ON EXTENSIONS OF NETS

DIETER JUNGNICKEL AND SHARAD S. SANE

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(s, r, μ)-nets are generalizations of the well-known Bruck nets; here any two nonparallel blocks intersect in μ points, any parallel class consists of s blocks and there are r parallel classes. We generalize the notion of transversals from the Bruck nets to the case of arbitrary μ. This notion is used to study extensions of a given net. We call a net step-t-extendable iff t new parallel classes can be adjoined. It is known that a symmetric (s, μ)-net (i.e., an $(s, s\mu, \mu)$-net whose dual is likewise an $(s, s\mu, \mu)$-net) is step-1-extendable; we show that it is step-2-extendable if and only if s divides μ and step-t-extendable (for $t \geq 3$) if and only if there exists an $(s, t, \mu/s)$-net. We then give an alternative, matrix-free proof for the results of Shrikhande and Bhagwandas on the completion of (s, r, μ)-nets with deficiency 1 or 2. We also construct an infinite series of $(4, r, \mu)$-nets of deficiency 2 that cannot be completed. We discuss a conjecture that would have interesting consequences for the possible parameters of affine 2-designs.

1. **Introduction.** An (s, r, μ)-net is an affine resolvable 1-design with r parallel classes of s blocks each such that any two nonparallel blocks intersect in μ points. The well-known Bruck nets are the case $\mu = 1$. These structures have been introduced by Drake and Jungnickel [8] but have in fact been studied for a long time either in the dual setting (called "transversal designs", see e.g., Hanani [9]) or in the guise of orthogonal arrays. It is well-known that r is bounded above by $(s^2\mu - 1)/(s - 1)$ with equality if and only if the net is in fact an (affine) 2-design. Such nets are called complete: a necessary condition here is obviously that $s - 1$ divides $\mu - 1$. If $s - 1$ does not divide $\mu - 1$, then there is a better bound due to Bose and Bush [1]. In this paper, we are concerned with the possibility of extending nets by adjoining new parallel classes of blocks. It turns out that a generalization of the notion of a transversal (well-known for Bruck nets) is helpful in this connection. Another useful tool (though only for nets with a large number of parallel classes) is the following: In any $(s, s\mu + t, \mu)$-net, the minimum number of blocks joining two points is t; and the joining number t induces an equivalence relation on the point set. This generalizes a result of Hine and Mavron [10] who had considered the case $t = 0$. These basic results are obtained in § 2 after reviewing the definitions and known facts.

In § 3, we study extensions of symmetric (s, μ)-nets, i.e., of (s, r, μ)-nets whose duals are likewise nets with the same parameters; so in particular $r = s\mu$ in this case. It is known that these are characterized among all $(s, s\mu, \mu)$-nets as those where the equivalence classes (as obtained by the result mentioned above) have all cardinality s. Symmetric nets have been studied quite extensively recently (see e.g., [7], [10], [11] and [12]) and are also related to generalized Hadamard matrices and to relative difference sets. Mavron [14] gives a good survey on these structures (called "hypernets" there). It is well-known (and trivial to see) that any symmetric net admits a step-1-extension (i.e., one may adjoin one parallel class). We show that a symmetric net admits a step-2-extension iff s divides μ and a step-t-extension $(t \geqq 3)$ iff there exists an $(s, t, \mu/s)$-net. This generalizes a result of Hine and Mavron [10] who have investigated the possibility of extending a symmetric net to a complete net.

In § 4, we consider values of s and μ for which a complete net could exist, i.e., $s - 1$ divides $\mu - 1$. Here an (s, r, μ)-net is said to be of deficiency $d = (s^2\mu - 1)/(s - 1) - r$. We then give a geometric matrix-free proof of the results of Shrikhande and Bhagwandas [19] about the completion of nets of deficiency 1 or 2 (there proved in the language of orthogonal arrays) and construct an infinite series of $(4, r, \mu)$-nets of deficiency 2 that cannot be completed; to our knowledge, these are the first known examples with $\mu \neq 1$. We also construct various other series of (s, r, μ)-nets for large values of r that are not extendable. Section 5 discusses a conjecture that would have interesting consequences on the parameters of affine 2-designs. We use the language and notations of Dembowski [5]; in particular, points are denoted by lower case and blocks by upper case letters. Also, $[x, y]$ is the number of blocks joining the points x and y and $[x]$ is the number of blocks through x; $[X, Y]$ and $[X]$ are defined dually. All incidence structures considered will be finite and there will be no repeated blocks; hence blocks may be considered to be point sets. The *line* through two points x, y $(x \neq y)$ is the intersection of all blocks through x and y. We call blocks X, Y with $X = Y$ or $[X, Y] = 0$ *parallel* and write $X \| Y; x \| y$ is defined dually for points.

2. Definitions and basic results.

DEFINITION 2.1 (Drake and Jungnickel [8]). An (s, r, μ)-net is an incidence structure satisfying

(2.1) ‖ is an equivalence relation on the set of blocks such that each parallel class partitions the point set;

(2.2) Any two nonparallel blocks intersect in μ points;

(2.3) There are $r \geq 3$ parallel classes, and some parallel class has precisely $s \geq 2$ blocks.

Here μ is an positive integer. A Bruck net then is just an $(s, r, 1)$-net. An (s, r, μ)-net with $r = s\mu$ will be called a *square* (s, μ)-net; a square (s, μ)-net whose dual is also a square (s, μ)-net is called *symmetric*.

We remark that the dual structure of a net is usually called a transversal design; references [9] to [12] use this setting. The use of the term "symmetric" as defined here agrees with that of [7], [11] and [12]; but Hine and Mavron [10] use "symmetric" in the sense of "square" and Mavron [14] calls symmetric nets "hypernets".

