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ON HEREDITARY RINGS AND NOETHERIAN V -RINGS

KARL ANDREW KOSLER

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The purpose of this paper is to examine conditions under which (1) a left noetherian left V -ring is left hereditary and (2) a left noetherian left V -ring is a two sided noetherian V -ring. For (1), left noetherian left V -rings which satisfy the restricted left minimum (RLM) condition are examined. The RLM condition is shown to be equivalent to $E(R)/R$ a semisimple left R -module. Consequently, hereditary is equivalent to $E(R)/R$ semisimple in the two sided case. Two sided noetherian V -rings which are critically nice are also examined. In this case, hereditary is shown to be equivalent to $E(R)/R$ injective and smooth. For (2), a theorem of Faith's concerning left QI -domains is extended to left noetherian left V -rings.

1. Introduction and definitions. A ring R is called a *left V -ring* provided every simple left R -module is injective. The definition of V -ring is due to Villamayor who has shown that a ring is a left V -ring if and only if every left ideal is the intersection of maximal left ideals. Consequently, all left V -rings are semiprime. Kaplansky has shown that a commutative ring is a V -ring if and only if it is regular. It follows that every commutative noetherian V -ring is semisimple artinian. Cozzens [4] showed that this result does not extend to the noncommutative case by producing an example of a nonartinian, two sided hereditary noetherian V -domain over which all cyclic modules are semisimple or free. This condition on cyclics forces every quasi-injective module to be injective. A ring with all its quasi-injective left R -modules injective will be called a *left QI -ring*. According to Boyle [1], a left QI -ring is left noetherian. Note that since a simple module is quasi-injective, a left QI -ring is a left V -ring.

As with Cozzens' example, all the known examples of left QI -rings are left hereditary. Cozzens and Johnson [5] produced examples of two sided noetherian V -rings which Boyle and Goodearl [3] demonstrated to be neither hereditary nor QI . Also, there is no known example of a one sided noetherian V -ring or QI -ring. In this paper, we will consider the problem of determining when a left noetherian left V -ring is left hereditary and when a left V -ring is a right V -ring.

Throughout, all rings will be associative with identity, all R -modules will be unitary left R -modules and maps between modules will be R -homomorphisms. If N is a submodule of a module M ,

then we will write $N \leq M$. In case $N \cap K \neq 0$ for all $0 \neq K \leq M$, then N is called *essential* in M and we will write $N \leq_e M$. For a module M , $E(M)$, $\text{Soc } M$ and $K \dim M$ will denote the injective hull, socle and Krull dimension of M respectively. It is assumed that the reader is familiar with the notions of singular, nonsingular and uniform modules as presented in [9]. We also use the notions of Krull dimension, critical module and smooth module as given in [11]. Throughout this paper, whenever we use the terms hereditary, noetherian, V -ring or QI -ring unqualified by "left" or "right", this will mean that the term applies to both the left and right.

In § 2, left noetherian left V -rings which satisfy the *restricted left minimum* (RLM) condition are examined. A module M satisfies the RLM condition provided M/K is artinian whenever $K \leq_e M$. It is shown that the RLM condition is equivalent to $E(R)/R$ semisimple. As a consequence, hereditary is equivalent to $E(R)/R$ semisimple in the two sided case.

The purpose of § 3 is to further investigate the role $E(R)/R$ plays in determining when a noetherian V -ring is hereditary. A necessary condition for hereditary is that R be *critically nice* (all finitely generated uniform modules are critical). In this case, R is hereditary iff $E(R)/R$ is injective and smooth.

In § 4, left-right symmetry is examined. A theorem of Faith's which states that a left QI -domain with the RLM condition is right QI iff it is right Goldie is extended to left noetherian left V -rings.

2. The restricted left minimum condition. A module M is said to satisfy the *restricted left minimum* condition, denoted RLM, provided M/K is artinian for all $K \leq_e M$. A ring R is said to satisfy the RLM condition provided the left R -module R satisfies the RLM condition. In this section, we investigate left noetherian left V -rings which satisfy the RLM condition.

LEMMA 2.1. *Let R be a semiprime ring with Krull dimension. Then R satisfies the RLM condition iff $K \dim R \leq 1$.*

Proof. By Gordon and Robson [11; 6.1], $K \dim R = \sup \{K \dim R/I + 1 \mid I \leq_e R\}$. If R satisfies the RLM condition, then $K \dim R/I \leq 0$ for all essential left ideals I . Thus, $K \dim R \leq 1$. Conversely, if $K \dim R \leq 1$, then $K \dim R/I \leq 0$ for all $I \leq_e R$. Hence, R/I is artinian for all essential left ideals I .

