Pacific Journal of Mathematics

DUALITY AND COHOMOLOGY FOR ONE-RELATOR GROUPS

Roger Fenn and Denis Karmen Sjerve

Vol. 103, No. 2 April 1982

PACIFIC JOURNAL OF MATHEMATICS
Vol. 103, No. 2, 1982

DUALITY AND COHOMOLOGY FOR ONE RELATOR GROUPS

ROGER FENN AND DENIS SJERVE

1. Introduction. Let G be a group having a one relator presentation and some fundamental integral class $[G] \in H_2(G)$. The object of this paper is to study the cap product homomorphism $[G] \cap \cdot$: $H^i(G; A) \to H_{2-i}(G; \bar{A})$ where A is a left G module and $\bar{A}$ is the right G module identified with A as an abelian group and whose scalar multiplication is given by $ag = g^{-1}a$ for $a \in A$, $g \in G$. If this homomorphism is an isomorphism we say that G satisfies *Poincaré duality* with respect to A.

For example consider the fundamental group G of an orientable surface M. In this case the homomorphism $[G] \cap \cdot$ is an isomorphism for all G modules A. Such a group is said to satisfy *Poincaré duality.* Recently Müller [11, 12] has shown that a one relator group satisfying Poincaré duality over A for all G modules A is isomorphic to the fundamental group of some orientable surface; thus answering a question of Johnson and Wall in [9]. Actually Müller's result is stronger than this since it applies to a larger class of groups than one relator groups. However, we will restrict our attention to one relator groups and study duality for fixed coefficients A.

In § 2 various preliminary work relating Fox derivatives and Magnus expansions is given and in § 3 there are some results for $\boldsymbol{Z}$ coefficients. In particular Theorem 3.4 proves that any group satisfying Poincaré duality over the integers has a presentation of the form $\{x_1, \cdots, x_{2g} | [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]W = 1\}$ where W lies in the third term of the lower central series of the free group on $x_1, \cdots, x_{2g}$. Note that if $W = 1$ then the presentation reduces to that of a surface group. This result has been proved independently by Ratcliffe, [15].

In § 4 an explicit formula for the homomorphism $[G] \cap \cdot$ on the chain level is given in terms of a Hessian matrix $\partial_i(\overline{\partial_j V})$ of Fox derivatives, where V is the relator.

Using the theory developed in this paper and results from [16] it is routine to verify the claims made in the following examples.

EXAMPLE. The group $G = \{x_1, x_2 | [x_1, x_2][x_2, [x_2, x_1]] = 1\}$ satisfies Poincaré duality over $\boldsymbol{Z}$. Now let A be the Laurent polynomial ring $\boldsymbol{Z}[\boldsymbol{Z}]$ on the generator t with the G module structure induced from the homomorphism $\phi: G \to \boldsymbol{Z}[t]$ defined by $\phi(x_1) = 1$, $\phi(x_2) = t$. If G were to satisfy Poincaré duality over A then it would be true that the ideal in A generated by the Fox derivatives $\phi(\partial V/\partial x_1)$, $\phi(\partial V/\partial x_2)$, where $V = [x_1, x_2][x_2, [x_2, x_1]]$, is the augmentation ideal $(1 - t)$. But a simple calculation gives $\phi(\partial V/\partial x_2) = 0$, $\phi(\partial V/\partial x_1) = 1 - t + (1 - t)^2$, and hence G does not satisfy duality with respect to A.

EXAMPLE. Consider the group $G = \{x_1, \cdots, x_4 \mid V = 1\}$, where $V = [x_1, x_2][x_3, x_4][x_1, [x_2, x_3]]$. Let A be the integral Laurent polynomial ring in variables $t_1, \cdots, t_4$ made into a G module by the homomorphism $\phi: Z[G] \to A$, $\phi(x_i) = t_i$. Then the ideal generated by the Fox derivatives $\phi(\partial_i V)$ is the augmentation ideal $(1 - t_1, \cdots, 1 - t_4)$ and hence $[G] \cap \cdot : H^2(G; A) \to H_0(G; \bar{A})$ is an isomorphism. A short calculation gives $H^0(G; A) = 0$, $H_2(G; \bar{A}) = 0$, and yet G does not satisfy Poincaré duality over A since if it did the matrix $[\phi\partial_i(\overline{\partial_j V})]$ would be invertible over A. But the ideal generated by the first row is $(t_2 - 1, 1 - 2t_3)$ and therefore this matrix is not invertible.