THEOREM 2.2. *An (s, r, μ)-net is a tactical configuration with parameter $v = s^2\mu$, $b = rs$, r and $k = s\mu$; also, each parallel class has precisely s blocks. Furthermore*

(2.4) $$r \leq (s^2\mu - 1)/(s - 1)$$

with equality if and only if the net is an (affine) 2-design with parameter $\lambda = (s\mu - 1)/(s - 1)$. An (s, r, μ)-net attaining the bound (2.4) is called complete; so a necessary condition for completeness is that $s - 1$ divides $\mu - 1$. As an $(s, r; \mu)$-net is the same as an affine $1 - (\mu s^2, \mu s, r)$-design with $r \geq 3$, we will extend our definition of (s, r, μ)-net to $r = 1$ and 2 by defining this also to be an affine $1 - (\mu s^2, \mu s, r)$-design.

These results have been proved independently by many authors; we refer the reader to [8, § 5] for simple proofs. The bound (2.4) may in fact be improved if $s - 1$ does not divide $\mu - 1$; this has been done (in the language of orthogonal arrays) by Bose and Bush [1]. The following result has been proved in the dual setting in [10, Theorem 2.2].

THEOREM 2.3 (*Hine and Mavron* [10]). *Let $\sum$ be an (s, r, μ)-net and assume the existence of parallel points (i.e., of points x, y with $[x, y] = 0$). Then*

(2.5) $$r \leq s\mu$$

with equality if and only if $[x, z] = [y, z]$ for all points $z \neq x, y,$

whenever $x \,\|\, y$. *Thus in a square* (s, μ)-*net*, $\|$ *is an equivalence relation on points; the parallel classes will be called cosets.*

Theorem 2.3 may be used to characterize the symmetric (s, μ)-nets:

COROLLARY 2.4 (*Hine and Mavron* [10]). *In a square* $(s, s\mu, \mu)$-*net* Σ, *every coset consists of* s *points iff the dual* Σ' *of* Σ *is resolvable: in which case* Σ *is symmetric and the cosets of* Σ *are the parallel classes of* Σ'.

For a proof (in the dual setting), see [10, Corollaries 2.3 and 2.4]. Essentially the same result has also been obtained in [11, § 3]. The following result will be crucial for our proofs in § 4. It generalizes Theorem 2.3 and can in fact be obtained quite easily from this result.

COROLLARY 2.5. *Let* Σ *be an* $(s, s\mu + t, \mu)$-*net, where* t *is a nonnegative integer. Then*

$$(2.6) \qquad\qquad [x, y] \geqq t \quad \textit{for all points } x, y \, ;$$

and $[x, y] = t$ *implies that* $[x, z] = [y, z]$ *for all* $z \neq x, y$.

Proof. Choose any two points x, y and let $[x, y] = u$. Now remove the u parallel classes determined by the u blocks joining x and y from Σ to obtain an $(s, s\mu + t - u, \mu)$-net Σ'. As we have $x \,\|\, y$ in Σ', we conclude $s\mu + t - u \leqq s\mu$ from Corollary 2.4 which yields (2.6). If in fact $t = u$, then $[x, z] = [y, z]$ for any $z \neq x, y$ in Σ' by Theorem 2.3. But then clearly the corresponding assertion holds in Σ, as the blocks of Σ through x that are not in Σ' are precisely the μ blocks joining x and y (by (2.1)).

It is of course also possible to prove Corollary 2.5 directly by a counting argument. We have not done so, as a very similar counting argument will be presented in the proof of Theorem 4.7. We note the following important consequence of Corollary 2.5.

COROLLARY 2.6. *Let* Σ *be an* $(s, s\mu + t, \mu)$-*net where* t *is a nonnegative integer. Then the points of* Σ *may be partitioned into disjoint classes such that any two points in the same class have joining number* t, *whereas any two points in distinct classes have joining number* $> t$. *Also, the joining number of two points in distinct sets depends only on the sets to which they belong.*

Again, we will call the equivalence classes on the point set determined by the minimal joining number *cosets*. The coset of the point x will be denoted by (x); similarly, the parallel class of the block X will be denoted by (X).

We now give some examples that show that it is in general impossible to say anything about the size of the cosets (x). More examples with still other coset sizes will be obtained in § 3.

EXAMPLES 2.7.

(a) By Corollary 2.4, in any symmetric (s, μ)-net, the cosets all have size s. Known examples include the values $s = p^i$, $\mu = p^j$ (p a prime) or $\mu = 2$ (cf. e.g. [11, § 2]).

(b) There are complete nets, i.e., affine designs, with the parameters of an affine space, that are not affine spaces (see e.g., Mavron [13] for a very general construction method). In these examples, there are lines of size m with $2 \leq m < s$ (this can be seen directly, but also follows from the result of Dembowski [4], that an affine design is in fact an affine space if all lines have size $s > 2$). Removing the $(s\mu - 1)/(s - 1)$ parallel classes of blocks determined by such a line L leaves a square (s, μ)-net with a coset of size m.

(c) Take any affine design (with parameters s, μ) and remove 1 resp. 2 parallel classes; the resulting cosets will then all have size $s\mu$ resp. μ.

The last example of course exhibits facts that have to be proved if one conversely wants to extend a net of "deficiency" 1 resp. 2 to a complete net which is the topic of § 4. We conclude this section by generalizing the concept of a transversal from Bruck nets to the case of arbitrary μ.

