$

for all  $C_{n,z} \in \mathcal{C}$ . (Here, the “ $\varepsilon$ ” of Theorem 2.1 is replaced by “ $\varepsilon\delta$ ”.)  
But

$$\int_{C_{n,z}} |s'(w)| d\nu = \int_{\sigma(C_{n,z})} \delta \cdot |\varphi'(z)| d\mu.$$

Thus (iii) implies (i).

All that remains is to prove (ii) implies (iii). We must find some constant  $c$  such that

$$\int_{\sigma(C_{n,z})} |\varphi'| d\mu \leq c.$$

Recall that  $\mu$  assigns zero mass to  $K \cup D_\delta$ . Let  $N_{n,z} = \sigma(C_{n,z}) \cap \{\xi: \delta \leq |\varphi(\xi)| \leq 1\}$ . Thus  $N_{n,z}$  is a component of  $\varphi^{-1}(R)$ , where  $R$  is the intersection of the closed annulus  $\{w: \delta \leq |w| \leq 1\}$  and the open ball  $B(\delta^{-1}z, (1 - \varepsilon\delta)/2\delta)$ . It is enough to show that

$$\int_{N_{n,z}} |\varphi'| d\mu \leq c.$$

We need the following lemma.

**LEMMA 3.2.** *Let  $C_{n,z_0} \in \mathcal{C}$ . Suppose  $\xi \in N_{n,z_0}$ ,  $\arg \varphi(\xi) = \arg z_0$ , and  $|\varphi(\xi)| = \delta$ . Then there is a constant  $c_1$ , independent of  $C_{n,z_0}$ ,*

such that

$$|\varphi'(z)| \leq c_1 |K_\xi(z)|^2 \frac{1 - |\xi|^2}{1 - |\varphi(\xi)|^2}$$

for all  $z \in N_{n, z_0}$ .

*Proof.* Let  $\varphi_\xi^{-1}$  denote the branch of  $\varphi^{-1}$  for which  $\varphi_\xi^{-1}(\varphi(\xi)) = \xi$ . Set  $T_\varepsilon$  equal to the circle of radius  $\varepsilon$  centered at the origin and let  $\alpha$  be the radial projection of  $\varphi(\xi)$  onto  $T_\varepsilon$ . Suppose  $\Omega$  is the simply connected region bounded by the unit circle and the line tangent to  $T_\varepsilon$  at  $\alpha$ . Let  $g: \Omega \rightarrow D$  be a conformal map of  $\Omega$  onto the disk such that  $g(\varphi(\xi)) = 0$ . Then  $f = \varphi_\xi^{-1} \circ g^{-1}$  maps the disk into itself. By the Schwarz-Pick theorem,

$$|g(\varphi(z))| \geq \left| \frac{\xi - z}{1 - \bar{\xi}z} \right|$$

for  $z \in \varphi_\xi^{-1}(\Omega)$ . Thus

$$1 - |g(\varphi(z))|^2 \leq \frac{(1 - |\xi|^2)(1 - |z|^2)}{|1 - \bar{\xi}z|^2}$$

and

$$\frac{|1 - (g(\varphi(z)))|^2}{1 - |\varphi(z)|^2} \cdot \frac{1 - |\varphi(z)|^2}{1 - |z|^2} \leq \frac{2}{1 - \delta} \cdot \left| \frac{1 - \overline{\varphi(\xi)}\varphi(z)}{1 - \bar{\xi}z} \right|^2 \cdot \frac{1 - |\xi|^2}{1 - |\varphi(\xi)|^2}$$

for  $z \in \varphi_\xi^{-1}(\Omega)$ . If  $z$  is restricted to  $N_{n, z_0}$  then  $(1 - |g(\varphi(z))|^2)/(1 - |\varphi(z)|^2)$  is bounded away from zero by a constant independent of  $z_0$ . Since  $|\varphi'(z)| \leq (1 - |\varphi(z)|^2)/(1 - |z|^2)$ , the lemma is proved.

To complete the proof of Theorem 3.1 observe that (ii) implies that for some constant  $c_2$ ,

$$\int \frac{1 - |\xi|^2}{1 - |\varphi(\xi)|^2} |K_\xi(z)|^2 d\mu(z) \leq c_2$$

for all  $\xi \in D$ . Choose  $C_{n, z_0} \in \mathcal{C}$  and  $\xi$  as in Lemma 3.2. Then

$$\int_{N_{n, z_0}} |\varphi'(z)| d\mu(z) \leq \int_{N_{n, z_0}} c_1 \frac{1 - |\xi|^2}{1 - |\varphi(\xi)|^2} |K_\xi(z)|^2 d\mu(z) \leq c_1 c_2 = c.$$

This completes the proof.