2. The free differential calculus and Magnus expansions. In this section we collect various results on Fox derivatives. Standard references are [4, 5, 6, 7, 8]. Throughout F will denote the free group on $x_1, \cdots, x_n$ and $\varepsilon: Z[F] \to Z$ will denote the augmentation homomorphism. The usual anti-automorphism $Z[F] \to Z[F]$ will be written $f \to \bar{f}$.

For $1 \leqq i \leqq n$ we let ∂_i be the Fox derivative $\partial/\partial x_i$ and for any finite sequence $I = (i_1, \cdots, i_r)$, where $1 \leqq i_k \leqq n$, we let ∂_I denote the higher order derivative $\partial_{i_1} \cdots \partial_{i_r}$. If $r = 0$ put $\partial_I = \mathrm{id}$ and set ε_I equal to the composite $\varepsilon\partial_I$ for any I.

If multiplication of sequences is by juxtaposition then induction on the length of a sequence will prove:

LEMMA 2.1. *For any sequence K and $f, g \in Z[F]$ we have $\varepsilon_K(fg) = \sum_{IJ=K} \varepsilon_I(f)\varepsilon_J(g)$, where the summation is over all ordered pairs (I, J), including (K, ϕ) and (ϕ, K), such that $IJ = K$.*

Thus it follows that $\varepsilon_i: F \to Z$ gives the exponent sum of x_i in a word and $\varepsilon_{ij}[g, h] = \varepsilon_i(g)\varepsilon_j(h) - \varepsilon_i(h)\varepsilon_j(g)$ for $g, h \in F$. Now let $\mathfrak{a}$ be the free associative power series ring on the noncommuting variables $a_1, \cdots, a_n$ and with coefficients in Z. For any sequence $I = (i_1, \cdots, i_r)$ let a_I denote the monomial $a_{i_1} \cdots a_{i_r}$, where $a_\phi = 1$ by convention. The Magnus expansion is defined to be the homomorphism $M: F \to \mathfrak{a}$, $x_i \to 1 + a_i$. Induction on word length easily proves:

LEMMA 2.2. *For any K and $f \in F$ we have $\varepsilon_K(f) = M_K(f)$.*

The following describes chain rules for Fox derivatives. Thus suppose F is free on $x_1, \cdots, x_n$ and G is free on $y_1, \cdots, y_p$. If $\phi: G \to F$ is a group homomorphism then

LEMMA 2.3. (a) $\varepsilon_i(\phi(g)) = \sum_{k=1}^{p} \varepsilon_i(\phi(y_k))\varepsilon_k(g)$,
(b) *for* $g \in [G, G]$ *we have* $\varepsilon_{ij}(\phi(g)) = \sum_{k,l=1}^{p} \varepsilon_i(\phi(y_k))\varepsilon_j(\phi(y_l))\varepsilon_{kl}(g)$.

As an example suppose G is free on $y_1, \cdots, y_{2g}$ and $W = [y_1, y_2] \cdots [y_{2g-1}, y_{2g}]$. Then

$$\varepsilon_{kl}(W) = \begin{cases} +1 & \text{if } (k, l) = (2i-1, 2i) \text{ for some } i, \ 1 \leqq i \leqq g \\ -1 & \text{if } (k, l) = (2i, 2i-1) \text{ for some } i, \ 1 \leqq i \leqq g \\ 0 & \text{otherwise}. \end{cases}$$

Thus the $2g$ by $2g$ matrix composed of the second order partials $\varepsilon_{kl}(W)$ is the skew symmetric matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

It is not a coincidence that this matrix is also the cup product matrix for the orientable surface of genus g.

3. Poincaré duality with untwisted Z-coefficients.

Throughout this section $K = \{x_1, \cdots, x_n \mid V = 1\}$ will denote a one relator presentation of the group G where the relator V belongs to $[F, F]$ and is assumed not to be a proper power.

If $1 \to R \to F \to G \to 1$ is the exact sequence of this presentation then the Hopf formula gives $H_2(K) \cong R/[R, F] \cong Z$ with generator $[K] = V \cdot [R, F]$. The other homology groups can be described as follows: $H_1(K)$ is free abelian on the cosets $\bar{x}_1, \cdots, \bar{x}_n \bmod [F, F]$, $H^1(K)$ is free abelian on the dual classes $x_1^*, \cdots, x_n^*$ and $H^2(K) \cong Z$ by evaluation $u \to \langle u, [K] \rangle$.

Define the *cup product matrix* $A = (a_{ij})$ over the integers by the formula

$$a_{ij} = \langle x_i^* \cup x_j^*, [K] \rangle = \langle x_i^*, [K] \cap x_j^* \rangle .$$

Now $[K] \cap \cdot$ is automatically an isomorphism for $i = 0, 2$ and so K satisfies Poincaré duality over Z if and only if $[K] \cap \cdot : H^1(K) \to H_1(K)$ is an isomorphism. This implies the following well known result.