DEFINITION 2.8. Let $\sum$ be an (s, r, μ)-net. A set T of points of $\sum$ is called a *transversal* for $\sum$ if T intersects every block of $\sum$ in precisely μ points.

Note that 2.8 implies that every transversal of an (s, r, μ)-net has precisely $s\mu$ points. We want to use transversals for extending nets. Two more definitions are needed:

DEFINITION 2.9. Let $\sum$ be an (s, r, μ)-net and $\sum'$ an (s, r', μ)-net on the same point set, where $r' > r$. $\sum$ is said to be *embeddable* into $\sum'$ if all blocks of $\sum$ are also blocks of $\sum'$; then $\sum'$ is

called an *extension* of $\sum$. $\sum$ is called *step-t-extendable*, if it can be embedded into an $(s, r + t, \mu)$-net $\sum'$. $\sum$ is called *completely extendable*, if it can be embedded into a complete net. $\sum$ is called *maximal*, if it is not extendable at all.

DEFINITION 2.10. Let $\sum$ be an (s, r, μ)-net and t a positive integer. Then a *system of ts transversals* of $\sum$ is a set $\{T_{ij} : i = 1, \cdots, t;\ j = 1, \cdots, s\}$ of ts transversals of $\sum$ satisfying

$$(2.7) \qquad |T_{ij} \cap T_{hk}| = \begin{cases} \mu & \text{if } i \neq h \\ 0 & \text{if } i = h \text{ and } j \neq k. \end{cases}$$

Then the following lemma is obvious:

LEMMA 2.11. *Let $\sum$ be an (s, r, μ)-net and t a positive integer. Then there is a one-to-one correspondence between the step-t-extensions of $\sum$ and the systems of ts transversals of $\sum$.*

3. Extensions of symmetric nets. The following observation about the transversals of a symmetric net is crucial for the study of extensions of such nets.

PROPOSITION 3.1. *Let $\sum$ be a symmetric (s, μ)-net. Then a set T of $s\mu$ points of $\sum$ is a transversal if and only if it is a union of (necessarily μ) cosets of $\sum$.*

Proof. In a symmetric net, any block meets any coset precisely once (by the dual of (2.1)); thus clearly any union of μ cosets is a transversal. Conversely, let T be a transversal and let x be any point of T; it will be sufficient to show that the number a of points of T not parallel to x is $s(\mu - 1)$, as each coset has precisely s points. Now count the number of flags (y, Y) with $x \neq y \in T$ and xIY to obtain the equation $0 \cdot (s\mu - a) + \mu a = s\mu(\mu - 1)$, i.e., $a = s(\mu - 1)$ as asserted. Here we have used the fact that any block meets T precisely μ times, that $[x] = s\mu$ and that $[x, y] = 0$ or μ in a symmetric net.

COROLLARY 3.2. *Let S, T be two distinct transversals of a symmetric (s, μ)-net and assume $|S \cap T| = q$. Then s divides q and $S \cap T$ is the union of q/s point classes.*

THEOREM 3.3. *Let $\sum$ be a symmetric (s, μ)-net with $\mu \neq 1$ and let t be an integer ≥ 3. Then every step-t-extension of $\sum$ induces an $(s, t, \mu/s)$-net on the set of cosets of $\sum$ whose blocks are the trans-*

versals used in the extension. Conversely, every $(s, t, \mu/s)$*-net can be obtained in this way (for given* $\sum$*).*

Proof. Let $\sum'$ be a step-t-extension of $\sum$; by Lemma 2.11, this gives a system $\{T_{ij}: i = 1, \cdots, t; \; j = 1, \cdots, s\}$ of ts transversals of $\sum$. By Proposition 3.1, each T_{ij} is the (disjoint) union of μ cosets of $\sum$. We now define an incidence structure $\prod$ on the set of cosets of $\sum$ (as points) with the transversals T_{ij} as blocks by

$$(3.1) \qquad\qquad (x) I T_{ij} \text{ if and only if } (x) \subset T_{ij} \,.$$

Clearly we obtain $s^2 m$ points and each block has sm points, where we put $m = \mu/s$. Because of 2.10 it is obvious that $\|$ is an equivalence relation satisfying (2.1) for $\prod$; in fact, the parallel classes are the sets $\{T_{ij}: j = 1, \cdots, s\}$ for $i = 1, \cdots, t$. By 2.10 and Corollary 3.2, nonparallel blocks intersect in m points of $\prod$. Hence $\prod$ is an (s, t, m)-net as asserted. The converse assertion is proved in [10, 2.7] and also in [14, 4.8. (ii)].

We remark that the basic idea of a net decomposing into two subnets one of which is defined on cosets as points is contained in Mavron's paper [13, 1.2], though Mavron deals only with complete nets. The following consequence of 3.3 generalizes results of [10] and [14].

COROLLARY 3.4. *Let* $\sum$ *be a symmetric* (s, μ)*-net. Then* $\sum$ *is always step-1-extendable; it is step-2-extendable iff s divides μ.* $\sum$ *is step-t-extendable for $t \geq 3$ iff there exists an* $(s, t, \mu/s)$*-net.*

The assertion on step-1-extensions is due to Hanani [9] who has in fact proved a slightly more general result in the dual setting. We now use 3.4 to construct some examples of square nets where all cosets have size 1.

