Pacific Journal of Mathematics

SHADOW AND INVERSE-SHADOW INNER PRODUCTS FOR A CLASS OF LINEAR TRANSFORMATIONS

GEORGE GOLIGHTLY

Vol. 103, No. 2 April 1982

PACIFIC JOURNAL OF MATHEMATICS
Vol. 103, No. 2, 1982

SHADOW AND INVERSE-SHADOW INNER PRODUCTS FOR A CLASS OF LINEAR TRANSFORMATIONS

GEORGE GOLIGHTLY

Suppose $\{H, (\cdot, \cdot)\}$ is a complete inner product space and H_1 is a dense subspace of H. In case T is a linear transformation from H_1 to H_1 (perhaps not bounded), a necessary and sufficient condition is obtained in Theorem 1 for the existence of an inner product $(\cdot, \cdot)_1$ for H_1 such that (i) the identity is continuous from $\{H_1, (\cdot, \cdot)_1\}$ to $\{H, (\cdot, \cdot)\}$ and (ii) T is bounded in $\{H_1, (\cdot, \cdot)_1\}$. When this condition holds, the inverse-shadow inner product is defined on H_1, for sufficiently large positive numbers β, by $(x, y)_{\beta,T} = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$. An extension of Theorem 1 provides a necessary and sufficient condition for the existence of an inner product $(\cdot, \cdot)_1$ for H_1 such that $\{H_1, (\cdot, \cdot)_1\}$ is complete and (i) and (ii) hold. This latter condition, stated in Theorem 5 in terms of a pair of inverse-shadow inner products, depends on a description of those complete inner product spaces $\{H_1, (\cdot, \cdot)_1\}$, with H_1 dense in H, for which (i) holds. According to this description, given in Theorem 4, each such inner product $(\cdot, \cdot)_1$ is a scalar-multiple of an inverse-shadow inner product $(\cdot, \cdot)_{\delta,C}$, where C is a bounded operator on H mapping H_1 to H_1 and $\delta = 1$.

This pattern was developed in an investigation, other results of which are in [4]. If H_1 is a linear subspace of H, $(\cdot, \cdot)_1$ is an inner product for H_1, and the identity is continuous from $\{H_1, (\cdot, \cdot)_1\}$ to $\{H, (\cdot, \cdot)\}$, $\{H_1, (\cdot, \cdot)_1\}$ is said in [6] to be continuously situated in $\{H, (\cdot, \cdot)\}$. The setting in Theorem 4 of a pair of complete inner product spaces, one continuously situated in the other, is discussed in [1], [2], [6], and [7]. Additional results in Theorems 2 and 3 relate the shadow inner product, the inner product $((1 - T^*T/\beta^2)\cdot, \cdot)'$ in those theorems, and the inverse-shadow inner product $(\cdot, \cdot)_{\beta,T}$. In contrast to Theorem 4, an example at the end of the paper shows that $\{H_1, (\cdot, \cdot)_{\beta,T}\}$ may be complete even when the closure in $H \times H$ of T is not a function.

Here is an example to which Theorem 1 applies (with $H = H_1$). Start with a complete infinite dimensional inner product space $\{H', (\cdot, \cdot)'\}$, a one-to-one (continuous) operator T on H' with range a dense, proper subspace of H', and a closed subspace Z of H' such that $Z \cap T(H')$ is $\{0\}$. Now, with P the orthogonal projection of H' onto $Z^{\perp}$, there is, by the Axiom of Choice, an algebraic complement H_1 of Z in H' of which $T(H')$ is a subspace and, with $(\cdot, \cdot)$ the inner product on H_1 such that $(x, y) = (Px, Py)'$, $\{H_1, (\cdot, \cdot)\}$ is complete and for x in H_1 $(x, x) \leqq (x, x)'$. Yet the restriction of T to H_1 is not continuous in $\{H_1(\cdot, \cdot)\}$. Of course, the above construction uses the Axiom of Choice, as the result of [8] implies it must. However, this use is not in constructing T but in selecting the subspace H_1 of H'.

Throughout the paper, $\{H, (\cdot, \cdot)\}$ is a complete infinite dimensional inner product space and H_1 a dense subspace of H. If some variation of the symbols '$(\cdot, \cdot)$' denotes an inner product for the space S, then the corresponding variation of '$\|\cdot\|$' denotes the corresponding norm for S. For instance, $\|x\|_{\beta,T} = [(x, x)_{\beta,T}]^{1/2}$. An operator on $\{H, (\cdot, \cdot)\}$ is a continuous linear transformation from all of H to (into) H. A closed operator in $\{H, (\cdot, \cdot)\}$ is a linear transformation from a dense subspace of H to H whose graph is closed in $H \times H$. If Z and Z' are two subspaces of H such that $Z \cap Z'$ is $\{0\}$ and H is the linear span of Z and Z', then Z is said to be an algebraic complement in H of Z' and that linear transformation ϕ on H such that ϕ is the identity 1 on Z and 0 on Z' is called the algebraic projection of H onto Z with kernel Z'. If Z is a subset of H, $\bar{Z}$ is the closure of Z in H.