The RLM condition has been shown by Faith [7] to be sufficient for a left QI ring to be left hereditary. Michler and Villamayor [12] have shown that $K \dim R \leq 1$ is sufficient for a left noetherian

left V -ring R to be left hereditary. Therefore, if all cyclic singular left R -modules are semisimple, then by 2.1, $K \dim R \leq 1$ and R is left hereditary. As the next result shows, $K \dim R \leq 1$ and the RLM condition on R are equivalent to all singular (cyclic) left R -modules semisimple.

THEOREM 2.2. *If R is a left noetherian left V -ring, then the following are equivalent:*

- (1) R satisfies the RLM condition.
- (2) $K \dim R \leq 1$.
- (3) All singular left R -modules are semisimple.

Furthermore, if (1)–(3) hold, then R is left hereditary.

Proof. (1) implies (3). Let M be a singular left R -module and let $0 \neq x \in M$. Then $Rx \cong R/I$ where $I \leq_e R$ and hence, Rx is artinian. Thus, $\text{Soc } Rx \leq_e Rx$. Since R is a left noetherian left V -ring, $\text{Soc } Rx$ is injective. Therefore, $\text{Soc } Rx$ is a direct summand of Rx . This is impossible unless $\text{Soc } Rx = Rx$. Therefore, every cyclic submodule of M and hence, M is semisimple.

(3) implies (1). Let I be an essential left ideal of R . Since R/I is singular, R/I is finitely generated semisimple. It follows that R/I is a finite direct sum of simple modules. Therefore, R/I is artinian.

The equivalence of (1) and (2) follows from 2.1.

According to 2.2, if a left noetherian left V -ring R satisfies the RLM condition, then $E(R)/R$ is a semisimple left R -module. In this case, $E(R)/R$ semisimple characterizes the RLM condition.

THEOREM 2.3. *Let R be a left noetherian left V -ring. Then R satisfies the RLM condition iff $E(R)/R$ is a semisimple left R -module.*

Proof. Suppose $E(R)/R$ is semisimple. Clearly, it suffices to show that every cyclic singular left R -module is semisimple. Let $I \leq_e R$. Then there is a regular $c \in I$ and $Rc \leq_e R$. The map $R \rightarrow Rc$ given by $r \rightarrow rc$ extends to an isomorphism $E(R) \rightarrow E(Rc)$. Since $E(R) = E(Rc)$, passing to the quotient yields an isomorphism $E(R)/R \cong E(R)/Rc$. Thus, $E(R)/Rc$ is semisimple. Now, $R/I \leq E(R)/I \cong (E(R)/Rc)/(I/Rc)$. Therefore, R/I is semisimple.

The converse follows from 2.2.

For a two sided noetherian V -ring R , Michler and Villamayor [12] have demonstrated that $K \dim R \leq 1$ and hereditary are equivalent. This result together with 2.3 allows us to characterize

hereditary in terms of the left R -module $E(R)/R$. This is in contrast to Boyle and Goodearl's result in [3] where $E(R)/R$ is required to be injective on both sides.

COROLLARY 2.4. *A noetherian V -ring R is hereditary iff $E(R)/R$ is a semisimple left R -module.*

Proof. According to Michler and Villamayor [12; 4.4], hereditary is equivalent to $K \dim R \leq 1$. The result follows from 2.2 and 2.3.

3. $E(R)/R$ and critically nice rings. A module U is called *critical* provided $K \dim U/K < K \dim U$ for every $0 \neq K \leq U$. Boyle [2] has shown that every finitely generated uniform left R -module over a left QI -ring is critical. Following Golan and Papp [8], we will call a ring over which every finitely generated uniform left R -module is *critically nice*. Since a hereditary noetherian V -ring is a QI -ring (Boyle [1; 5]), critically nice is necessary for a noetherian V -ring to be hereditary. Our purpose here will be to examine $E(R)/R$ when R is critically nice and extend some of our previous results.

LEMMA 3.1. *If R is a left noetherian ring, then the following are equivalent:*

- (1) R is critically nice.
- (2) If $A \neq 0$ is finitely generated, then every finitely generated submodule of $E(A)/A$ has Krull dimension strictly less than the Krull dimension of A .