PROPOSITION 3.5. *Assume the existence of a symmetric* (s, μ)*-net with $\mu \neq 1$. Then there is a square* (s, μ)*-net for which all cosets have size 1.*

Proof. Extend the symmetric (s, μ)-net by adjoining one parallel class as in 3.3. As $\mu \neq 1$, it is clear that this extended net has the same cosets as the symmetric net. Now omit one of the original parallel classes of the symmetric net; it is easily seen that still all point pairs are joined, as each block of the symmetric net meets each coset only once and as points in the same coset are joined

exactly once after the extension process. The reader might compare this with Examples 2.7. We also remark that the $(s, s\mu + 1, \mu)$-net constructed in Theorem 3.3 is in fact a divisible partial design on 3 associate classes with parameters $N_3 = s$, $N_2 = \mu$, $N_1 = s$, $\lambda_1 = 1$, $\lambda_2 = \mu + 1$ and $\lambda_3 = \mu$. See Raghavarao [15, § 8] for the parameters used; we use the shorter term "partial design" instead of "partially balanced incomplete block design".

We now mention three more corollaries to Theorem 3.3.

COROLLARY 3.6 (*Hine and Mavron* [10]). *A necessary condition for the completion of a symmetric* (s, μ)-*net with* $\mu \neq 1$ *is that* $s-1$ *divides* $\mu - 1$ *and that* s *divides* μ. *A necessary and sufficient condition is the existence of an* $(s, (s\mu - 1)/(s - 1), \mu/s)$-*net, i.e., of a complete net for* s *and* μ/s.

COROLLARY 3.7. *Let* $\sum$ *be a symmetric* (s, μ)-*net with* $\mu \neq s$. *Then the following assertions are equivalent:*
  (i) *s divides* μ;
  (ii) $\sum$ *is step-2-extendable;*
  (iii) $\sum$ *is step-7-extendable.*

Proof. (iii) trivially implies (ii), and (i) and (ii) are equivalent by Corollary 3.4. But if s divides μ, then μ/s is an integer ≥ 2 by assumption; hence by a result of Hanani [9], there exists an $(s, 7, \mu/s)$-net. Thus (i) implies (iii) by Corollary 3.4.

COROLLARY 3.8. *A symmetric* (s, s)-*net has a step-t-extension* $(t \geq 3)$ *if and only if there exists a Bruck net with parameters* $(s, t, 1)$. *In particular, there always is a step-3-extension.*

We remark, that it is possible to generalize Theorem 3.3 as follows: Assume the existence of an equivalence relation $\sim$ on the point set of the given symmetric net $\sum$ such that $x \| y$ implies $x \sim y$ and that all $\sim$-classes have the same cardinality, say ns. Then a system of ts transversals of $\sum$ consisting of complete $\sim$-classes corresponds to an $(s, t, \mu/ns)$-net. As we have no applications for this result, we have only stated the case $\sim = \|$ in Theorem 3.3.

It is worthwhile to have a name for the nets constructed in Theorem 3.3.

DEFINITION 3.9. Let $\sum$ be a symmetric (s, μ)-net and let $\sum'$ be

a step-t-extension of $\sum$ (with $t \geqq 3$). Then the net $\prod$ induced on the cosets of $\sum$ (as in the proof of Theorem 3.3) is called the *net induced by $\sum'$*.

It is then easy (though lengthy) to prove the following extension of Theorem 3.3.

THEOREM 3.10. *Let $\sum$ be a symmetric (s, μ)-net with $\mu \neq 1$ and let $\sum'$ be a step-t-extension of $\sum$ with $t \geqq 3$. Then $\sum'$ is step-u-extendable (u a positive integer) if and only if the net induced by $\sum'$ is step-u-extendable. In particular, $\sum'$ is maximal if and only if the net induced by $\sum'$ is maximal.*

COROLLARY 3.11. *Let $\sum$ be a symmetric (s, μ)-net. If s divides μ, then $\sum$ can be extended to a maximal $(s, s\mu + t, \mu)$-net if and only if there exists a maximal $(s, t, \mu/s)$-net. Here necessarily $t \geqq 7$. On the other hand, for s not a divisor of μ the maximal extensions of $\sum$ are precisely the step-1-extensions.*

Proof. Apply 3.10 together with 3.3, 3.4 and 3.7.

Corollary 3.11 may be used in the recursive construction of series of maximal nets for large values of r. In particular, we get the following improved version of [11, Theorem 3.4]:

THEOREM 3.12. *Assume the existence of the series of symmetric (s, ts^n)-nets where s does not divide t and n is any nonnegative integer. Then there exists a maximal $(s, t(s^{n+1} + s^n + \cdots + s) + 1, ts^n)$-net for all n.*

Proof. We use induction on n. The case $n = 0$ is true by Corollary 3.11 as s does not divide t. Assume the theorem is true for a given value of n; then it is also true for $n + 1$, as is seen by using the induction hypothesis and a symmetric (s, ts^{n+1})-net in Corollary 3.11.

COROLLARY 3.13. *There exists a maximal $(s, t(s^{n+1} + s^n + \cdots + s + 1, ts^n)$-net in the following cases:*
   (i) *$s = p^i$, $t = p^j$, $i > j$, p a prime;*
   (ii) *s an odd prime power, $t = 2$;*
   (iii) *both s and $t = s - 1$ prime powers;*
   (iv) *$s = 3$, $t = 4$.*
In fact, this holds whenever s is a prime power provided there exists a generalized Hadamard matrix $GH(s, t)$ (s does not divide t) in the elementary abelian group of order s.

Proof. (i) to (iv) follow from the general case by using known GH-matrices. If s is a prime power, there is a $GH(s, 1)$ in the elementary abelian group of order s; then the well-known Kronecker product construction yields $GH(s, st)$, $GH(s, s^2t)$, etc. But any $GH(s, \mu)$ gives rise to a symmetric (s, μ)-net (with a nice collineation group, in fact). Proofs of these assertions as well as a definition of GH-matrices can be found in [11]. A list of known GH-matrices is in [12, Examples 6.11].