THEOREMS AND EXAMPLES

THEOREM 1. *Suppose that T is a linear transformation from H_1 to H_1. In order that there be a norm $\|\cdot\|_1$ for H_1 such that* (i) *there is a positive number c such that $\|\cdot\| \leqq c\|\cdot\|_1$ on H_1 and* (ii) *T is continuous in $\{H_1, \|\cdot\|_1\}$ it is necessary and sufficient that there be a positive number β such that for x in H_1 $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges. In case there is such a norm $\|\cdot\|_1$, if β is a number exceeding the operator-norm for T in $\{H_1, \|\cdot\|_1\}$ then for x and y in H_1 the formula $(x, y)_{\beta,T} = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ defines an inner product $(x, y)_{\beta,T}$ for H_1 such that*

(1) *there is a positive number d such that for x in H_1 $\|x\| \leqq \|x\|_{\beta,T} \leqq d\|x\|_1$,*

(2) *for x in H_1 $\lim_{p\to\infty} \|(T/\beta)^p x\|_{\beta,T} = 0$, and*

(3) *for x and y in H_1 $(Tx, Ty)_{\beta,T} = \beta^2[(x, y)_{\beta,T} - (x, y)]$.*

Proof. In case there is a positive number β for which $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges on H_1, we have for x and y in H_1 and n a positive integer,

$$\sum_{p=0}^{n} |((T/\beta)^p x, (T/\beta)^p y)|$$

$$\leqq \sum_{p=0}^{n} \|(T/\beta)^p x\| \, \|(T/\beta)^p y\|$$

$$\leqq \left(\sum_{p=0}^{n} \|(T/\beta)^p x\|^2\right)^{1/2} \left(\sum_{p=0}^{n} \|(T/\beta)^p y\|^2\right)^{1/2},$$

so that $\sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ converges absolutely. Moreover, the formula $(x, y)_{\beta,T} = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ defines as inner product for H_1.

Suppose that there is a norm $\|\cdot\|_1$ for H_1 for which (i) and (ii) hold. Suppose n is a positive integer, β is a positive number, and r is a number greater than 1 such that for x in H_1 $r\|Tx\|_1 \leqq \beta\|x\|_1$. Then for x and y in H_1

$$\begin{aligned} &\sum_{p=0}^{n} |((T/\beta)^p x, (T/\beta)^p y)| \\ &\quad \leqq \sum_{p=0}^{n} \|(T/\beta)^p x\| \, \|(T/\beta)^p y\| \\ \text{(A)} &\quad \leqq c^2 \sum_{p=0}^{n} \|(T/\beta)^p x\|_1 \|(T/\beta)^p y\|_1 \\ &\quad \leqq c^2 \sum_{p=0}^{n} \|x\|_1 \|y\|_1 (1/r^{2p}) \\ &\quad = c^2 \|x\|_1 \|y\|_1 r^2/(r^2 - 1). \end{aligned}$$

Thus, for x and y in H_1 the series $\sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ converges absolutely and, replacing y by x in (A), we have

$$\text{(B)} \qquad \sum_{p=0}^{n} \|(T/\beta)^p x\|^2 \leqq c^2(\|x\|_1)^2 r^2/(r^2 - 1).$$

Note that (1) follows from (B) with $d = cr/(r^2 - 1)^{1/2}$. To establish (2), observe that for x in H_1

$$(\|(T/\beta)^p x\|_{\beta,T})^2 = \sum_{q=0}^{\infty} \|(T/\beta)^{p+q} x\|^2 \longrightarrow 0 \quad \text{as } p \longrightarrow \infty,$$

since $\sum_{q=0}^{\infty} \|(T/\beta)^q x\|^2$ converges. The equality (3) is established by noting that

$$\begin{aligned} &(Tx, Ty)_{\beta,T} \\ &\quad = \sum_{p=0}^{\infty} ((T/\beta)^p Tx, (T/\beta)^p Ty) \\ &\quad = \beta^2 \sum_{p=1}^{\infty} ((T/\beta)^p x, (T/\beta)^p y) \\ &\quad = \beta^2 \left[\sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y) - (x, y)\right] \\ &\quad = \beta^2 [(x, y)_{\beta,T} - (x, y)]. \end{aligned}$$

The following example is offered in connection with Lemma 1. This lemma is useful in the proof of Theorems 3 and 4.

EXAMPLE 1. Suppose that S is the subspace of $L^2[0, 1]$ of all absolutely continuous f on $[0, 1]$ such that f' is in $L^2[0, 1]$ and for such f $Tf = f'$, so that T is a closed operator in $L^2[0, 1]$. Suppose H_1 is the set of all f in S such that for $p \geqq 0$ $T^p f$ is in S and $\sum_{p=0}^{\infty} \int_0^1 |T^p f|^2$ converges. Then H_1 is a dense subspace of $L^2[0, 1]$ and, with $\beta = 1$ and $(f, g)_{\beta,T} = \sum_{p=0}^{\infty} \int_0^1 [T^p f][T^p g]^*$ on H_1, $\{H_1, (\cdot, \cdot)_{\beta,T}\}$ is complete.

