UNITARY ANALOGS OF GENERALIZED RAMANUJAN SUMS

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The multiplicative properties of a certain type of generalized Ramanujan sum have been studied by several authors. In this paper we investigate the multiplicative properties of the unitary analog of this function.

Cohen [2] defined the unitary product of two arithmetic functions $f$ and $g$, by

$$f \times g(n) = \sum_{d \mid n} f(d)g(n/d),$$

where $d \mid n$ indicates that $d$ is a unitary divisor of $n$, i.e., $d \mid n$ and $(d, n/d) = 1$. He also defined a unitary analog of Ramanujan’s sum $c_k(n)$ by

$$c_k^*(n) = \sum_{(j, k) = 1 \atop j \equiv i \pmod{k}} \exp \left( 2\pi i jn/k \right)$$

where $(j, k)_*$ denotes the largest divisor of $j$ which is a unitary divisor of $k$. Cohen then demonstrated that, paralleling the Dirichlet product result, we have

$$c_k^*(n) = \sum_{d \mid (n, k)_*} d \mu^*(k/d).$$

Here $\mu^*$ is the unitary Möbius function and $\mu^* = 1^{-1}$ with respect to the unitary product ($1(n) = 1$ for all $n$). The function $\mu^*$ is multiplicative and $\mu^*(1) = 1$, $\mu^*(p^k) = -1$ for all primes $p$ and positive integers $k$. It is easy to see that (3) may be rewritten

$$c_k^*(n) = \sum_{d \mid (n, k)_*} d \mu^*(k/d).$$

Cohen also defined $\phi^*(n) = c_k^*(0)$, and paralleling the Dirichlet case showed that $\phi^*(n)$ counts the number of integers unitarily semi-prime to $n$, i.e., the number of integers $k$ such that $(k, n)_* = 1$. He also showed that $\phi^*(n) = i \times \mu^*(n)$, where $i$ is the identity function, which is also analogous to the well known Dirichlet result.

Anderson and Apostol [1] defined a more general Ramanujan type sum by

$$s_k(n) = \sum_{d \mid (n, k)} f(d)g(k/d),$$

and studied the multiplicative properties of this new function. In
this paper we study the multiplicative properties of the unitary analog of \( s_k(n) \), defined as follows.

**Definition 1.** For arithmetic functions \( f \) and \( g \), let

\[
 s^*_k(n) = \sum_{d \mid (n, k^*)} f(d)g(k/d).
\]

The proof of the following lemma is straightforward.

**Lemma 2.** If \( (a, k) = (b, m) = 1 \) then \((ab, mk)_* = (a, m)_*(b, k)_*\) and \((a, m)_*(b, k)_* = 1\).

**Theorem 3.** If \( f \) and \( g \) are multiplicative then \( s^*_k(n) \) has the following multiplicative properties:

(i) \( s^*_k(ab) = s^*_k(a)s^*_k(b) \) whenever \((a, k) = (b, m) = 1\)

(ii) \( s^*_k(ab) = s^*_k(a) \) whenever \((b, m) = 1\)

(iii) \( s^*_k(a) = s^*_m(a)g(k) \) whenever \((a, k) = 1\).

**Proof.** Suppose \((a, k) = (b, m) = 1\). Then

\[
 s^*_k(ab) = \sum_{d \mid (ab, mk)_*} f(d)g(mk/d) = \sum_{d \mid (a, m)_*(b, k)_*} f(d)g(mk/d),
\]

by Lemma 2,

\[
 = \sum_{d_j \mid (n, m)_*} f(d_1)g(m/d_1) \sum_{d_2 \mid (b, k)_*} f(d_2)g(k/d_2), \quad \text{since} \quad (d_1, d_2) = 1
\]

\[
 = s^*_m(a)s^*_k(b).
\]

This proves (i). Now let \( k = 1\).

\[
 s^*_k(ab) = s^*_m(a)s^*_k(b) = s^*_m(a) \quad \text{which is (ii).}
\]

Not let \( b = 1 \) in (i)

\[
 s^*_k(a) = s^*_m(a)s^*_k(a) = s^*_m(a)g(k).
\]

The function \( s^*_k(n) \) is multiplicative in another sense.