We complete this section by characterizing Carleson measures for  $M^1$  in terms of the growth of the function

$$h(\xi) = \int \frac{1 - |\xi|^2}{|1 - \bar{\xi}z|^2} d\mu(z).$$

**THEOREM 3.2.** *Let  $\varphi$  be an inner function and suppose  $D_\varepsilon$  is*

connected. If  $\mu$  is a measure on  $\bar{D}$  which assigns zero mass to  $K$ , then the following properties are equivalent:

- (i)  $\mu$  is a Carleson measure for  $M^1$ .
- (ii) There is a constant  $c$  such that

$$h(\xi) \leq \frac{c}{1 - |\varphi(\xi)|}$$

for all  $\xi \in D$ .

- (iii) (a)  $\mu_\varepsilon$  is a Carleson measure for  $D$ , where  $\varepsilon < \delta < 1$ , and
- (b) There is a constant  $c$  such that

$$\int_{N_{n,z_0}} |\varphi'(z)| d\mu \leq c$$

for all  $N_{n,z_0}$ .

*Proof.* We have already shown that (i) and (iii) are equivalent. That (i) implies (ii) follows easily from the inequality

$$\int |K_\xi(z)|^2 d\mu(z) \leq c \cdot \frac{1 - |\varphi(\xi)|^2}{1 - |\xi|^2}.$$

We turn to the proof that (ii) implies (iii) (a).

Let  $I$  be an arc on  $T$  of length  $|I|$ . We must find  $c$  such that

$$\mu(R(I) \cap D_\delta) \leq c|I|.$$

For  $\xi \in D$ ,  $\xi \neq 0$ , define  $I_\xi$  to be the arc on  $T$  with center  $\xi/|\xi|$  and length  $2(1 - |\xi|)$ . There is a constant  $\gamma$  such that

$$(2) \quad \frac{1 - |\xi|^2}{|1 - \bar{\xi}z|^2} \geq \gamma(1 - |\xi|)^{-1}$$

for all  $z \in R(I_\xi)$ , and all  $\xi \in D$ .

Let  $S_1 = R(I) \cap D_\delta$  and set  $\alpha_1 = \max_{\xi \in S_1} (1 - |\xi|)$ . Choose  $\xi_1 \in S_1$  such that  $1 - |\xi_1| \geq 7/8 \alpha_1$ . Proceeding inductively, suppose  $S_1, S_2, \dots, S_n$  and  $\xi_1, \xi_2, \dots, \xi_n$  have been chosen. Let  $S_{n+1} = S_n \setminus R(I_{\xi_n})$  and set  $\alpha_{n+1} = \max_{\xi \in S_{n+1}} (1 - |\xi|)$ . Choose  $\xi_{n+1}$  such that

$$1 - |\xi_{n+1}| \geq 7/8 \alpha_{n+1}.$$

In this fashion we obtain a sequence  $\{\xi_n\}$  such that

$$S_1 \subseteq \bigcup_{n=1}^{\infty} R(I_{\xi_n})$$

and

$$\sum_{n=1}^{\infty} 1 - |\xi_n| \leq c_1 |I|,$$



where  $c_1$  is a constant independent of  $I$ .

Condition (ii) and inequality (2) yield

$$\begin{aligned} \mu(S_1) &\leq \sum_n \mu(R(I_{\xi_n})) \leq \frac{1}{2^\gamma} \sum_{n=1}^\infty \int_{R(I_{\xi_n})} \frac{(1 - |\xi_n|)^2}{|1 - \xi_n z|^2} d\mu(z) \\ &\leq \frac{c}{\gamma} \sum_{n=1}^\infty (1 - |\xi_n|) \leq \frac{c \cdot c_1}{\gamma} |I|. \end{aligned}$$

Thus  $\mu_\delta$  is a Carleson measure.

To show that (ii) implies (iii) (b), observe that with  $\xi$  and  $N_{n,z_0}$  related as in Lemma 3.2,

$$\frac{1 - |\xi|^2}{|1 - \xi z|^2} \leq c \cdot |\varphi'(z)|$$

for all  $z \in N_{n,z_0}$ . Thus (iii) (b) is an easy consequence of property (ii). This completes the proof.

4. Perhaps the most representative example occurs when  $\varphi(z) = \exp(-(1+z)/(1-z))$ . In this case,  $D_\varepsilon$  is a disk tangent to  $T$  at the point 1, and Theorems 3.1 and 3.2 are in force.