LEMMA 1. *Suppose that T is a closed operator in $\{H, (\cdot, \cdot)\}$ and $\beta > 0$. Then the set H_2 of all x in H such that for $p > 0$ x is in the domain of T^p and $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges is a linear space such that $T(H_2)$ lies in H_2. Also, if $(\cdot, \cdot)_{\beta,T}$ is the inner product for H_2 given, as in Theorem 1, by $(x, y)_{\beta,T} = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ then $\{H_2, (\cdot, \cdot)_{\beta,T}\}$ is complete. In case T is self-adjoint in $\{H, (\cdot, \cdot)\}$, then the restriction of T to H_2 is self-adjoint in $\{H_2, (\cdot, \cdot)_{\beta,T}\}$.*

The following argument is offered. In general (when T is only closed and not defined everywhere), H_2 need not be dense in H. Suppose x is in H_2. Then $\sum_{p=0}^{\infty} \|(T/\beta)^p Tx\|^2 = \beta^2 \sum_{p=1}^{\infty} \|(T/\beta)^p x\|^2$, so that Tx is in H_2. To show that H_2 is a linear space, suppose S_1 is the linear space of all H-valued sequences, S_2 is the subspace of S_1 to which z belongs only in case $\sum_{p=0}^{\infty} \|z_p\|^2$ converges, and for z and w in S_2 $\langle z, w\rangle = \sum_{p=0}^{\infty} (z_p, w_p)$, so that $\{S_2, \langle\cdot, \cdot\rangle\}$ is a complete inner product space. Suppose D is the set of all x in H such that for $p > 0$ x is in the domain of T^p and $\tilde{T}$ the linear transformation from D to S_1 such that for $p \geqq 0$ $(\tilde{T}x)_p = (T/\beta)^p x$. Note that $H_2 = \tilde{T}^{-1}(S_2)$, a linear space, and that $\tilde{T}$, restricted to H_2, is a linear isometry from $\{H_2, (\cdot, \cdot)_{\beta,T}\}$ onto a subspace of S_2. Suppose y is a convergent sequence in $\{H_2, (\cdot, \cdot)_{\beta,T}\}$. Then $\tilde{T}y$ is convergent in S_2, with limit z in S_2. Since, for $p \geqq 0$ the sequence $\{(T/\beta)^p y, (T/\beta)^{p+1} y\}$ has values in the closed transformation T/β and limit $\{z_p, z_{p+1}\}$ in $H \times H$, $z_{p+1} = (T/\beta)z_p$. Thus, for $p \geqq 0$ $z_p = (T/\beta)^p z_0$, so that $z = \tilde{T}z_0$. Since $\tilde{T}$ is an isometry, y has limit z_0 in $\{H_2, (\cdot, \cdot)_{\beta,T}\}$. Suppose T is self-adjoint in $\{H, (\cdot, \cdot)\}$. Then for x and y in H_2

$$\begin{aligned}(Tx, y)_{\beta,T} &= \sum_{p=0}^{\infty} ((T/\beta)^p Tx, (T/\beta)^p y) \\ &= \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p Ty) = (x, Ty)_{\beta,T} ,\end{aligned}$$

so that T is self-adjoint on the complete space $\{H_2, (\cdot, \cdot)_{\beta,T}\}$.

EXAMPLE 2. This example shows that in case $\{H, (\cdot, \cdot)\}$ is separable the set of linear transformations T with domain H and range lying in H for which there is a positive number β such that $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges on H is not a linear space.

Suppose y is in H, $\|y\| = 1$, and Y is the linear span of $\{y\}$. Suppose $\{e_m\}_1^\infty$ is a complete orthonormal sequence in $H \ominus Y$. Suppose for $m > 0$ $u_m = e_m + (m!)y$. The linear span U of $\{u_m\}_1^\infty$ is dense in H. One sees this by noting that $y = \lim_{m\to\infty}(u_m/m!)$. Hence, for $p > 0$ $e_p = u_p - (p!)y$ is in $\bar{U}$. Thus, the linear space $\bar{U}$ includes both Y and $H \ominus Y$. Suppose that Z is an algebraic complement of Y in H of which U is a subspace. Suppose ϕ is the algebraic projection of H onto Z with kernel Y and that C is the operator on H such that $Cy = 0$ and for m a positive integer $Ce_m = e_{m+1}$. Since the operator-norm of C is 1, $\sum_{p=0}^{\infty} \|(C/2)^p x\|^2$ converges on H. Since for $p > 0 (\phi - 1)^p = (-1)^{p+1}(\phi - 1)$, $\sum_{p=0}^{\infty} \|[(\phi - 1)/2]^p x\|^2$ converges on H.

Suppose T is $C + (\phi - 1)$ and m is the number-sequence such that $m_1 = 1$ and for $n > 0$ $m_{n+1} = (n+1)! - m_n$. Then for $n > 0$ $T^n(e_1) = e_{n+1} + m_n y$ and $\|T^n e_1\|^2 = 1 + m_n^2$. Note that for $n \geqq 1$ $n! - (n-1)! \leqq m_n \leqq n!$, so that $m_{n+1} \geqq n!$. Thus, for $\beta > 0$ $\sum_{p=0}^{\infty} \|(T/\beta)^p e_1\|^2$ diverges.