THEOREM 3.1. *Using the notation above K satisfies Poincaré duality over Z if and only if* $A \in GL_n(Z)$.

See for example [15].

Suppose now that $n = 2g$ and $V = [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]$ so that K is a surface. Then it is easily checked that the cup product matrix (a_{ij}) is equal to the matrix (ε_{ij}) defined in the previous section. This is also a consequence of the following general result.

THEOREM 3.2. *Suppose* $K = \{x_1, \cdots, x_n \mid V = 1\}$ *is such that* $V \in [F, F]$ *is not a proper power. Then the cup product matrix* $a_{ij} = \langle x_i^* \cup x_j^*, [K] \rangle = \varepsilon_{ij}(V)$.

Proof. See Porter [14] or Fenn, Sjerve [3].

COROLLARY. *K satisfies Poincaré duality over $\boldsymbol{Z}$ if and only if the $n \times n$ matrix $\varepsilon_{ij}(V)$ is invertible over $\boldsymbol{Z}$.*

There are several effective procedures for computing $\varepsilon_{ij}(V)$. For example we can use the Magnus expansion or if $V = [U_1, V_1] \cdots [U_g, V_g]$ then

$$\varepsilon_{ij}(V) = \sum_{k=1}^{g} \{\varepsilon_i(U_k)\varepsilon_j(V_k) - \varepsilon_i(V_k)\varepsilon_j(U_k)\} .$$

It follows that if we write V in the form
$V = \prod_{1 \leq i < j \leq n} [x_i, x_j]^{a_{ij}} V'$, where $V' \in [F, [F, F]] \cdots$ *

then
$$\varepsilon_{ij}(V) = \begin{cases} a_{ij} & \text{if } i < j \\ 0 & \text{if } i = j \\ -a_{ji} & \text{if } i > j . \end{cases}$$

This together with 3.2 gives the following result due to Labute and Shapiro-Sonn, [10] and [17].

THEOREM 3.3. *Suppose* $K = \{x_1, \cdots, x_n \mid V = 1\}$ *where V is written in the form given by *. Then the cup product matrix for K is given by the skew symmetric matrix*

$$A = \begin{bmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & & & \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{bmatrix} .$$

If K satisfies Poincaré duality over $\boldsymbol{Z}$ then the following theorem, which has been proved independently by Ratcliffe [15], shows that the relator V can be made almost like that of a surface.

THEOREM 3.4. *Suppose K satisfies Poincaré duality over $\boldsymbol{Z}$.*

Then K has the homotopy type of

$$L = \{x_1, \cdots, x_{2g} \mid [x_1, x_2] \cdots [x_{2g-1}, x_{2g}]V'\}$$

where $V' \in [F, [F, F]]$.

Proof. If $N \in \mathrm{Aut}(F)$ is an automorphism then the complex $\{x_1, \cdots, x_n \mid V = 1\}$ has the homotopy type of $\{x_1, \cdots, x_n \mid N(V) = 1\}$. Let A, B be the respective cup product matrices. Then there exists $U \in GL_n(Z)$ such that $B = UAU^T$. Conversely if B is congruent to A then there is an $N \in \mathrm{Aut}(F)$ such that B is the cup product matrix of $\{x_1, \cdots, x_n \mid N(V) = 1\}$ as can be seen using routine calculations with Nielsen transformations.

Now if K satisfies Poincaré duality then A is a nonsingular skew symmetric matrix and so by well known results in linear algebra is congruent to

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \qquad \text{see e.g. [13]}.$$

By using the above argument K may be made into the required form.

Finally we note the following corollary to the above results.

THEOREM 3.5. *Let $U_1, V_1, \cdots, U_g, V_g$ be words in the free group on $x_1, \cdots, x_{2g}$. Then $\{x_1, \cdots, x_{2g} \mid [U_1, V_1] \cdots [U_g, V_g] = 1\}$ satisfies Poincaré duality with respect to Z-coefficients if and only if, the group $\{x_1, \cdots, x_{2g} \mid U_1 = V_1 = \cdots = U_g = V_g = 1\}$ is perfect.*

Thus there exists a correspondence between presentations of perfect groups on an even number of generators with defect zero and group presentations satisfying Poincaré duality over Z. For example the binary icosahedral group I^* has the defect zero presentation $\{x_1, x_2 \mid U = V = 1\}$ where $U = x_1x_2x_1x_2^{-4}$ and $V = x_1^{-2}x_2x_1x_2$. Therefore the group presentation

$$K = \{x_1, x_2 \mid x_1x_2x_1x_2^{-4}x_1^{-2}x_2x_1x_2^5x_1^{-1}x_2^{-1}x_1^{-1}x_2^{-1}x_1^{-1}x_2^{-1}x_1^2\}$$

of the group G satisfies Poincaré duality with Z coefficients. Notice that K cannot possibly satisfy duality for twisted coefficients since this would force G to be isomorphic to $Z \oplus Z$ and there is a homomorphism of G onto the binary icosahedral group.