**Theorem 4.** If \( f \) and \( g \) are multiplicative then \( s^*_k(n) \) is multiplicative in \( k \) for each fixed \( n \).

**Proof.** Suppose \((k, m) = 1\) and \( n \) is fixed. Then

\[
 s^*_k(n)s^*_m(n) = \sum_{d_1 \mid (n, k)_*} f(d_1)g(k/d_1) \sum_{d_2 \mid (n, m)_*} f(d_2)g(m/d_2)
\]

\[
 = \sum_{d_1 \mid (n, k)_*} \sum_{d_2 \mid (n, m)_*} f(d_1)d_2g(km/d_1d_2) = \sum_{d \mid (n, km)_*} f(d)g(km/d)
\]

\[
 = s^*_m(n).
\]

The case \( s^*_k(n) = c^*_k(n) \) was proved by Cohen [2].

**Theorem 5.** If \( f \) and \( g \) are multiplicative, and \( g(n) = \pm 1 \) for all \( n \), then for fixed \( k \) the function \( g(k)s^*_k(n) \) is multiplicative in the variable \( n \).
Proof. Choose \( (n, m) = 1 \) and fix \( k \). Now

\[
g(k)s^*_*(n)g(k)s^*_*(m) = s^*_*(n)s^*_*(m), \quad \text{since} \quad g^*(k) = 1.
\]

Since both sides of the equality

\[
s^*_*(n)s^*_*(m) = g(k)s^*_*(nm)
\]

are multiplicative in \( k \) (by the previous theorem), it is enough to prove the same when \( k \) is a prime power.

\[
s^*_*(n)s^*_*(m) = \sum_{d \mid (n, k)_*} f(d_1)g(k/d_1) \sum_{d_2 \mid (m, k)_*} f(d_2)g(k/d_2)
\]

but since \( k \) is a prime power either \( d_1 \) or \( d_2 \) is 1, so \( g(k/d_1)g(k/d_2) = g(k)g(k/d_1d_2) \) and

\[
s^*_*(n)s^*_*(m) = \sum_{d_1d_2 \mid (nm, k)_*} f(d_1d_2)g(k/d_1d_2)
\]

\[
= g(k) \sum_{d \mid (nm, k)_*} f(d)g(k/d) = g(k)s^*_*(nm).
\]

In particular,

**Corollary 6.** For fixed \( k \), the function \( \mu^*(k)c^*_*(n) \) is multiplicative in the variable \( n \).

The Dirichlet analog of Corollary 6 was proved by Donovan and Rearick [4].

Theorem 4 is also useful in the proof of another unitary version of a Dirichlet result [1]. A somewhat weaker theorem of this type was proved by V. Sitah Ramaiah [6].

**Theorem 7.** Suppose \( g \) and \( f \) are multiplicative and \( (n, k)_* = 0 \) for all \( n \). Then

\[
(5) \quad s^*_*(n) = \frac{F(k)g(N)}{F(N)}
\]

where \( N = k/(n, k)_* \).

**Proof.** After Theorem 4 it is sufficient to show that the right hand side of (5) is multiplicative in \( k \) and demonstrate the equality when \( k \) is a prime power. But \( F \) is multiplicative [2, Theorem 2.1]. Using this and the fact that \( (n, k)_*(n, m)_* = (n, km)_* \) if \( (k, m) = 1 \), it is easy to see that the right hand side of (5) is indeed multiplicative. So without loss of generality we may assume \( k = p^r = P \), a prime power. If \( P \mid n \), then \( (n, P)_* = 1 \) and \( F(k)g(N)/F(N) \) reduces to \( g(P) \). If \( P \nmid n \) then \( (n, P)_* = P \) and the right hand side
of (5) reduces to \( f(1)g(P) + f(P)g(1) \). In either case the value obtained is the value of \( s^*_r(n) \), thus establishing the theorem.

**Corollary 8.** \( c^*_r(n) = \phi^*(k)\mu^*(k/(n, k^*))/\phi^*(k/(n, k^*)) \).

**Proof.** As stated earlier \( \phi^*(k) = i \times \mu^*(k) \).

This particular special case of Theorem 7 has been proved by several authors [3], [5], and [7].

**References**


Received February 27, 1980 and in revised form August 24, 1981.

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