THEOREM 2. *Suppose that* $\{H', (\cdot, \cdot)'\}$ *is a complete inner product space,* T *is an operator on* $\{H', (\cdot, \cdot)'\}$, *and* H_1 *is a dense subspace of* H' *such that* $T(H_1)$ *lies in* H_1. *Suppose, moreover, that there is a positive number* β *such that for each of* x *and* y *in* H_1 $(x, y)' = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$. *Then* (i) β *is not less than the operator-norm for* T *in* $\{H', (\cdot, \cdot)'\}$, (ii) *with* T^* *the adjoint of* T *in* $\{H', (\cdot, \cdot)'\}$ *and* x *and* y *in* H_1 $(x, y) = ((1 - T^*T/\beta^2)x, y)'$, *and* (iii) *in case* $H' \neq H_1$ *and* $\{H_1, (\cdot, \cdot)\}$ *is complete, so that* $H = H_1$, *then* β *is the operator-norm for* T *in* $\{H', (\cdot, \cdot)'\}$ *and for* T *on* H_1 *in* $\{H_1, (\cdot, \cdot)'\}$.

Proof. Since H_1 is dense in H' and T continuous on H', the operator-norm for T in $\{H', (\cdot, \cdot)'\}$ is the operator-norm for T on H_1 in $\{H_1, (\cdot, \cdot)'\}$. Suppose that for x and y in H_1 $(x, y)' = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$. Then for x in H_1

$$(\|Tx\|')^2 = \beta^2[(\|x\|')^2 - \|x\|^2] \leqq \beta^2(\|x\|')^2 .$$

Thus, β is not less than the operator-norm for T in $\{H', (\cdot, \cdot)'\}$. Also, on H_1

$$\begin{aligned}(x, y) &= (x, y)' - ((T/\beta)x, (T/\beta)y)' \\ &= ((1 - T^*T/\beta^2)x, y)' ,\end{aligned}$$

so that (ii) is established.

To prove (iii), note that, since $H' \neq H_1$, H_1 is not closed in H'.

Also, the identity function from $\{H_1, (\cdot, \cdot)'\}$ to $\{H_1, (\cdot, \cdot)\}$ is continuous. Since $\{H_1, (\cdot, \cdot)\}$ is complete, the identity function from $\{H_1, (\cdot, \cdot)\}$ to $\{H_1, (\cdot, \cdot)'\}$ is not continuous. By the Closed Graph theorem, the set Z of all $\|\cdot\|'$-limits in H' of H_1-sequences having $\|\cdot\|$-limit 0 is nondegenerate. Since Z is the kernel of $(1 - T^*T/\beta^2)^{1/2}$, there is a nonzero point x of H' such that $x = (T^*T/\beta^2)x$. Thus, $(\|Tx\|')^2 = \beta^2(\|x\|')^2$. In view of (i), (iii) is established.

REMARK. Here I will describe why I call an inner product, $((1 - T^*T/\beta^2)\cdot, \cdot)'$, a shadow inner product. The point of view taken by the author is that one starts with $\{H, (\cdot, \cdot)\}$, a linear transformation T from H to H, not continuous in $\{H, (\cdot, \cdot)\}$, and a positive number β such that $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges on H. (T might be the transformation $\phi - 1$ of Example 2 with $\beta = 2$). One builds the space $\{H, (\cdot, \cdot)_{\beta,T}\}$ with a completion $\{H'(\cdot, \cdot)'\}$ so that H is a proper subspace of H', dense in H'. Now T has continuous linear extension to H', also denoted by T, with adjoint T^* in $\{H', (\cdot, \cdot)'\}$. Then by Theorem 2, $(x, y) = ((1 - T^*T/\beta^2)x, y)'$ on H. The identity function from $\{H, (\cdot, \cdot)'\}$ to $\{H, (\cdot, \cdot)\}$ is continuous. If $\{H, (\cdot, \cdot)\}$ is complete, by Note 5 of [4], the set Z of all $\|\cdot\|'$-limits in H' of sequences in H with $\|\cdot\|$-limit 0 is closed in H' and also an algebraic complement of H in H', and if P is the orthogonal projection of H' onto $Z^\perp$ then $(\cdot, \cdot)$ is equivalent on H to $(P\cdot, P\cdot)'$. That is, the inner product $((1 - T^*T/\beta^2)x, y)'$ on H is equivalent to the inner product $(Px, Py)'$ on H, the inner product in H' of the shadow of x in $Z^\perp$ with the shadow in $Z^\perp$ of y. Another point of view, starting with a complete space $\{H', (\cdot, \cdot)'\}$, an operator T on $\{H', (\cdot, \cdot)'\}$, and a dense, proper subspace H_1 of H', and yielding a shadow inner product $((1 - T^*T)\cdot, \cdot)'$ for H_1 such that $\{H_1, ((1 - T^*T)\cdot, \cdot)'\}$ is complete, will be pursued in Example 3.