One calculates that

$$|\varphi'(z)| = \frac{2|\varphi(z)|}{|1-z|^2}.$$

Suppose  $\mu$  is a measure on  $T$  which assigns zero mass to  $\{1\}$ . It follows from Theorem 3.2 that  $\mu$  is a Carleson measure for  $M^\perp$  if and only if

$$\int_{I_n} |\varphi'| d\mu \leq c$$

for all arcs  $I_n$  of the form

$$I_n = (e^{i\pi/n} + 1, e^{i\pi/n}),$$

where  $n = \pm 1, \pm 2, \dots$ .

Simple estimates show that this is the case if and only if

$$\int_{I_n} d\mu \leq \frac{c}{n^2}.$$

This leads to the following result. For  $f \in M^\perp$ ,

$$\sum_{n=-\infty}^\infty \max_{z \in I_n} |f(z)|^2 \cdot \frac{1}{n^2} \leq c \cdot \|f\|_2^2.$$

This may be regarded as a generalization of a theorem of Clark; see [3], pages 176-177.

More generally, let  $E$  be a closed compact subset of  $D$  with zero capacity. Let  $\varphi: D \rightarrow D \setminus E$  be an analytic universal covering map. Then  $\varphi$  is an inner function; see [6]. If  $E \subseteq \{z: |z| < \varepsilon\}$ , then it is not hard to show that  $D_\varepsilon$  is connected. Thus Theorems 3.1 and 3.2 apply to this class of inner functions.

Now suppose that  $\varphi(z) = z^n$ . Then  $M^\perp$  is the span of the functions  $1, z, z^2, \dots, z^{n-1}$ . Let  $u \in L^1(T)$ ,  $u \geq 0$ , and suppose  $u$  has the Fourier expansion  $\sum_{-\infty}^\infty c_n e^{in\theta}$ . If  $f(z) = \sum_{m=0}^{n-1} a_m z^m$ , then  $f \in M^\perp$  and

$$\int_T |f(z)|^2 u(z) |dz| = \sum_{k,m}^{n-1} c_{m-n} a_k \bar{a}_m.$$

The expression on the right is a finite section Toeplitz operator. If we take the supremum over all  $\{a_0, a_1, \dots, a_{n-1}\}$  such that  $\sum |a_k|^2 = 1$ , then we obtain the largest eigenvalue of the form. On the other hand,

$$\sup_{f \in M^\perp, \|f\|_2=1} \int_T |f|^2 u d\theta$$

is the ‘‘Carleson constant’’ for the Carleson measure for  $M^\perp$ ,  $u d\theta$ . Observe that for any  $\varepsilon, 0 < \varepsilon < 1$ ,  $\{z: |z|^n < \varepsilon\}$  is connected. If we choose  $\varepsilon = 1/4$  and  $\delta = 1/2$ , then applying Theorem 3.1 we see that if  $\mathcal{I}_n$  is the collection of arcs  $I$ ,

$$I = (e^{i\theta\pi}, e^{i\theta\pi + i\pi/n})$$

then for a constant  $c$ , independent of  $n$ ,

$$\int_T |f|^2 u d\theta \leq c \cdot \gamma_n$$

where  $\gamma_n = \sup_{I \in \mathcal{I}_n} \int_I u \cdot n d\theta$  and  $f$  ranges over all  $(z^n H^2)^\perp$  functions with norm less than 1.

Thus we obtain order of magnitude estimates for the largest eigenvalue of finite section Toeplitz operators. These results can be compared with the asymptotic estimates, in the case where  $u$  satisfies more restricted hypotheses, found on page 72 of Grenander and Szegö, [7].

### REFERENCES

1. P. R. Ahern and D. N. Clark, *On functions orthogonal to invariant subspaces*, Acta Math., **124** (1970), 191-204.
2. L. Carleson, *Interpolations by bounded analytic functions and the corona problem*, Ann. of Math., **76** (1962), 547-559.
3. D. N. Clark, *One dimensional perturbations of restricted shifts*, J. Analyse Math., **25** (1972), 169-191.

4. R. G. Douglas, H. S. Shapiro and A. L. Shields, *Cyclic vectors and invariant subspaces for the backwards shift*, Ann. Inst. Fourier, Grenoble, **20**, 1, (1970), 37-76.
5. P. L. Duren, *Theory of  $H^p$  Spaces*, Academic Press, 1970.
6. O. Frostman, *Potential d'equilibre et capacite des ensembles avec quelques applications a la theorie des fonctions*, Meddel. Lunds Univ. Mat. Sem., **3** (1935), 1-118.
7. V. Grenander and G. Szego, *Toeplitz Forms and Their Applications*, University of California Press, 1958.
8. M. Tsuji, *Potential Theory in Modern Function Theory*, Chelsea Publishing Company, 1975.

