

Pacific Journal of Mathematics

UNITARY ANALOGS OF GENERALIZED RAMANUJAN SUMS

KENNETH RICHARD JOHNSON

UNITARY ANALOGS OF GENERALIZED RAMANUJAN SUMS

KENNETH R. JOHNSON

The multiplicative properties of a certain type of generalized Ramanujan sum have been studied by several authors. In this paper we investigate the multiplicative properties of the unitary analog of this function.

Cohen [2] defined the unitary product of two arithmetic functions f and g , by

$$(1) \quad f \times g(n) = \sum_{d||n} f(d)g(n/d),$$

where $d||n$ indicates that d is a unitary divisor of n , i.e., $d|n$ and $(d, n/d) = 1$. He also defined a unitary analog of Ramanujan's sum $c_k(n)$ by

$$(2) \quad c_k^*(n) = \sum_{\substack{(j,k)_*=1 \\ j \bmod k}} \exp(2\pi i j n / k)$$

where $(j, k)_*$ denotes the largest divisor of j which is a unitary divisor of k . Cohen then demonstrated that, paralleling the Dirichlet product result, we have

$$(3) \quad c_k^*(n) = \sum_{\substack{d|n \\ d||k}} d\mu^*(k/d).$$

Here μ^* is the unitary Möbius function and $\mu^* = 1^{-1}$ with respect to the unitary product ($1(n) = 1$ for all n). The function μ^* is multiplicative and $\mu^*(1) = 1$, $\mu^*(p^k) = -1$ for all primes p and positive integers k . It is easy to see that (3) may be rewritten

$$(4) \quad c_k^*(n) = \sum_{d|(n,k)_*} d\mu^*(k/d).$$

Cohen also defined $\phi^*(n) = c_n^*(0)$, and paralleling the Dirichlet case showed that $\phi^*(n)$ counts the number of integers unitarily semi-prime to n , i.e., the number of integers k such that $(k, n)_* = 1$. He also showed that $\phi^*(n) = i \times \mu^*(n)$, where i is the identity function, which is also analogous to the well known Dirichlet result.

Anderson and Apostol [1] defined a more general Ramanujan type sum by

$$s_k(n) = \sum_{d|(n,k)} f(d)g(k/d),$$

and studied the multiplicative properties of this new function. In

this paper we study the multiplicative properties of the unitary analog of $s_k(n)$, defined as follows.

DEFINITION 1. For arithmetic functions f and g , let

$$s_k^*(n) = \sum_{d \mid \mid (n, k)_*} f(d)g(k/d) .$$

The proof of the following lemma is straightforward.

LEMMA 2. If $(a, k) = (b, m) = 1$ then $(ab, mk)_* = (a, m)_*(b, k)_*$ and $((a, m)_*, (b, k)_*) = 1$.

THEOREM 3. If f and g are multiplicative then $s_k^*(n)$ has the following multiplicative properties:

- (i) $s_{mk}^*(ab) = s_m^*(a)s_k^*(b)$ whenever $(a, k) = (b, m) = 1$
- (ii) $s_m^*(ab) = s_m^*(a)$ whenever $(b, m) = 1$
- (iii) $s_{mk}^*(a) = s_m^*(a)g(k)$ whenever $(a, k) = 1$.

Proof. Suppose $(a, k) = (b, m) = 1$. Then

$$\begin{aligned} s_{mk}^*(ab) &= \sum_{d \mid \mid (ab, mk)_*} f(d)g(mk/d) = \sum_{d \mid \mid (a, m)_*(b, k)_*} f(d)g(mk/d), \text{ by Lemma 2,} \\ &= \sum_{d_1 \mid \mid (a, m)_*} f(d_1)g(m/d_1) \sum_{d_2 \mid \mid (b, k)_*} f(d_2)g(k/d_2), \text{ since } (d_1, d_2) = 1 \\ &= s_m^*(a)s_k^*(b). \end{aligned}$$

This proves (i). Now let $k = 1$.

$$\begin{aligned} s_m^*(ab) &= s_m^*(a)s_1^*(b) = s_m^*(a) \text{ which is (ii). Not let } b = 1 \text{ in (i)} \\ s_{mk}^*(a) &= s_m^*(a)s_k^*(a) = s_m^*(a)g(k) . \end{aligned}$$

The function $s_k^*(n)$ is multiplicative in another sense.

THEOREM 4. If f and g are multiplicative then $s_k^*(n)$ is multiplicative in k for each fixed n .