THEOREM 3. *Suppose, as in Theorem 2, that $\{H', (\cdot, \cdot)'\}$ is a complete inner product space, that H_1 is a dense subspace of H', and that T is an operator on $\{H', (\cdot, \cdot)'\}$ such that $T(H_1)$ lies in H_1. Suppose that β is a positive number and that, with T^* the adjoint of T in $\{H', (\cdot, \cdot)'\}$,* (i) *β is not less than the operator-norm for T in $\{H', (\cdot, \cdot)'\}$ and* (ii) *$1 - T^*T/\beta^2$ is a one-to-one transformation on H_1. Then for x and y in H_1 the formula $(x, y)'' = ((1 - T^*T/\beta^2)x, y)'$ defines an inner product $(\cdot, \cdot)''$ for H_1 such that if $(\cdot, \cdot)$ denotes $(\cdot, \cdot)''$ on H_1 then for x in H_1 $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2$ converges, with limit not exceeding $(\|x\|')^2$. In case $\lim_{p\to\infty} (\|(T/\beta)^p x\|') = 0$ on H_1, then on H_1 $(x, y)' = (x, y)_{\beta,T}$ and if, in addition, $\{H_1, (\cdot, \cdot)\}$ is complete, so that $(1 - T^*T/\beta^2)^{1/2}(H_1)$ is closed in H', and $H' \neq H_1$ then the restriction of T to H_1 is not continuous in $\{H_1, (\cdot, \cdot)\}$. (Despite the conven-*

tion of the introduction, here $(\cdot, \cdot)$ *is not given beforehand*).

Proof. Note that, since $1 - T^*T/\beta^2$ is a one-to-one function when restricted to H_1, $\{H_1, (\cdot, \cdot)''\}$ is isometrically isomorphic to the subspace $(1 - T^*T/\beta^2)^{1/2}(H_1)$ of $\{H', (\cdot, \cdot)'\}$. Thus, writing $(\cdot, \cdot)$ in place of $(\cdot, \cdot)''$, $\{H_1, (\cdot, \cdot)\}$ is complete if and only if $(1-T^*T/\beta^2)^{1/2}(H_1)$ is closed in H'. Suppose n is a positive integer and each of x and y is in H_1. We have

$$\tag{C} \begin{aligned} &\sum_{p=0}^{n} ((T/\beta)^p x, (T/\beta)^p y) \\ &\quad = \sum_{p=0}^{n} ((T/\beta)^p x, (T/\beta)^p y)' \\ &\quad\quad - \sum_{p=0}^{n} ((T/\beta)^{p+1} x, (T/\beta)^{p+1} y)' \\ &\quad = (x, y)' - ((T/\beta)^{n+1} x, (T/\beta)^{n+1} y)' . \end{aligned}$$

Hence, in case $\lim_{p\to\infty} \|(T/\beta)^p x\|' = 0$ on H_1 then on H_1 $(x, y)' = (x, y)_{\beta,T}$. Now for x in H_1 the number-sequence $\{\|(T/\beta)^p x\|'\}_{p=0}^{\infty}$ is non-increasing with limit α_x. By (C), for x in H_1

$$\begin{aligned} &\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2 \\ &\quad = (\|x\|')^2 - (\alpha_x)^2 \leqq (\|x\|')^2 . \end{aligned}$$

Suppose $H' \neq H_1$, $(x, y)' = (x, y)_{\beta,T}$ on H_1, and $\{H_1, (\cdot, \cdot)\}$ is complete. Then, by Lemma 1, in case T on H_1 is continuous in $\{H_1, (\cdot, \cdot)\}$, $\{H_1, (\cdot, \cdot)'\}$ is complete, so that H_1 is closed in H'. Since H_1 is dense in H' and $H_1 \neq H'$, H_1 is not closed in H'. Hence, T on H_1 is not continuous in $\{H_1, (\cdot, \cdot)\}$.

EXAMPLE 3. Suppose that on l^2 $\langle f, g\rangle = \sum_{p=0}^{\infty} f_p g_p^*$ and that y is the point of l^2 such that $y_0 = 1$ and for $p > 0$ $y_p = 0$. Suppose Y is the linear span of $\{y\}$, P the orthogonal projection of l^2 onto $Y^\perp$, and T the operator on l^2 such that $T(c)$ is the sequence d, with $d_0 = \sum_{p=1}^{\infty} c_p/2^{p+1}$, $d_1 = c_0$, and for $p > 1$ $d_p = c_{p-1}/2^{2p-1}$. Now $T^*(c)$ is the sequence e such that $e_0 = c_1$ and for $p > 0$ $e_p = c_0/2^{p+1} + c_{p+1}/2^{2p+1}$ and $T^*T(c)$ the sequence f such that $f_0 = c_0$ and for $p > 0$ $f_p = [\sum_{p=1}^{\infty} c_q/2^{q+1}]/2^{p+1} + c_p/2^{4p+2}$. Hence,

$$\begin{aligned} &\langle (1 - T^*T)c, c\rangle \\ &\quad = \sum_{p=1}^{\infty} [1 - 1/2^{4p+2}]|c_p|^2 - \sum_{p=1}^{\infty} \left\{\left[\sum_{q=1}^{\infty} c_q/2^{q+1}\right] c_p^*/2^{p+1}\right\} \\ &\quad = \sum_{p=1}^{\infty} [1 - 1/2^{4p+2}]|c_p|^2 - \left|\sum_{p=1}^{\infty} c_p/2^{p+1}\right|^2 \\ &\quad \geqq (63/64) \sum_{p=1}^{\infty} |c_p|^2 - \left[\sum_{p=1}^{\infty} |c_p|^2\right]\left[\sum_{p=1}^{\infty} 1/2^{2p+2}\right] \end{aligned}$$

$$\geqq (1/2) \sum_{p=1}^{\infty} |c_p|^2 .$$

By the above inequality,

$$\text{(D)} \qquad \langle Pc, Pc\rangle \geqq \langle (1 - T^*T)c, c\rangle \geqq (1/2)\langle Pc, Pc\rangle .$$