At this point, it should be emphasized that GH-matrices are *not* equivalent to nets (except for $s=2$, see Drake [7]): a GH-matrix is a stronger concept, actually being equivalent to a net admitting a certain type of collineation group, see [11].

REMARK 3.14. In extending a symmetric (s, μ)-net as in Theorem 3.3, the coset sizes will all remain equal to s as long as $t < \mu$. As soon as $t = \mu$ this will in general not be true any longer. For instance, take an (s, s^d)-net and extend it by a symmetric (s, s^{d-1})-net (so $t = \mu = s^d$); it is easily seen that this yields coset size s^2. This may be used recursively to construct (s, r, s^d)-nets with coset size s^i, $i = 1, \cdots, d + 2$ cf. Examples 2.7. This construction method in fact yields divisible partial designs. We just state the simplest case and leave it to the reader to construct more partial designs (on more than 2 classes) by a recursive method similar to that in Theorem 3.12.

PROPOSITION 3.15 (cf. [14, 4.8. (iii)]). *Assume the existence of symmetric $(s, s\mu)$-and (s, μ)-nets. Then there exists an $(s, s^2\mu + s\mu, s\mu)$-net with coset size s^2, that is simultaneously a divisible design with parameters $\lambda_1 = s\mu$ and $A_2 = s\mu + \mu$ (i.e., points are joined λ_1 times if they are in the same coset and λ_2 times otherwise).*

Values for s and μ that may be used are in fact given by (i) to (iv) of Corollary 3.13; more values may be found in the list of GH-matrices in [12] (note that s may divide μ here).

4. **Nets with small deficiency.** In this section we are only concerned with pairs (s, μ) for which a complete net could exist, i.e., we assume that $s - 1$ divides $\mu - 1$. We will abbreviate the value of λ in a complete net by c, i.e.,

(4.1) $$c = (s\mu - 1)/(s - 1) ;$$

the maximum possible value of r (the value in a complete net) then is

(4.2) $r_{\max} = s\mu + c$

by Theorem 2.2. We recall

DEFINITION 4.1. Let $\sum$ be an (s, r, μ)-net where $s - 1$ divides $\mu - 1$. Then the *deficiency* d of $\sum$ is defined by

(4.3) $d = r_{\max} - r = s\mu + c - r$.

Before dealing with possible completions of nets of small deficiency, we will construct series of maximal nets with small d showing that there is not much hope for very strong completion theorems.

THEOREM 4.2. *Let s be a prime power and assume the existence of a maximal Bruck net of deficiency $d \neq 0$. Then there also exist maximal nets with parameters s, $\mu = s^n$ and deficiency d for all nonnegative integers n.*

Proof. We use induction on n; the case $n = 0$ is true by assumption. If the theorem holds for a particular value of n, it also holds for $n + 1$: To see this, use the induction hypothesis and a symmetric (s, s^n)-net (which exists, as s is a prime power) in Corollary 3.11. Note that this indeed leaves the value of d invariant.

Maximal Bruck nets with small deficiency have found some interest, motivated by Bruck's completion theorem [2]. We now use Theorem 4.2 together with results of Bruen [3] and the table in Drake [6, § 3] (who used the language of orthogonal Latin squares instead of nets) to obtain some infinite series of maximal nets with small deficiency.

EXAMPLES 4.3. Let n be any nonnegative integer. Then there exists a maximal (s, r, s^n)-net of deficiency d in all of the following cases:
  (i) $d = 2$, $s = 4$;
  (ii) $d = 3$, $s = 5$ or 9;
  (iii) $d = 4$, $s = 7, 8$ or 25;
  (iv) $d = 5$, $s = 7$ or 8;
  (v) s an arbitrary prime power $\equiv 1 \pmod 4$, $d = s - 2$.
To prove (v), one uses the following result of Mann which is discussed in Drake [6, § 2]: Let x be any positive integer; then there exist Latin squares of order $4x + 1$, respectively, with a subsquare of order x which do not have an orthogonal mate. To our knowledge, series (i) above yields the first examples of maximal nets with $d = 2$ and $\mu \neq 1$. In the remainder of this section, we consider

nets with deficiency 1 or 2 and give alternative matrixfree proofs
of the results of Shrikhande and Bhagwandas [19].

LEMMA 4.4. *Let x and y be two distinct points of an $(s, s\mu + t, \mu)$-net with deficiency d (so $t = c - d$). Then:*

(4.4) $$t \leq [x, y] \leq 2\mu - t + 2(t - 1)/s \;;$$

(4.5)   *If $d = 1$, then $[x, y] = c - 1$ or c ;*

(4.6)   *If $d = 2$, then $c - 2 \leq [x, y] \leq c$.*

Proof. (4.4) is the dual of Theorem 8.5.9 in Raghavarao [15];
the lower bound follows also by our Theorem 2.5. For $d = 1$, $t = c - 1$; then after some simplification, (4.4) implies $[x, y] \leq c + (s-2)/s$.
But $(s - 2)/s < 1$, which yields (4.5). (4.6) is a rather difficult
involving a lot of computations and has recently been proved by
Shrikhande and Singhi [20].

PROPOSITION 4.5 (*Shrikhande and Bhagwandas* [19]). *Any net
of deficiency 1 can be completed.*

As the proof of 4.5 is both well-known and easy, it will be
omitted.

We now consider the case of deficiency 2. In view of Example
2.7.(c) the cosets here should have size μ; we first prove this fact
using (4.6).