Proof. (1) implies (2). Let $A \neq 0$ be finitely generated and let $F \leq E(A)/A$ be finitely generated. There are U_1, \dots, U_n uniform submodules of A such that $U_1 \oplus \dots \oplus U_n \leq_s A$. Then F is an epimorphic image of a finitely generated $F' \leq E(U_1)/U_1 \oplus \dots \oplus E(U_n)/U_n$. Since F' is finitely generated, there are finitely generated $F_i/U_i \leq E(U_i)/U_i$ such that $F' \leq F_1/U_1 \oplus \dots \oplus F_n/U_n$. Therefore, $K \dim F \leq K \dim F' \leq K \dim (F_1/U_1 \oplus \dots \oplus F_n/U_n) = K \dim F_j/U_j$ for some j . Since $U_j \leq A$, $K \dim U_j \leq K \dim A$. Also, F_j is critical. Thus, $K \dim F \leq K \dim F_j/U_j < K \dim F_j = K \dim U_j \leq K \dim A$.

(2) implies (1). Let $U \neq 0$ be finitely generated and uniform, and let $0 \neq K \leq U$. Then $K \dim K \leq K \dim U$. Since $U/K \leq E(U)/K = E(K)/K$, $K \dim U/K < K \dim K \leq K \dim U$.

A module M is called *smooth* provided $K \dim F = K \dim H$ for all nonzero finitely generated submodules F, H of M . According to 2.4 and 2.2, R hereditary implies that $E(R)/R$ is smooth and injective. In case R is critically nice, the following result shows that

the reverse implication holds.

Note that by [7; 2, 3], we may freely use the hypothesis that our ring is a simple ring.

THEOREM 3.2. *A simple noetherian V-ring R is hereditary iff R is critically nice and $E(R)/R$ is smooth and injective.*

Proof. Sufficiency. According to 2.4, it suffices to show that $E(R)/R$ is semisimple. Consequently, it suffices to show that every cyclic submodule of $E(R)/R$ is injective. Let $0 \neq C \leq E(R)/R$ be cyclic. Then $C \cong R/I$ where $I \leq_e R$. As in the proof of 2.3, $E(R)/R \cong E(R)/Rc$ where $c \in I$ is regular. Thus, $E(R)/Rc = E(R/Rc) \oplus E'$. Also, $E(R)/R \cong (E(R)/Rc)/(R/Rc) \cong E(R/Rc)/(R/Rc) \oplus E'$. Thus, if $0 \neq F \leq E(R/Rc)/(R/Rc)$ is finitely generated, then $K \dim F < K \dim R/Rc$ by 3.1. However, F imbeds in $E(R)/R \cong E(R)/Rc$ and hence, $K \dim F = K \dim R/Rc$ which is a contradiction. It follows that $E(R/Rc) = R/Rc$. Thus, $R/Rc = E(I/Rc) \oplus E''$. Now, $R/I \cong (R/Rc)/(I/Rc) \cong E(I/Rc)/(I/Rc) \oplus E''$. Thus, if $0 \neq K \leq E(I/Rc)/(I/Rc)$ is finitely generated, then $K \dim K < K \dim I/Rc$ by 3.1. However, since R/I imbeds in $E(R)/R \cong E(R)/Rc$ and K imbeds in R/I , $K \dim R/I = K \dim K = K \dim R/Rc = K \dim I/Rc$ which is a contradiction. It follows that $E(I/Rc) = I/Rc$. Therefore, $R/I \cong E''$ is injective.

Necessity follows from the remark prior to 3.1 and by 2.4.

Since a QI -ring is critically nice by Boyle [2], we immediately obtain the following corollary.

COROLLARY 3.3. *A QI -ring R is hereditary iff $E(R)/R$ is smooth and injective.*

4. Left-right symmetry. In this section, we examine the question of symmetry for left noetherian left V -rings which satisfy the RLM condition. We determine that right Goldie is equivalent to the ring being a right noetherian right V -ring. As a corollary to this result, we obtain a theorem of Faith's.