Since $\langle c, c\rangle - \langle Tc, Tc\rangle \geqq 0$ on l^2, the operator-norm for T does not exceed 1. However, $T^2(c) = g$, where $g_0 = c_0/4 + \sum_{p=2}^{\infty} (c_{p-1})/2^{3p}$, $g_1 = \sum_{p=1}^{\infty} c_p/2^{p+1}$, $g_2 = c_0/8$, and for $p > 2$ $g_p = (c_{p-2})/2^{4p-4}$. Computation reveals that the operator-norm for T^2 does not exceed 1/2. Hence, $\lim_{p\to\infty} \langle T^p c, T^p c\rangle$ is 0 on l^2. Note that $T(l^2) \cap Y$ is $\{0\}$. Also, with z the l^2-sequence such that for $p \geqq 0$ z_p is the sequence w with $w_q = 2^{p+1}$ or 0 accordingly as $q = p$ or not, Tz has limit y in l^2. Hence, y is in $\overline{T(l^2)}$. Since $\overline{PT(l^2)}$ is $Y^{\perp}$, we conclude that $T(l^2)$ is dense in l^2.

Suppose H_1 is an algebraic complement of Y in l^2 and $T(l^2)$ is a subspace of H_1. Then the formula $(x, y)'' = \langle Px, Py\rangle$ defines an inner product for H_1 such that $\{H_1, (\cdot, \cdot)''\}$ is complete. By (D), the formula $(x, y) = \langle (1 - T^*T)x, y\rangle$ defines an inner product for H_1 equivalent to $(\cdot, \cdot)''$. Of course, with $\beta = 1$, by Theorem 3 $\langle \cdot, \cdot\rangle = (\cdot, \cdot)_{\beta,T}$ on H_1. It is of interest to note that $[(x, y)'']_{\beta,T}$ $(=\sum_{p=0}^{\infty} \langle PT^p x, PT^p y\rangle)$ is equivalent to $\langle \cdot, \cdot\rangle$ on H_1. For

$$(1/2)[\|x\|'']^2 \leqq \|x\|^2 \leqq [\|x''\|^2]$$

implies

$$(1/2)[(x, x)'']_{\beta,T} \leqq (x, x)_{\beta,T} \leqq [(x, x)'']_{\beta,T}$$

on H_1.

Note 1. An argument for most of the following, known to the author through work of MacNerney [6], may be found in [1] (Lemma, p. 316), in which it is partly attributed to Friedrichs [3]. No argument will be offered here.

Suppose $\{H_1, (\cdot, \cdot)'\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$, in the sense that H_1 lies in H and there is a positive number c such that $\|\cdot\| \leqq c\|\cdot\|'$ on H_1, that H_1 is dense in H, and that B is the adjoint of the identity function from $\{H_1, (\cdot, \cdot)'\}$ to $\{H, (\cdot, \cdot)\}$, so that B is that linear transformation from H to H_1 such that for x in H_1 and y in H $(x, y) = (x, By)'$. Suppose C is an operator on $\{H, (\cdot, \cdot)\}$. Then

(1) B is positive definite in $\{H, (\cdot, \cdot)\}$ and the operator-norm for B in $\{H, (\cdot, \cdot)\}$ does not exceed c;

(2) with $B^{1/2}$ the positive definite square-root of B in $\{H, (\cdot, \cdot)\}$

and $B^{-1/2} = (B^{1/2})^{-1}$, $H_1 = B^{1/2}(H)$ and $(\cdot, \cdot)' = (B^{-1/2}\cdot, B^{-1/2}\cdot)$ on H_1;

(3) if $C(H)$ lies in H_1 then C is continuous from $\{H, (\cdot, \cdot)\}$ to $\{H_1, (\cdot, \cdot)'\}$;

(4) if $CB = BC$, then $CB^{1/2} = B^{1/2}C$ so that $C(H_1)$ lies in H_1 and for x and y in H, with $x \neq 0$, $\|CB^{1/2}x\|'/\|B^{1/2}x\|' = \|Cx\|/\|x\|$ and $(CB^{1/2}x, B^{1/2}y)' = (Cx, y)$; hence, the operator-norm in $\{H_1, (\cdot, \cdot)'\}$ for the restriction C_1 of C to H_1 is the operator-norm for C in $\{H, (\cdot, \cdot)\}$ and if C is nonnegative in $\{H, (\cdot, \cdot)\}$ C_1 is nonnegative in $\{H_1, (\cdot, \cdot)'\}$; and (5) if $C(H)$ is dense in H and C is one-to-one the formula $(x, y)'' = (C^{-1}x, C^{-1}y)$ defines an inner product for $C(H)$ such that $\{C(H), (\cdot, \cdot)''\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$ and the adjoint of the identity function from $\{C(H), (\cdot, \cdot)''\}$ to $\{H, (\cdot, \cdot)\}$ is CC^* on H, where C^* is the adjoint of C as an operator of H into itself. Moreover, for the adjoint $C^+ \colon C(H) \to H$ of $C \colon H \to C(H)$ we have $CC^* = C^+C$ (or $C^+ = CC^*C^{-1}$).