Received September 9, 1981.

WAYNE STATE UNIVERSITY  
DETROIT, MI 48202

# PACIFIC JOURNAL OF MATHEMATICS

## EDITORS

DONALD BABBITT (Managing Editor)

University of California  
Los Angeles, CA 90024

HUGO ROSSI

University of Utah  
Salt Lake City, UT 84112

C. C. MOORE and ARTHUR AGUS

University of California  
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics  
University of Southern California  
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University  
Stanford, CA 94305

## ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

## SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF AAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

---

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies,

---

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. 39. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

---

The *Pacific Journal of Mathematics* is issued monthly as of January 1966, Regular subscription rate: \$114.00 a year (6 Vol., 12 issues). Special rate: \$57.00 a year to individual members of supporting institution.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).  
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

# Pacific Journal of Mathematics

Vol. 103, No. 2

April, 1982

<b>Alberto Alesina and Leonede De Michele</b> , A dichotomy for a class of positive definite functions .....	251
<b>Kahtan Alzubaidy</b> , Rank <sub>2</sub> $p$ -groups, $p > 3$ , and Chern classes .....	259
<b>James Arney and Edward A. Bender</b> , Random mappings with constraints on coalescence and number of origins .....	269
<b>Bruce C. Berndt</b> , An arithmetic Poisson formula .....	295
<b>Julius Rubin Blum and J. I. Reich</b> , Pointwise ergodic theorems in l.c.a. groups .....	301
<b>Jonathan Borwein</b> , A note on $\varepsilon$ -subgradients and maximal monotonicity .....	307
<b>Andrew Michael Brunner, Edward James Mayland, Jr. and Jonathan Simon</b> , Knot groups in $S^4$ with nontrivial homology .....	315
<b>Luis A. Caffarelli, Avner Friedman and Alessandro Torelli</b> , The two-obstacle problem for the biharmonic operator .....	325
<b>Aleksander Całka</b> , On local isometries of finitely compact metric spaces .....	337
<b>William S. Cohn</b> , Carleson measures for functions orthogonal to invariant subspaces .....	347
<b>Roger Fenn and Denis Karmen Sjerve</b> , Duality and cohomology for one-relator groups .....	365
<b>Gen Hua Shi</b> , On the least number of fixed points for infinite complexes .....	377
<b>George Golightly</b> , Shadow and inverse-shadow inner products for a class of linear transformations .....	389
<b>Joachim Georg Hartung</b> , An extension of Sion's minimax theorem with an application to a method for constrained games .....	401
<b>Vikram Jha and Michael Joseph Kallaher</b> , On the Lorimer-Rahilly and Johnson-Walker translation planes .....	409
<b>Kenneth Richard Johnson</b> , Unitary analogs of generalized Ramanujan sums .....	429
<b>Peter Dexter Johnson, Jr. and R. N. Mohapatra</b> , Best possible results in a class of inequalities .....	433
<b>Dieter Jungnickel and Sharad S. Sane</b> , On extensions of nets .....	437
<b>Johan Henricus Bernardus Kemperman and Morris Skibinsky</b> , On the characterization of an interesting property of the arcsin distribution .....	457
<b>Karl Andrew Kosler</b> , On hereditary rings and Noetherian $V$ -rings .....	467
<b>William A. Lampe</b> , Congruence lattices of algebras of fixed similarity type. II .....	475
<b>M. N. Mishra, N. N. Nayak and Swadeenananda Pattanayak</b> , Strong result for real zeros of random polynomials .....	509
<b>Sidney Allen Morris and Peter Robert Nickolas</b> , Locally invariant topologies on free groups .....	523
<b>Richard Cole Penney</b> , A Fourier transform theorem on nilmanifolds and nil-theta functions .....	539
<b>Andrei Shkalikov</b> , Estimates of meromorphic functions and summability theorems .....	569
<b>László Székelyhidi</b> , Note on exponential polynomials .....	583
<b>William Thomas Watkins</b> , Homeomorphic classification of certain inverse limit spaces with open bonding maps .....	589
<b>David G. Wright</b> , Countable decompositions of $E^n$ .....	603
<b>Takayuki Kawada</b> , Correction to: "Sample functions of Pólya processes" .....	611
<b>Z. A. Chanturia</b> , Errata: "On the absolute convergence of Fourier series of the classes $H^\omega \cap V[v]$ " .....	611