LEMMA 4.1. *Let R be a simple right Goldie ring. Then R is right noetherian iff R satisfies the ascending chain condition on finitely generated essential right ideals.*

Proof. The forward implication is trivial. For the reverse implication, let $U \neq 0$ be a uniform right ideal of R . By [10; 1.2], there is a 1 - 1 map $R \rightarrow U^n$ where U^n is a direct sum of n copies of U for some n . Thus, if every submodule of U is finitely gener-

ated, then U and hence R is right noetherian. Therefore, it suffices to show that every uniform right ideal of R is finitely generated. If not, then there is a uniform right ideal U with an infinite ascending chain $K_1 < K_2 < \dots < U$ where each K_i is finitely generated. Since R is right Goldie, there are finitely generated right ideals U_1, \dots, U_m such that $U \oplus U_1 \oplus \dots \oplus U_m \leq_e R$. Thus, if $F_i = K_i \oplus U_1 \oplus \dots \oplus U_m$, then $F_i \leq_e R$ for all i and $F_1 < F_2 < \dots < R$ is an infinite ascending chain which is a contradiction. Therefore, every uniform right ideal of R is finitely generated.

THEOREM 4.2. *Let R be a simple left noetherian left V -ring which satisfies the RLM condition. Then R is a right noetherian right V -ring iff R is right Goldie.*

Proof. Sufficiency. By 2.2, R is left hereditary. If R is right noetherian, then by Small [13], R is right hereditary, and by Boyle and Goodearl [3; 2], R is a right V -ring. Thus, it suffices to show that R is right noetherian.

Let $0 \neq I_1 \leq I_2 \leq \dots \leq R$ where each I_i is a finitely generated essential right ideal of R . Since R is right Goldie, there is a regular $c \in I_1$ and hence, $I_0 = cR$ is essential in R . For every i , let $I_i^* = \text{Hom}_R(I_i, R)$ and let $g_i: I_i^* \rightarrow I_{i-1}^*$ be given by $g_i(f) = f|_{I_{i-1}}$. Since each I_i is essential in R and R is nonsingular, g_i is $1 - 1$ for all i . Define a $1 - 1$ map $h_i: I_i^* \rightarrow R$ for all i by $h_i(f) = f(c)$. Let $J_i = h_i(I_i^*)$ for all i . It is easily verified that each J_i is a left ideal and that since each g_i is $1 - 1$, $R \supseteq J_0 \supseteq J_1 \supseteq \dots$. Also, since each I_i contains the inclusion map, $c \in J_i$ for all i . Thus, since R/Re is artinian, there is an n for which $J_n = J_{n+k}$ for all k . It is well known that this forces $I_n = I_{n+k}$ for all k . By 4.1, R is right noetherian.

Necessity is trivial.

COROLLARY 4.3 [6; 22]. *Let R be a left QI -domain which satisfies the RLM condition. If R is right Goldie, then R is a right QI -ring.*

Proof. By 4.2, R is a right noetherian right V -ring. By Small [13], R is right hereditary. According to Boyle [1; 5], R is a right QI -ring.

REFERENCES

1. A. K. Boyle, *Hereditary QI -rings*, Trans. A.M.S., **192** (1974), 115-120.
2. ———, *Injectives containing no proper quasi-injective submodules*, Comm. in Alg., **4**(8), (1976), 775-785.

3. A. K. Boyle and K. R. Goodearl, *Rings over which certain modules are injective*, Pacific. J. Math., **58** No. 1, (1975), 43-53.
4. J. H. Cozzens, *Homological properties of the ring of differential polynomials*, Bull. A.M.S., **76** (1970), 75-79.
5. J. H. Cozzens and J. Johnson, *Some applications of differential algebra to ring theory*, Proc. A.M.S., **31** No. 2, (1972), 354-356.
6. C. Faith, *When are proper cyclics injective?* Pacific. J. Math., **45** No. 1, (1973), 97-111.
7. ———, *On hereditary rings and Boyles conjecture*, Archiv Der Mathematik., **27** No. 2, (1976), 113-119.
8. J. S. Golan and Z. Papp, *Cocritically nice rings and Boyles conjecture*, Preprint.
9. K. R. Goodearl, *Ring Theory*, Marcel Dekker, 1976.
10. K. R. Goodearl, D. Handelman, and J. Lawrence, *Strongly Prime and Completely Torsion Free Rings*, Carleton University Mathematical Series, No. 109, June 1974.
11. R. Gordon and J. C. Robson, *Krull Dimension*, Memoirs A.M.S., No. 133, 1973.
12. G. O. Michler and O. E. Villamayor, *On rings whose simple modules are injective*, J. Algebra, **25** (1973), 185-201.
13. L. W. Small, *Semiheditary rings*, Proc. Nat. Acad. Sci. U.S.A., **55** (1966), 25-27.

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