Proof. Suppose $(k, m) = 1$ and n is fixed. Then

$$\begin{aligned} s_k^*(n)s_m^*(n) &= \sum_{d_1 \mid \mid (n, k)_*} f(d_1)g(k/d_1) \sum_{d_2 \mid \mid (n, m)_*} f(d_2)g(m/d_2) \\ &= \sum_{d_1 \mid \mid (n, k)_*} \sum_{d_2 \mid \mid (n, m)_*} f(d_1 d_2)g(km/d_1 d_2) = \sum_{d \mid \mid (n, km)_*} f(d)g(km/d) \\ &= s_{km}^*(n) . \end{aligned}$$

The case $s_k^*(n) = c_k^*(n)$ was proved by Cohen [2].

THEOREM 5. If f and g are multiplicative, and $g(n) = \pm 1$ for all n , then for fixed k the function $g(k)s_k^*(n)$ is multiplicative in the variable n .

Proof. Choose $(n, m) = 1$ and fix k . Now

$$g(k)s_k^*(n)g(k)s_k^*(m) = s_k^*(n)s_k^*(m), \text{ since } g^2(k) = 1.$$

Since both sides of the equality

$$s_k^*(n)s_k^*(m) = g(k)s_k^*(nm)$$

are multiplicative in k (by the previous theorem), it is enough to prove the same when k is a prime power.

$$s_k^*(n)s_k^*(m) = \sum_{d_1 | (n, k)_*} f(d_1)g(k/d_1) \sum_{d_2 | (m, k)_*} f(d_2)g(k/d_2)$$

but since k is a prime power either d_1 or d_2 is 1, so $g(k/d_1)g(k/d_2) = g(k)g(k/d_1d_2)$ and

$$\begin{aligned} s_k^*(n)s_k^*(m) &= \sum_{d_1d_2 | (nm, k)_*} f(d_1d_2)g(k)g(k/d_1d_2) \\ &= g(k) \sum_{d | (nm, k)_*} f(d)g(k/d) = g(k)s_k^*(nm). \end{aligned}$$

In particular,

COROLLARY 6. *For fixed k , the function $\mu^*(k)c_k^*(n)$ is multiplicative in the variable n .*

The Dirichlet analog of Corollary 6 was proved by Donovan and Rearick [4].

Theorem 4 is also useful in the proof of another unitary version of a Dirichlet result [1]. A somewhat weaker theorem of this type was proved by V. Sitah Ramaiah [6].

THEOREM 7. *Suppose g and f are multiplicative and $F(n) = f \times g(n) \neq 0$ for all n . Then*

$$(5) \quad s_k^*(n) = \frac{F(k)g(N)}{F(N)}$$

where $N = k/(n, k)_*$.

Proof. After Theorem 4 it is sufficient to show that the right hand side of (5) is multiplicative in k and demonstrate the equality when k is a prime power. But F is multiplicative [2, Theorem 2.1]. Using this and the fact that $(n, k)_*(n, m)_* = (n, km)_*$ if $(k, m) = 1$, it is easy to see that the right hand side of (5) is indeed multiplicative. So without loss of generality we may assume $k = p^v = P$, a prime power. If $P \nmid n$, then $(n, P)_* = 1$ and $F(k)g(N)/F(N)$ reduces to $g(P)$. If $P | n$ then $(n, P)_* = P$ and the right hand side

of (5) reduces to $f(1)g(P) + f(P)g(1)$. In either case the value obtained is the value of $s_p^*(n)$, thus establishing the theorem.

COROLLARY 8. $c_p^*(n) = \phi^*(k)\mu^*(k/(n, k)_*)/\phi^*(k/(n, k)_*)$.

Proof. As stated earlier $\phi^*(k) = i \times \mu^*(k)$.

This particular special case of Theorem 7 has been proved by several authors [3], [5], and [7].

REFERENCES

1. D. R. Anderson and T. M. Apostol, *The evaluation of Ramanujan's sum and generalizations*, Duke Math. J., **20** (1952), 211-216.
2. E. Cohen, *Arithmetical functions associated with the unitary divisors of an integer*, Math. Zeit., **74** (1960), 66-80.
3. ———, *Unitary functions (mod r), II*, Publications Mathematicae, **9** (1962), 94-104.
4. G. S. Donovan and D. Rearick, *On Ramanujan's sum*, Det Kgl. Norske Vidensk Selsk. Fordhandler, **39** (1966), 1-2.
5. P. J. McCarthy, *Some more remarks on arithmetrical identities*, Portugal Math., **21** (1962), 45-57.
6. V. Sita Ramaiah, *Arithmetical sums in regular convolutions*, J. Reine Angew. Math., **304** (1976), 265-283.
7. D. Suryanarayana, *A property of the unitary analogue of Ramanujan's sum*, Elem. Math., **25** (1970), 114.

Received February 27, 1980 and in revised form August 24, 1981.