LEMMA 4.6. *Every coset of an $(s, s\mu + c - 2, \mu)$-net has size μ.*

Proof. Let x_i be the number of points y with $[x, y] = i$; here
$c - 2 \leq i \leq c$ by (4.6). Clearly

(4.7) $$\sum_{i=c-2}^{c} x_i = s^2\mu - 1 \;;$$

counting flags (y, Y) with $y \neq xIY$ in two ways yields

(4.8) $$\sum_{i=c-2}^{c} ix_i = (s\mu + c - 2)(s\mu - 1) \;;$$

and counting triples (y, X, Y) with x, yIX, Y and $y \neq x$, $Y \neq X$
yields

(4.9) $$\sum_{i=c-2}^{c} i(i - 1)x_i = (s\mu + c - 2)(s\mu + c - 3)(\mu - 1) .$$

Multiplying (4.7) by c and subtracting (4.8) from it yields

(4.10) $$2x_{c-2} + x_{c-1} = 2(s\mu - 1) \; ;$$

similarly, (4.8) and (4.9) yield

(4.11) $$2(c - 2)x_{c-2} + (c - 1)x_{c-1} = 2(s\mu + c - 2)(\mu - 1) \; .$$

(4.10) and (4.11) yield the assertion.

The following result is in case $s = 2$ and 3 due to Shrikhande and Bhagwandas [19]; the general case is due to Shrikhande and Singhi [20] who proved (4.6) above and then could just quote a result of [19], called Theorem B in [20]. This result has been proved using matrix arguments in [19]; we here give an alternative matrix-free proof using the geometric notions developed in this paper.

THEOREM 4.7 (*Shrikhande, Bhagwandas, Singhi*). *Any* $(s, s\mu + c - 2, \mu)$-*net* $\sum$ *with* $\mu \neq 1$ *and* $s \neq 4$ *can be completed.*

Proof. We first want to define a 2-class-association scheme (i.e., a strongly regular graph) on the s^2 cosets of $\sum$. From now on, we will denote cosets by capital letters P, Q, R. For distinct cosets P, Q, we define $[P, Q] = [x, y]$, where $x \in P$ and $y \in Q$ are arbitrary; note that this makes sense by Corollary 2.6. Call P and Q first associates (written $P \sim Q$) if $[P, Q] = c - 1$ and second associates otherwise. Now for any point x in P, the number of points y with $[x, y] = c - 1$ is precisely $2(s - 1)\mu$; this follows with $x_{c-2} = \mu - 1$ from (4.10). But whenever $[x, y] = c - 1$, then also $[x, z] = c - 1$ for all $z \in (y)$. As each coset has μ points, it follows that the number of first associates of every coset P is $2(s - 1)$.

Now let $P \sim Q$ and put $A(P): = \{R: R \sim P, R \neq Q\}$ and $A(Q): = \{S: S \sim Q, S \neq P\}$. Clearly $|A(P)| = |A(Q)| = 2(s-1) - 1$. We want to show that always $|A(P) \cap A(Q)| = s - 2$. To this end, choose $x \sim P$ and $y \sim Q$ and put $a_z = [x, z]$ and $b_z = [y, z]$ for every point $z \neq x, y$. First count all flags (z, Z) with xIZ to obtain

(4.12) $$\sum a_z = \sum b_z = (c - 1)(s\mu - 2) + (s\mu - 1)^2 \; .$$

Then count triples (z, Z, W) with $x, zIZ, W, Z \neq W$; this gives

(4.13) $$\sum a_z(a_z - 1) = \sum b_z(b_z - 1) = (c - 1)(c - 2)(\mu - 2) \\ + (s\mu - 1)(\mu - 1)(2(c - 1) + s\mu - 2) \; .$$

Finally, counting all triples (z, Z, W) with zIZ, W, xIZ, yIW (but not necessarily $Z \neq W$) gives

(4.14) $$\sum a_z b_z = (c - 1)(s\mu - 2) + (c - 1)(c - 2)(\mu - 2) \\ + 2(c - 1)(s\mu - 1)(\mu - 1) + (s\mu - 1)(s\mu - 2)\mu \; .$$

Using (4.12) to (4.14), we obtain

(4.15) $\sum (a_z - b_z)^2 = 2(\mu - 1) + 2(s - 1)\mu$.

Up to now, all summations were over all $z \neq x, y$. But if $z \in P$, then $a_z = c - 2$ and $b_z = c - 1$, i.e., $(a_z - b_z)^2 = 1$; a similar argument holds for $z \in Q$. Thus we obtain from (4.15)

(4.16) $\sum_{z \notin P \cup Q} (a_z - b_z)^2 = 2(s - 1)\mu$.

Now let $R \neq P, Q$ and put $a_R = [P, R]$ and $b_R = [Q, R]$. For $z \in R$ we have a_z and b_z equal to $c - 1$ or c, thus $(a_z - b_z)^2 = 0$ or 1; but this is independent of the particular choice of $z \in R$ by Theorem 2.5. As each R contains μ points z, we conclude from (4.16)

(4.17) $\sum_{R \neq P, Q} (a_R - b_R)^2 = 2(s - 1)$.

Here each summand of (4.17) is either 0 or 1. Also, a summand 1 occurs if and only if $[P, R] \neq [Q, R]$, i.e., iff $R \in (A(P) \backslash A(Q)) \cup (A(Q) \backslash A(P))$. But as $|A(P)| = |A(Q)|$, it is clear that $|A(P) \backslash A(Q)| = |A(Q) \backslash A(P)|$. Hence by (4.17), $2|A(P) \backslash A(Q)| = 2(s - 1)$, i.e., $|A(P) \backslash A(Q)| = s - 1$. But then clearly $|A(P) \cap A(Q)| = s - 2$, as asserted.