4. Poincaré duality with twisted coefficients. As in the previous section $K = \{x_1, \cdots, x_n \mid V = 1\}$ will denote a presentation of the group G such that $V \in [F, F]$ is not a proper power.

The presenting homomorphism $\phi: F \to G$ induces a ring homomorphism $\phi: ZF \to ZG$ also denoted by ϕ.

In this section we will obtain necessary and sufficient conditions for G to satisfy Poincaré duality with respect to a fixed G module A. To do this we need the duality map on the chain level. Thus let $\Lambda = Z[G]$ and let C_* denote the usual chain complex associated to the Lyndon resolution, i.e., C_* is

$$0 \longrightarrow \Lambda \xrightarrow{d_2} \underbrace{\Lambda \oplus \cdots \oplus \Lambda}_{n \text{ copies}} \xrightarrow{d_1} \Lambda \longrightarrow 0 ,$$

where

$$d_2(\lambda) = (\lambda\phi(\partial_1 V), \cdots, \lambda\phi(\partial_n V))$$
$$d_1(\lambda_1, \cdots, \lambda_n) = \lambda_1(\phi(x_1) - 1) + \cdots + \lambda_n(\phi(x_n) - 1) .$$

Now define $D: \operatorname{Hom}_\Lambda(C_i, A) \to \bar{A} \otimes_\Lambda C_{2-i}$ as follows:

$i = 2$, $D: A \longrightarrow \bar{A}$ is $D: a \longrightarrow -a$

$i = 0$, $D: A \longrightarrow \bar{A}$ is $D: a \longrightarrow a$

$i = 1$, $D: A \oplus \cdots \oplus A \longrightarrow \bar{A} \oplus \cdots \oplus \bar{A}$ is given by the formula

$$D(a_1, \cdots, a_n) = (\cdots, \underbrace{-\sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})a_j}_{i\text{th coordinate}}, \cdots) .$$

THEOREM 4.1. *$D: \operatorname{Hom}_\Lambda(C_*, A) \to \bar{A} \otimes_\Lambda C_*$ is a chain map.*

Proof. We must verify the commutativity of the diagram

$$(4.2)\qquad \begin{array}{ccccccccc} 0 \longrightarrow & \operatorname{Hom}_\Lambda(C_0, A) & \xrightarrow{d_1^*} & \operatorname{Hom}_\Lambda(C_1, A) & \xrightarrow{d_2^*} & \operatorname{Hom}_\Lambda(C_2, A) & \longrightarrow 0 \\ & \downarrow D & & \downarrow D & & \downarrow D & \\ 0 \longrightarrow & \bar{A} \otimes_\Lambda C_2 & \xrightarrow{d_2} & \bar{A} \otimes_\Lambda C_1 & \xrightarrow{d_1} & \bar{A} \otimes_\Lambda C_0 & \longrightarrow 0 . \end{array}$$

Thus

$$\begin{aligned} (d_1 \circ D)(a_1, \cdots, a_n) &= d_1(\cdots, -\sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})a_j, \cdots) \\ &= -\sum_i \sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})a_j(\phi(x_i) - 1) \\ &= -\sum_i \sum_j (\phi(x_i^{-1}) - 1)\phi(\overline{\partial_i(\overline{\partial_j V})})a_j . \end{aligned}$$

But

$$\begin{aligned} \sum_i (\phi(x_i^{-1}) - 1)\phi(\overline{\partial_i(\overline{\partial_j V})}) &= \phi \sum_i (x_i^{-1} - 1)\overline{\partial_i(\overline{\partial_j V})} = \phi \sum_i \overline{\partial_i(\overline{\partial_j V})(x_i - 1)} \\ &= \phi(\overline{\overline{\partial_j V} - \varepsilon(\overline{\partial_j V})}) = \phi(\partial_j V) . \end{aligned}$$

Therefore

$$(d_1 \circ D)(a_1, \cdots, a_n) = -\sum_j \phi(\partial_j V)a_j = (D \circ d_2^*)(a_1, \cdots, a_n) .$$

On the other hand

$$\begin{aligned}(D \circ d_1^*)(a) &= D((\phi(x_1) - 1)a, \cdots, (\phi(x_n) - 1)a) \\ &= (\cdots, -\sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})(\phi(x_j) - 1)a, \cdots) .\end{aligned}$$