THEOREM 4. *Suppose that H_1 is a dense subspace of H. Then in order that $(\cdot, \cdot)_1$ be such an inner product for H_1 that $\{H_1, (\cdot, \cdot)_1\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$ it is necessary and sufficient that for some operator C on $\{H, (\cdot, \cdot)\}$ and positive number d H_1 is the set of all x in H such that $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges and, if each of x and y is in H_1, $(x, y)_1 = d\sum_{p=0}^{\infty} (C^p x, C^p y)$.*

Proof. The sufficiency of the condition follows from Lemma 1. To argue necessity, let e be a number such that for x in H_1 $\|x\|^2 \leq e(\|x\|_1)^2$ and $(\cdot, \cdot)'$ be $e(\cdot, \cdot)_1$ on H_1. Then the complete inner product space $\{H_1, (\cdot, \cdot)'\}$ is continuously situated in $\{H, (\cdot, \cdot)\}$ and the operator-norm for the identity function from $\{H_1, (\cdot, \cdot)'\}$ to $\{H, (\cdot, \cdot)\}$ does not exceed 1. Hence, with B as in Note 1, the operator-norm for B in $\{H, (\cdot, \cdot)\}$ does not exceed 1. Suppose that C is $(1 - B)^{1/2}$ on H, so that $B = 1 - C^2$. Since $BC = CB$, by Note 1 $C(H_1)$ lies in H_1, the restriction of C to H_1 is nonnegative in $\{H_1, (\cdot, \cdot)'\}$, and the operator-norm for this restriction in $\{H_1, (\cdot, \cdot)'\}$, does not exceed 1. By Theorem 3, $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges on H_1. (Note that $\{H', (\cdot, \cdot)'\}$ in Theorem 3 is replaced by $\{H_1, (\cdot, \cdot)'\}$ here and that $T = C$, $1 - T^*T = B$, $((1 - C^2)x, y)' = (Bx, y)' = (x, y)$.) Suppose that $\{H'', (\cdot, \cdot)''\}$ is the complete inner product space of all x in H for which $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges with $(x, y)'' = \sum_{p=0}^{\infty} (C^p x, C^p y)$. Note that, since H_1 lies in H'', H'' is dense in H and $(1 - C^2)(H)$ lies in H''. Also, by Lemma 1, $C(H'')$ lies in H'' and the restriction of C to H'' is self-adjoint in H''. By Note 1, $1 - C^2$ is continuous from $\{H, (\cdot, \cdot)\}$ to $\{H'', (\cdot, \cdot)''\}$. Suppose each of x and y is in H''. Then, by Theorem 2, $(x, y) = (x, (1 - C^2)y)''$. (The $\{H', (\cdot, \cdot)'\}$ of Theorem 2 is $\{H'', (\cdot, \cdot)''\}$ now, $\beta = 1$ and $T = C$; the H_1 of Theorem 2 is H'' now.)

Suppose z is in H, x is in H'', and y is a sequence in H'' with limit z in H. Then

$$(x, z) = \lim (x, y) = \lim (x, (1 - C^2)y)'' = (x, (1 - C^2)z)'' ,$$

so that $1 - C^2$ is the adjoint of the identity function from $\{H'', (\cdot, \cdot)''\}$ to $\{H, (\cdot, \cdot)\}$. Hence, $H'' = (1 - C^2)^{1/2}(H) = H_1$ and for x and y in H_1, by Note 1,

$$\begin{aligned}(x, y)_1 &= (1/e)(x, y)' \\ &= (1/e)((1 - C^2)^{-1/2}x, (1 - C^2)^{-1/2}y) \\ &= (1/e)(x, y)'' \\ &= (1/e)\sum_{p=0}^{\infty}(C^p x, C^p y) .\end{aligned}$$

The theorem is established, taking d as $1/e$.

It may be noted that an argument for Theorem 4 could be based on a theorem, Theorem 2 of [5], of the author and Note 1. The argument given above is more closely related to the other theorems of this paper.

THEOREM 5. *Suppose that H_1 is a dense subspace of H and T is a linear transformation from H_1 to H_1. Then in order that there be an inner product $(\cdot, \cdot)_1$ for H_1 such that $\{H_1, (\cdot, \cdot)_1\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$ and T is continuous in $\{H_1, (\cdot, \cdot)_1\}$ it is necessary and sufficient that for some pair, β and γ, of positive numbers and some operator C on $\{H, (\cdot, \cdot)\}$ H_1 is the set of all x in H for which $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges and for x in H_1 $\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2 \leqq \gamma \sum_{p=0}^{\infty} \|C^p x\|^2$.*

Proof. To argue necessity, suppose b is the operator-norm for T in $\{H_1, (\cdot, \cdot)_1\}$ and $\beta = 2b$. By Theorem 4, there is an operator C in $\{H, (\cdot, \cdot)\}$ and a positive number d such that H_1 is the set of all x in H for which $\sum_{p=0}^{\infty} \|C^p x\|^2$ converges, with limit $(1/d)(\|x\|_1)^2$. Now, with $e = (1/d)^{1/2}$, $\|x\| \leqq e\|x\|_1$ and

$$\begin{aligned}\sum_{p=0}^{\infty} \|(T/\beta)^p x\|^2 &\leqq e^2 \sum_{p=0}^{\infty} (\|(T/\beta)^p x\|_1)^2 \\ &\leqq e^2(4/3)(\|x\|_1)^2 = (4/3)\sum_{p=0}^{\infty} \|C^p x\|^2 ,\end{aligned}$$

on H_1, so that the condition follows with $\gamma = 4/3$.