Thus we have proved that $\sim$ defines an association scheme with parameters $v = s^2$, $n_1 = 2(s - 1)$ and $p_{11}^1 = s - 2$ on the set of cosets of $\sum$. But a well-known result of Shrikhande [16] implies that this is an L_2-association scheme (note that $s \neq 4$ by assumption), i.e., the cosets can be labelled P_{ij} ($i, j = 1, \cdots, s$) such that $P_{ij} \sim P_{hk}$ if and only if $i = h$ or $j = k$ and $(i, j) \neq (h, k)$. We want to show that $T_1, \cdots, T_s$ with $T_j = \bigcup_i P_{ij}$ is a system of s transversals of $\sum$; then this may be used to extend $\sum$ to a net of deficiency 1 by Lemma 2.11, which may then be completed using Corollary 3.4 (alternatively, $T_1' = \bigcup_j P_{ij}$ defines another system of s transversals and $T_1, \cdots, T_s, T_1', \cdots, T_s'$ together are a system of $2s$ transversals).

Let X be any block of $\sum$, P any coset and $y \in P$ a point. Let y_i denote the number of points $\neq y$ on X that are joined to y by precisely i blocks. Then clearly $y_{c-2} + y_{c-1} + y_c = s\mu - 1$, if we assume yIX first. Counting all flags (z, Z) with $y \neq zIZ \neq X$ yields the further equation $(c - 3)y_{c-2} + (c - 2)y_{c-1} + (c - 1)y_c = (\mu - 1)(s\mu + c - 3)$. Both equations together yield

(4.18) $2y_{c-2} + y_{c-1} = 2(\mu - 1)$ for yIX .

A similar argument gives

(4.19) $2y_{c-2} + y_{c-1} = 2\mu$ for yIX .

But (4.18) and (4.19) show that in any case

(4.20) $$2|X \cap P| + \sum_{Q \sim P} |X \cap Q| = 2\mu .$$

In the following summations, all indices range from 1 to s. Put $r_{ij} = |X \cap P_{ij}|$. (4.20) yields

(4.21) $$2r_{ij} + \sum_{k \neq j} r_{ik} + \sum_{m \neq i} r_{mj} = 2\mu$$

and

(4.22) $$2r_{ij'} + \sum_{k \neq j'} r_{ik} + \sum_{m \neq i} r_{mj'} = 2\mu$$

where i, j, j' are fixed and $j \neq j'$. Then subtracting (4.22) from (4.21) gives

$$2(r_{ij} - r_{ij'}) + (r_{ij'} - r_{ij}) + \sum_{m \neq i} (r_{mj} - r_{mj'}) = 0 ,$$

i.e., $\sum_m r_{mj} = \sum_m r_{mj'}$ for all j, j'. Hence $s \sum_m r_{mj} = \sum_{m,j} r_{mj} = |X| = s\mu$ and so finally $\sum_m r_{mj} = \mu$ for all j. But this shows that every T_j meets every block X precisely μ times; hence $T_1, \cdots, T_s$ is indeed a system of s transversals of $\sum$ and the theorem is proved.

We remark that the case of deficiency 2 for $\mu = 1$ had already been settled by Shrikhande [17]. It might also be mentioned that a net of deficiency 3 can be completed provided that $s \geqq 104$ (Shrikhande and Singhi [21]); for $s = 2$ nets of deficiency $\leqq 6$ may be completed (Verheiden [22]). We also remark that counting arguments in analogy to (4.13) and (4.15) may be used to give a direct proof of Corollary 2.5 without referring to the results of Hine and Mavron (There of course one assumes that $[x, y] = t$.) We finally note the following result, which is an immediate consequence of Theorem 3.10 and the completion theorem for Bruck nets (see Bruck [2]).

PROPOSITION 4.8. *Let $\sum$ be an (s, r, s)-net of deficiency d that contains a symmetric net. Then $\sum$ can be completed provided that*

(4.23) $$s > \frac{1}{2}(d - 1)^4 + (d - 1)^3 + (d - 1)^2 + \frac{3}{2}(d - 1) = p(d - 1) .$$

Of course then, the examples of maximal nets of small deficiency obtained from Theorem 4.2 necessarily have $s \leqq p(d - 1)$.

5. **A conjecture.** In this final section we consider connections between symmetric and complete nets. We first note

PROPOSITION 5.1. *Let μ be an odd positive integer for which a*

Hadamard matrix of order 4μ exists. Then there exists a complete $(2, \mu)$-net that does not contain a symmetric net.[1]

Proof. It is well-known that the existence of a Hadamard matrix of order 4μ implies the existence of a Hadamard-3-design Σ on 4μ points, which is in our terminology a complete $(2, \mu)$-net (see e.g., Hanani [9, §2.1]). If Σ contains a symmetric $(2, \mu)$-net $\prod$, then $\prod$ can be completed; hence $s = 2$ divides μ by Corollary 3.6 a contradiction.