However

$$\begin{aligned}\sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})(\phi(x_j) - 1) &= \phi \sum_j \overline{\partial_i(\overline{\partial_j V})}(x_j - 1) \\ &= \phi\, \overline{\sum_j (x_j^{-1} - 1)\partial_i(\overline{\partial_j V})} = \phi\, \overline{\sum_j \partial_i[(x_j^{-1} - 1)\overline{\partial_i V}]}\end{aligned}$$

since

$$\begin{aligned}\partial_i[(x_j^{-1} - 1)\overline{\partial_j V}] &= \partial_i(x_j^{-1} - 1)\varepsilon(\overline{\partial_j V}) + (x_j^{-1} - 1)\partial_i(\overline{\partial_j V}) \\ &= (x_j^{-1} - 1)\partial_i(\overline{\partial_j V})\end{aligned}$$

(recall that $\varepsilon(\overline{\partial_j V}) = \varepsilon(\partial_j V) = \varepsilon_j(V) = 0$ because $V \in [F, F]$). Hence

$$\begin{aligned}\sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})(\phi(x_j) - 1) &= \phi\overline{\partial_i(\sum_j (x_j^{-1} - 1)\overline{\partial_j V})} = \phi\overline{\partial_i(\overline{\sum_j \partial_j(V)(x_j - 1)})} \\ &= \phi\overline{\partial_i(\bar{V} - 1)} = \phi\overline{\partial_i(\bar{V})} = \phi(\overline{\partial_i(V^{-1})}) \\ &= \phi(\overline{-V^{-1}\partial_i(V)}) = -\phi(\overline{\partial_i(V)}) \text{ since } \phi(V) = 1 .\end{aligned}$$

This shows that $(Dd_1^*)(a) = (\cdots, \phi(\overline{\partial_i V})a, \cdots) = (d_2 D)(a)$. □

The chain transformation $D\colon \mathrm{Hom}_\Lambda(C_*, A) \to \bar{A} \otimes_\Lambda C_*$ is clearly natural in A and so the induced map in homology $D_*\colon H^*(G; A) \to H_*(G; \bar{A})$ is functional in A. The cap product homomorphism $[G] \cap \cdot\colon H^*(G; A) \to H_*(G; \bar{A})$ is also functorial in A. In the next theorem we prove that $D_* = [G] \cap \cdot$, but first we compare D_*, $[G] \cap \cdot$ for the special case $H^1(G) \to H_1(G)$. We have

$$\begin{aligned}D_*(x_k^*) &= D_*(0, \cdots, 0, 1, 0, \cdots, 0) = (\cdots, -\sum_j \phi(\overline{\partial_i(\overline{\partial_j V})})\delta_{jk}, \cdots) \\ &= -\sum_i \phi(\overline{\partial_i(\overline{\partial_k V})})\bar{x}_i = -\sum_i \varepsilon(\overline{\partial_i(\overline{\partial_k V})})\bar{x}_i\end{aligned}$$

(since the module structure on the coefficients is given by augmentation). Now $-\varepsilon(\overline{\partial_i(\overline{\partial_k V})}) = -\varepsilon\, \partial_i(\overline{\partial_k V}) = \varepsilon\, \partial_i\partial_k(V)$ because $\varepsilon\, \partial_i(\bar{f}) = -\varepsilon\, \partial_i(f)$ for $f \in F$. Therefore

$$D_*(x_k^*) = \sum_i \varepsilon_{ik}(V)\bar{x}_i = \sum_i \langle x_i^* \cup x_k^*, [G]\rangle \bar{x}_i$$

according to (3.2). But we also have

$$[G] \cap x_k^* = \sum_i \langle x_i^*, [G] \cap x_k^* \rangle \bar{x}_i = \sum_i \langle x_i^* \cup x_k^*, [G] \rangle \bar{x}_i \, .$$

Thus we proved that

$$D_* = [G] \cap \cdot : H^1(G; Z) \longrightarrow H_1(G; Z) \, .$$

THEOREM 4.3. *$D_* = [G] \cap \cdot : H^*(G; A) \to H_*(G; \bar{A})$ for any A.*

Proof. The method of proof is modelled on some of the proofs in [1, 2]. For any A the homomorphism $D_*: H^2(G; A) \to H_0(G; \bar{A})$ is induced by the chain map $D: \mathrm{Hom}_\Lambda(C_2, A) \to \bar{A} \otimes_\Lambda C_0$, $D: a \to -a$. Thus $D_*: H^2(G; A) \to H_0(G; \bar{A})$ is the homomorphism

$$A/(\sum \lambda_i \phi(\partial_i V)) \longrightarrow A/(\sum \lambda_i(\phi(x_i) - 1)) \quad \text{induced by} \quad a \longrightarrow -a \, .$$