To argue the sufficiency of the condition, suppose $(x, y)_1 = \sum_{p=0}^{\infty} (C^p x, C^p y)$ on H_1, so that $\{H_1, (\cdot, \cdot)_1\}$ is complete and continuously situated in $\{H, (\cdot, \cdot)\}$, and set $(x, y)_2 = \sum_{p=0}^{\infty} ((T/\beta)^p x, (T/\beta)^p y)$ on

H_1. Now T on H_1 is continuous in $\{H_1, (\cdot, \cdot)_1\}$ and $\|x\|_2 \leqq \gamma^{1/2}\|x\|_1$ on H_1. Suppose T is not continuous in $\{H_1, (\cdot, \cdot)_1\}$. Then, by the Closed Graph theorem, there is an H_1-sequence x with limit 0 in $\{H_1, (\cdot, \cdot)_1\}$ such that Tx has limit $y \neq 0$ in $\{H_1, (\cdot, \cdot)_1\}$. Since $\|z\|_2 \leqq \gamma^{1/2}\|z\|_1$ on H_1, x has limit 0, and Tx limit y, in $\{H_1, (\cdot, \cdot)_2\}$. But Tx has limit 0 in $\{H_1, (\cdot, \cdot)_2\}$. Thus, $y = 0$. This is a contradiction.

EXAMPLE. There is a dense subspace H_1 of H and a linear transformation T on H_1 such that $T(H_1)$ lies in H_1, the formula $(x, y)_1 = \sum_{p=0}^{\infty}(T^p x, T^p y)$ defines on H_1 an inner product such that $\{H_1, (\cdot, \cdot)_1\}$ is complete, and yet T is not a closed operator in $\{H, (\cdot, \cdot)\}$.

Suppose C is an operator on H such that the set H_2 of all x in H for which $\sum_{p=0}^{\infty}\|C^p x\|^2$ converges is a dense proper subspace of H. Suppose y is not in H_2, H_1 is the linear span of $\{y\}$ and H_2, and ϕ is the algebraic projection of H_1 onto H_2 with kernel the linear span Y of $\{y\}$. Suppose T is $C\phi + 1/2(1 - \phi)$ on H_1. Since $C(H_2)$ lies in H_2, T^p is C^p on H_2. Since the set of all x for which $\sum_{p=0}^{\infty}\|T^p x\|^2$ converges is a linear space including both Y and H_2, this set is H_1. Define $(x, y)_1$ to be $\sum_{p=0}^{\infty}(T^p x, T^p y)$ on H_1. Then H_2 is a complete subspace of $\{H_1, (\cdot, \cdot)_1\}$. Since Y is one-dimensional, $\{H_1, (\cdot, \cdot)_1\}$ is complete. Now, since y is not in H_2, $Cy \neq (1/2)y$ so that T does not lie in C. Yet the closure of T in $H \times H$ includes C. Hence, the closure of T in $H \times H$ is not a function.

REFERENCES

1. R. Adams, N. Aronszajn, and M. S. Hanna, *Theory of Bessel potentials, III*, Ann. Inst. Fourier (Grenoble), **19** (1969), fasc. **2** (1970), 279-338.

2. P. A. Fillmore and J. P. Williams, *On operator ranges*, Advances in Math., **7** (1971), 254-281.

3. K. O. Friedrichs, *Spektraltheorie halbbeschränkter Operatoren und Anwendung auf die Spektralzerlegung von Differentialoperatoren*, Math. Ann., **109** (1934), 465-487, 685-713. Errata: Ibid., **110** (1935), 777-779.

4. George O. Golightly, *Graph-dense linear transformations*, Pacific J. Math., **82**, No. 2 (1979), 371-377.

5. ———, *A characterization of the range of a bounded linear transformation in Hilbert space*, Proc. Amer. Math. Soc., **79**, no. 4 (1980), 591-592.

6. J. S. Mac Nerney, *Continuous embeddings of Hilbert spaces*, Rend. Circ. Mat. Palermo, (2) **19** (1970), 109-112.

7. ———, *Dense embeddings of Hilbert spaces*, Proc. Amer. Math. Soc., **24**, No. 1 (1970), 92-94.

8. J. D. Maitland-Wright, *All operators on a Hilbert space are bounded*, Bull. Amer. Math. Soc., **79** (1973), 1247-1250.

AUBURN UNIVERSITY
AL 36849

b

Mathematical parers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd unmbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966, Regular subscription rate: $114.00 a year (6 Vol., 12 issues). Special rate: $57.00 a year to individual members of supporting institution.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address shoud be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 103, No. 2 April, 1982