Proposition 5.1 provides infinitely many complete nets with $s=2$ not containing a symmetric net. But no example of this type is known for $s \neq 2$. This leads us to the following

CONJECTURE 5.2.   *Any complete net with $s \neq 2$ contains a symmetric net.*

We conclude this paper with a discussion of Conjecture 5.2. First, it would yield the following interesting result:

THEOREM 5.3.   *Assume the validity of Conjecture 5.2. Then a complete (s, μ)-net with $s \neq 2$ exists if and only if $\mu = s^d$ (d a nonnegative integer) and s is the order of an affine plane.*

Proof. First let Σ be a complete (s, μ)-net with $s \neq 2$. By assumption, Σ contains a symmetric (s, μ)-net $\prod$. As $\prod$ can be completed, we conclude that s divides μ (unless $\mu = 1$, in which case Σ is an affine plane; the assertion holds trivially then) and that there exists a complete $(s, \mu/s)$-net by Corollary 3.6. The assertion follows by induction. Conversely, let s be the order of an affine plane and let $\mu = s^d$. We use induction on d to show the existence of a complete (s, s^d)-net. The case $d = 0$ is trivial. So assume the existence of a complete (s, s^d)-net for a particular value of d. Using this net and the affine plane of order s, one may construct a symmetric (s, s^{d+1})-net by a result of Mavron [13, Theorem 1.4] which then may be completed by Corollary 3.6 (also by [13, Theorem 1.3]).

Thus Conjecture 5.2 yields a weaker version of Shrikhande's Conjecture 2 in [18], where the condition "s is the order of an affine plane" has been replaced by "s is a prime power". In view of all the work that has already been done on the question whether or not a projective plane has to have prime power order, there

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[1] Note that the value of r is determined for a complete net by s and μ; we thus define a *complete (s, μ)-net* as an $(s, (s^2\mu-1)/(s-1), \mu)$-net.

seems more hope to prove the restricted version of Shrikhande's conjecture. Conjecture 5.2 proposes a possible method of attack. We next give a reformulation of Conjecture 5.2. Now any line in a complete net with parameters (s, μ) has at most s points and the parallelism on the block set induces a parallelism on the line set (cf. Dembowski [5, 2.1.19 and 2.2.11]). Also, a *parallel class* is a set of pairwise parallel lines partitioning the point set (this need not be the case). Cf. Dembowski [5, 2.1.19 and 2.2.11]. We then have the following result that is essentially due to Mavron [13, Theorem 1.2]. His proof applies if one realizes that the cosets of a symmetric subnet of a complete net form a parallel class of lines.

PROPOSITION 5.4. *Let* $\sum$ *be a complete* (s, μ)-*net. Then the symmetric* (s, μ)-*nets contained in* $\sum$ *are in a one-to-one correspondence with those parallel classes of lines of* $\sum$ *containing only lines of* s *points each.*

COROLLARY 5.5. *Conjecture 5.2 holds if and only if each complete* (s, μ)-*net with* $s \neq 2$ *contains at least one parallel class of lines of size* s.

THEOREM 5.6. *Let* $\sum$ *be a complete* (s, μ)-*net with* $s \neq 2$ *and* $\mu \neq 1$. *Then* $\sum$ *contains at most* $(s^2\mu - 1)/(s - 1)$ *symmetric* (s, μ)-*nets and equality holds if and only if* $\sum$ *is isomorphic to the system of points and hyperplanes of an affine space.*

Proof. By Proposition 5.4, the number of symmetric (s, μ)-nets contained in $\sum$ equals the number of parallel classes of lines of size s of $\sum$. As any parallel class partitions the point set, this is just the number of lines through a given point P of $\sum$; as there are $s^2\mu - 1$ points $\neq p$ and as each of the lines under consideration has $s - 1$ points $\neq p$, we obtain the upper bound of the assertion. Also, if equality holds, then every line has to be of size s. But this implies that $\sum$ is an affine space by Dembowski's theorem [4]. Finally, if $\sum$ is an affine space, then it is easily seen that we have the desired number of parallel classes of lines (which then all have size s).

We finally remark, that Conjecture 5.2 does not say anything about the net-completion problem of § 4. To obtain results in this direction using the containment of a symmetric net one would have to strengthen Conjecture 5.2 considerably; in fact, the necessary induction process would require that any net with more than $s\mu$

parallel classes contains a symmetric net. But this is completely false; the following result gives counterexamples of very small deficiency.

PROPOSITION 5.7. *For any prime power s and for any positive integer n, there exists an (s, r, s^n)-net of deficiency $d = n + 2$ that does not contain a symmetric net.*

Proof. Consider the affine space $AG(n + 2, s)$; its points and hyperplanes form a complete (s, s^n)-net. Now choose $n + 2$ linear subspaces of dimension $n + 1$ which intersect in 0 only. Then removing the $n + 2$ parallel classes determined by these subspaces yields a net $\sum$ of deficiency $n + 2$. If $\sum$ contained a symmetric net $\prod$, then the cosets of $\prod$ would form a parallel class of lines of size s of $AG(n + 2, s)$; in particular, the coset of 0 would have to be contained in the removed $n + 2$ subspaces, which is absurd.

We state one more result in this direction.

PROPOSITION 3.8. *Any $(s, s\mu + t, \mu)$-net $\sum$ with $t < \mu$ contains at most one symmetric net.*

Proof. Assume that $\prod$ is a symmetric net contained in $\sum$. It is then easily seen that the cosets of $\prod$ are the cosets of $\sum$ and that two points in distinct cosets have $> \mu$ blocks in common in $\sum$. Any other square net $\prod'$ contained in $\sum$ is obtained by removing t parallel classes of $\sum$, at least one of which belongs to $\prod$. But then it is obvious that any two points of $\prod'$ are still joined at least once, as $\mu > t$ and so $\prod$ is not symmetric.

Note added in proof. Recent results of S. J. Dow ("Partial projective planes". Ph. D. thesis, Univ. of Florida, Gainesville, 1982) and of the first author (D. Jungnickel: "Maximal partial spreads and translation nets of small deficiency", to appear) on Bruck nets of small deficiency allow to add the following cases to Examples 4.3: (vi) $s = q^2$ a prime power, $d = q$; (vii) $s = p^2$, p an odd prime, $d = p - 1$.

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