It follows that $D_*: H^2(G; Z) \to H_0(G; Z)$ is an isomorphism. Since both of these groups are infinite cyclic and $[G] \cap \cdot : H^2(G; Z) \to H_0(G; Z)$ is also an isomorphism we must have

$$D_* = e \cap \cdot : H^2(G; Z) \longrightarrow H_0(G; Z) \, , \quad \text{where} \quad e = \pm[G] \, .$$

Now consider the coefficient sequence $0 \to I[G] \to \Lambda \xrightarrow{\varepsilon} Z \to 0$ of left Λ modules. Conjugating we get the exact sequence $0 \to I[G] \to \bar{\Lambda} \xrightarrow{\varepsilon} Z \to 0$ of right Λ modules. Then the functoriality of D_* and $e \cap \cdot$ gives the commutative diagram

$$\begin{array}{ccccccc}
\cdots \longrightarrow & H^2(G; I[G]) & \longrightarrow & H^2(G; \Lambda) & \xrightarrow{\varepsilon_*} & H^2(G; Z) & \longrightarrow 0 \\
& D_* \downarrow \quad \downarrow e \cap \cdot & & D_* \downarrow \quad \downarrow e \cap \cdot & & \downarrow D_* = e \cap \cdot & \\
\cdots \longrightarrow & H_0(G; I[G]) & \longrightarrow & H_0(G; \bar{\Lambda}) & \xrightarrow{\varepsilon_*} & H_0(G; Z) & \longrightarrow 0 \, .
\end{array}$$

But $\varepsilon_*; H_0(G; \bar{\Lambda}) \to H_0(G; Z)$ is a monomorphism since the homomorphism $H_0(G; I[G]) \to H_0(G; \bar{\Lambda})$ may be identified with the homomorphism

$$I[G]/I[G] \cdot I[G] \longrightarrow \Lambda/\Lambda \cdot I[G] \quad \text{induced by} \quad I[G] \subseteq \Lambda \, .$$

Chasing around the second square in the diagram now gives

$$D_* = e \cap \cdot : H^2(G; \Lambda) \longrightarrow H_0(G; \bar{\Lambda}) \, .$$

The group G admits a finite resolution of Z by finitely generated free Λ modules and hence the functor $H^*(G; \cdot)$ commutes with direct sums. From this fact it follows that

$$D_* = e \cap \cdot : H^2(G; M) \longrightarrow H_0(G; \bar{M}) \quad \text{for any free module } M \, .$$

Given any module A we choose some presentation $0 \to N \to M \xrightarrow{\phi} A \to 0$. By naturality there is a commutative diagram

$$\begin{array}{ccc} H^2(G; M) & \xrightarrow{\psi_*} & H^2(G; A) \longrightarrow 0 \\ \Big\downarrow D_* = e \cap \cdot & & D_* \Big\downarrow \quad \Big\downarrow e \cap \cdot \\ H_0(G; \bar{M}) & \xrightarrow{\bar{\psi}_*} & H_0(G; \bar{A}) \longrightarrow 0 \,. \end{array}$$

Note that $\psi_*: H^2(G; M) \to H^2(G; A)$ is an epimorphism since G has cohomological dimension 2. Commutativity of this diagram now implies that

$$D_* = e \cap \cdot : H^2(G; A) \longrightarrow H_0(G; \bar{A}) \quad \text{for any module} \quad A \,.$$

Now consider the commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^1(G; M) & \longrightarrow & H^1(G; A) & \longrightarrow & H^2(G; N) & \longrightarrow \cdots \\ & D_* \Big\downarrow \quad \Big\downarrow e \cap \cdot & & D_* \Big\downarrow \quad \Big\downarrow e \cap \cdot & & \Big\downarrow D_* = e \cap \cdot & \\ \cdots \longrightarrow & H_1(G; \bar{M}) & \longrightarrow & H_1(G; \bar{A}) & \longrightarrow & H_0(G; \bar{N}) & \longrightarrow \cdots \,. \end{array}$$

$\bar{M}$ is a free right module and so $H_1(G; \bar{M}) = 0$. Therefore $H_1(G; \bar{A}) \to H_0(G; \bar{N})$ is a monomorphism, and this implies that

$$D_* = e_* \cap \cdot : H^1(G; A) \longrightarrow H_1(G; \bar{A}) \quad \text{for all} \quad A \,.$$