NORTH DAKOTA STATE UNIVERSITY
FARGO, ND 58105

PACIFIC JOURNAL OF MATHEMATICS

EDITORS

DONALD BABBITT (Managing Editor)

University of California
Los Angeles, CA 90024

HUGO ROSSI

University of Utah
Salt Lake City, UT 84112

C. C. MOORE and ARTHUR AGUS

University of California
Berkeley, CA 94720

J. DUGUNDJI

Department of Mathematics
University of Southern California
Los Angeles, CA 90007

R. FINN and J. MILGRAM

Stanford University
Stanford, CA 94305

ASSOCIATE EDITORS

R. ARENS

E. F. BECKENBACH

B. H. NEUMANN

F. WOLF

K. YOSHIDA

SUPPORTING INSTITUTIONS

UNIVERSITY OF ARIZONA

UNIVERSITY OF BRITISH COLUMBIA

CALIFORNIA INSTITUTE OF TECHNOLOGY

UNIVERSITY OF CALIFORNIA

MONTANA STATE UNIVERSITY

UNIVERSITY OF NEVADA, RENO

NEW MEXICO STATE UNIVERSITY

OREGON STATE UNIVERSITY

UNIVERSITY OF OREGON

UNIVERSITY OF SOUTHERN CALIFORNIA

STANFORD UNIVERSITY

UNIVERSITY OF AAWAII

UNIVERSITY OF TOKYO

UNIVERSITY OF UTAH

WASHINGTON STATE UNIVERSITY

UNIVERSITY OF WASHINGTON

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies,

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966, Regular subscription rate: \$114.00 a year (6 Vol., 12 issues). Special rate: \$57.00 a year to individual members of supporting institution.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Copyright © 1982 by Pacific Journal of Mathematics
Manufactured and first issued in Japan

Pacific Journal of Mathematics

Vol. 103, No. 2

April, 1982

Alberto Alesina and Leonede De Michele , A dichotomy for a class of positive definite functions	251
Kahtan Alzubaidy , Rank ₂ p -groups, $p > 3$, and Chern classes	259
James Arney and Edward A. Bender , Random mappings with constraints on coalescence and number of origins	269
Bruce C. Berndt , An arithmetic Poisson formula	295
Julius Rubin Blum and J. I. Reich , Pointwise ergodic theorems in l.c.a. groups	301
Jonathan Borwein , A note on ε -subgradients and maximal monotonicity	307
Andrew Michael Brunner, Edward James Mayland, Jr. and Jonathan Simon , Knot groups in S^4 with nontrivial homology	315
Luis A. Caffarelli, Avner Friedman and Alessandro Torelli , The two-obstacle problem for the biharmonic operator	325
Aleksander Całka , On local isometries of finitely compact metric spaces	337
William S. Cohn , Carleson measures for functions orthogonal to invariant subspaces	347
Roger Fenn and Denis Karmen Sjerve , Duality and cohomology for one-relator groups	365
Gen Hua Shi , On the least number of fixed points for infinite complexes	377
George Golightly , Shadow and inverse-shadow inner products for a class of linear transformations	389
Joachim Georg Hartung , An extension of Sion's minimax theorem with an application to a method for constrained games	401
Vikram Jha and Michael Joseph Kallaher , On the Lorimer-Rahilly and Johnson-Walker translation planes	409
Kenneth Richard Johnson , Unitary analogs of generalized Ramanujan sums	429
Peter Dexter Johnson, Jr. and R. N. Mohapatra , Best possible results in a class of inequalities	433
Dieter Jungnickel and Sharad S. Sane , On extensions of nets	437
Johan Henricus Bernardus Kemperman and Morris Skibinsky , On the characterization of an interesting property of the arcsin distribution	457
Karl Andrew Kosler , On hereditary rings and Noetherian V -rings	467
William A. Lampe , Congruence lattices of algebras of fixed similarity type. II	475
M. N. Mishra, N. N. Nayak and Swadeenananda Pattanayak , Strong result for real zeros of random polynomials	509
Sidney Allen Morris and Peter Robert Nickolas , Locally invariant topologies on free groups	523
Richard Cole Penney , A Fourier transform theorem on nilmanifolds and nil-theta functions	539
Andrei Shkalikov , Estimates of meromorphic functions and summability theorems	569
László Székelyhidi , Note on exponential polynomials	583
William Thomas Watkins , Homeomorphic classification of certain inverse limit spaces with open bonding maps	589
David G. Wright , Countable decompositions of E^n	603
Takayuki Kawada , Correction to: "Sample functions of Pólya processes"	611
Z. A. Chanturia , Errata: "On the absolute convergence of Fourier series of the classes $H^\omega \cap V[v]$ "	611