Finally we look at the commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & H^0(G; M) & \longrightarrow & H^0(G; A) & \longrightarrow & H^1(G; N) & \longrightarrow \cdots \\ & D_* \Big\downarrow \quad \Big\downarrow e \cap \cdot & & D_* \Big\downarrow \quad \Big\downarrow e \cap \cdot & & \Big\downarrow D_* = e \cap \cdot & \\ \cdots \longrightarrow & H_2(G; \bar{M}) & \longrightarrow & H_2(G; \bar{A}) & \longrightarrow & H_1(G; \bar{N}) & \longrightarrow \cdots \,. \end{array}$$

$H_2(G; \bar{M}) = 0$ as $\bar{M}$ is free and therefore

$$D_* = e \cap \cdot : H^0(G; A) \longrightarrow H_2(G; \bar{A}) \quad \text{for all} \quad A \,.$$

To prove that $e = [G]$ we use the functoriality of D_* and $[G] \cap \cdot$ with respect to the variable G, while keeping the coefficients fixed at Z. If G has the presentation $\{x_1, \cdots, x_n \mid V = [U_1, V_1] \cdots [U_g, V_g] = 1\}$ let π be the surface group $\{y_1, \cdots, y_{2g} \mid [y_1, y_2] \cdots [y_{2g-1}, y_{2g}] = 1\}$. We also have the obvious degree 1 map $\phi: \pi \to G$. Then there are classes $e_G \in H_2(G)$, $e_\pi \in H_2(\pi)$ and a commutative diagram

$$\begin{array}{ccc} H^2(G) & \xrightarrow{D_* = e_G \cap \cdot} & H_0(G) \\ \Big\downarrow \phi^* & & \Big\uparrow \phi_* \\ H^2(\pi) & \xrightarrow{D_* = e_\pi \cap \cdot} & H_0(\pi) \,. \end{array}$$

It has already been noted that $D_* = [\pi] \cap \cdot : H^1(\pi) \to H_1(\pi)$. This coupled with the fact that $D_*: H^1(\pi) \to H_1(\pi)$ is an isomorphism implies that $e_\pi = [\pi]$. If $[G]^*$, $[\pi]^*$ are the cohomology classes dual to $[G]$, $[\pi]$ respectively then

$$\varepsilon D_*([G]^*) = \varepsilon\phi_* D_* \phi^*([G]^*) = \varepsilon\phi_* D_*([\pi]^*) \quad (\text{as } \phi^*([G]^*) = [\pi]^*)$$

where $\varepsilon: H_0(\cdot) \to Z$ is the augmentation. Hence

$$\varepsilon D_*([G]^*) = \varepsilon\phi_*([\pi] \cap [\pi]^*) = \langle [\pi]^*, [\pi] \rangle = 1$$

and therefore $\langle [G]^*, e_G \rangle = \varepsilon e_G \cap [G]^* = \varepsilon D_*([G]^*) = 1$. This proves that $e_G = [G]$.

By chasing around diagram 4.2 we prove the following theorem.

THEOREM 4.4. *With the notation above, G satisfies Poincaré duality with respect to A if, and only if, $D: \bigoplus_1^n A \to \bigoplus_1^n \bar{A}$ is an isomorphism.*

As an example of this theorem consider the case $A = Z$ with the trivial module structure. Then

$$\phi(\overline{\partial_i(\overline{\partial_j V})})a = \varepsilon(\overline{\partial_i(\overline{\partial_j V})})a = \varepsilon(\partial_i(\overline{\partial_j V}))a .$$

But for any $f \in F$ we have

$$\varepsilon\partial_i(\bar{f}) = \varepsilon\,\partial_i(f^{-1}) = \varepsilon(-f^{-1}\partial_i(f)) = -\varepsilon\,\partial_i(f) .$$

Therefore $-\phi(\overline{\partial_i(\overline{\partial_j V})})a = \varepsilon\,\partial_i\partial_j(V)a = \varepsilon_{ij}(V)a$. This means that the cap product map $D: \mathrm{Hom}_\Lambda(C_1, Z) \to Z \otimes_\Lambda C_1$, that is $D: Z \oplus \cdots \oplus Z \to Z \oplus \cdots \oplus Z$, becomes

$$D(a_1, \cdots, a_n) = (\cdots, \sum_j \varepsilon_{ij}(V)a_j, \cdots) .$$

In other words D is the $n \times n$ matrix $[\varepsilon_{ij}(V)]$, a result in agreement with 3.2.

As another example consider the Λ module $Z[G_{ab}]$, where the Λ module structure is induced by the abelianization homomorphism $\alpha: G \to G_{ab}$. For convenience set $t_i = \alpha\phi(x_i)$, $1 \leqq i \leqq n$. Then $Z[G_{ab}]$ is the Laurent polynomial ring on the variables $t_1, \cdots, t_n$. If $p(t_1, \cdots, t_n)$ is a Laurent polynomial then the module structure is given by

$$\phi(x_i^{\pm 1}) \cdot p(t_1, \cdots, t_n) = t_i^{\pm 1} p(t_1, \cdots, t_n) , \quad 1 \leqq i \leqq n .$$

THEOREM 4.5. *G satisfies duality for $Z[G_{ab}]$ coefficients if, and only if, the matrix $[\alpha\,\partial_i(\overline{\partial_j V})]$ is invertible over $Z[G_{ab}]$.*

Proof. Since $\phi: F \to G$ induces an isomorphism $F_{ab} \cong G_{ab}$ we have

$$-\phi(\overline{\partial_i(\overline{\partial_j V})})p(t_1, \cdots, t_n) = -\alpha(\overline{\partial_i(\overline{\partial_j V})})p(t_1, \cdots, t_n)$$

where $\alpha: F \to F_{ab}$ again denotes abelianization. But $\alpha(\bar{f}) = -\alpha(f)$ and so the duality map $D: \boldsymbol{Z}[G_{ab}] \oplus \cdots \oplus \boldsymbol{Z}[G_{ab}] \to \boldsymbol{Z}[G_{ab}] \oplus \cdots \oplus \boldsymbol{Z}[G_{ab}]$ may be identified with the matrix $[\alpha\, \partial_i(\overline{\partial_j V})]$. □

We can generalize this result by replacing G_{ab} by an abelian group J and letting $\alpha: G \to J$ be some homomorphism. Then G satisfies duality for $\boldsymbol{Z}[J]$ coefficients if, and only if, the $n \times n$ matrix $[\beta\, \partial_i(\overline{\partial_j V})]$ is invertible over $\boldsymbol{Z}[J]$, where $\beta = \alpha\phi: F \to J$.

REFERENCES

1. R. Bieri. *Gruppen mit Poincaré-Dualität*, Comm. Math. Helv., **47** (1972), 373–396.
2. R. Bieri and B. Eckmann, *Groups with homological duality generalizing Poincaré duality*, Invent. Math., **20** (1973), 103–124.
3. R. Fenn and D. Sjerve, *Elementary complexes and Massey products*, to appear.
4. R. H. Fox, *Free differential calculus I*, Ann. of Math., **57** (1953), 547–560.
5. ———, *Free differential calculus II*, Ann. of Math., **59** (1954), 196–210.
6. ———, *Free differential calculus III*, Ann. of Math., **64** (1956), 407–419.
7. R. H. Fox, K. T. Chen and R. C. Lyndon, *Free differential calculus IV*, Ann. of Math., **68** (1958), 81–95.
8. R. H. Fox, *Free differential calculus V*, Ann. of Math., **71** (1960), 408–422.
9. F. E. A. Johnson and C. T. C. Wall, *Groups satisfying Poincaré duality*, Ann. of Math., **96** (1972), 592–598.
10. J. Labute, *Classification of Demushkin groups*, Canad. J. Math., **19** (1967), 106–121.
11. H. Müller, *Groupes et paires de groupes à dualité de Poincaré de dimension* 2, C.R. Acad. Sc. Paris, **289** (1979), 373–374.
12. ———, *Two Dimensional Poincaré Duality Groups and Pairs*, preprint F.I.M., E.T.H. Zürich, 1979.
13. M. Newman, *Integral Matrices*, v. 45 in Pure and Applied Mathematics, Academic Press, 1972.
14. R. Porter, *Milnor's $\bar{\mu}$-invariants and Massey products*, T.A.M.S., **257** (1980), 39–71.
15. J. Ratcliffe, *On one relator groups which satisfy Poincaré duality*, Math. Z., **177** (1981), 425–438.
16. J. Schafer, *Poincaré complexes and one relator groups*, J. Pure and Applied Algebra, **10** (1977), 121–126.
17. J. Shapiro and I. J. Sonn, *Free factors of one relator groups*, Duke Math. J., **41** (1974), 83–88.

Received August 14, 1980. Research partially supported by N.S.E.R.C. contract A 7218 and Nato Research Grant 102.80.

UNIVERSITY OF SUSSEX
FALMER, BRIGHTON
ENGLAND
AND
UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, B.C. V6T 1W5 CANADA

b

Mathematical parers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd unmbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966, Regular subscription rate: $114.00 a year (6 Vol., 12 issues). Special rate: $57.00 a year to individual members of supporting institution.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address shoud be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 103, No. 2 April, 1982