Pacific Journal of Mathematics

# ON HEREDITARY RINGS AND NOETHERIAN $V$-RINGS

Karl Andrew Kosler

Vol. 103, No. 2 April 1982

PACIFIC JOURNAL OF MATHEMATICS
Vol. 103, No. 2, 1982

# ON HEREDITARY RINGS AND NOETHERIAN $V$-RINGS

KARL A. KOSLER

**The purpose of this paper is to examine conditions under which (1) a left noetherian left $V$-ring is left hereditary and (2) a left noetherian left $V$-ring is a two sided noetherian $V$-ring. For (1), left noetherian left $V$-rings which satisfy the restricted left minimum (RLM) condition are examined. The RLM condition is shown to be equivalent to $E(R)/R$ a semisimple left $R$-module. Consequently, hereditary is equivalent to $E(R)/\mathrm{R}$ semisimple in the two sided case. Two sided noetherian $V$-rings which are critically nice are also examined. In this case, hereditary is shown to be equivalent to $E(R)/R$ injective and smooth. For (2), a theorem of Faith's concerning left $QI$-domains is extended to left noetherian left $V$-rings.**

**1. Introduction and definitions.** A ring $R$ is called a *left $V$-ring* provided every simple left $R$-module is injective. The definition of $V$-ring is due to Villamayor who has shown that a ring is a left $V$-ring if and only if every left ideal is the intersection of maximal left ideals. Consequently, all left $V$-rings are semiprime. Kaplansky has shown that a commutative ring is a $V$-ring if and only if it is regular. It follows that every commutative noetherian $V$-ring is semisimple artinian. Cozzens [4] showed that this result does not extend to the noncommutative case by producing an example of a nonartinian, two sided hereditary noetherian $V$-domain over which all cyclic modules are semisimple or free. This condition on cyclics forces every quasi-injective module to be injective. A ring with all its quasi-injective left $R$-modules injective will be called a *left $QI$-ring.* According to Boyle [1], a left $QI$-ring is left noetherian. Note that since a simple module is quasi-injective, a left $QI$-ring is a left $V$-ring.

As with Cozzens' example, all the known examples of left $QI$-rings are left hereditary. Cozzens and Johnson [5] produced examples of two sided noetherian $V$-rings which Boyle and Goodearl [3] demonstrated to be neither hereditary nor $QI$. Also, there is no known example of a one sided noetherian $V$-ring or $QI$-ring. In this paper, we will consider the problem of determining when a left noetherian left $V$-ring is left hereditary and when a left $V$-ring is a right $V$-ring.

Throughout, all rings will be associative with identity, all $R$-modules will be unitary left $R$-modules and maps between modules will be $R$-homomorphisms. If $N$ is a submodule of a module $M$, then we will write $N \leqq M$. In case $N \cap K \neq 0$ for all $0 \neq K \leqq M$, then $N$ is called *essential* in $M$ and we will write $N \leqq_e M$. For a module $M$, $E(M)$, $\operatorname{Soc} M$ and $K \dim M$ will denote the injective hull, socle and Krull dimension of $M$ respectively. It is assumed that the reader is familiar with the notions of singular, nonsingular and uniform modules as presented in [9]. We also use the notions of Krull dimension, critical module and smooth module as given in [11]. Throughout this paper, whenever we use the terms hereditary, noetherian, $V$-ring or $QI$-ring unqualified by "left" or "right", this will mean that the term applies to both the left and right.

In § 2, left noetherian left $V$-rings which satisfy the *restricted left minimum* (RLM) condition are examined. A module $M$ satisfies the RLM condition provided $M/K$ is artinian whenever $K \leqq_e M$. It is shown that the RLM condition is equivalent to $E(R)/R$ semisimple. As a consequence, hereditary is equivalent to $E(R)/R$ semisimple in the two sided case.

The purpose of § 3 is to further investigate the role $E(R)/R$ plays in determining when a noetherian $V$-ring is hereditary. A necessary condition for hereditary is that $R$ be *critically nice* (all finitely generated uniform modules are critical). In this case, $R$ is hereditary iff $E(R)/R$ is injective and smooth.

In § 4, left-right symmetry is examined. A theorem of Faith's which states that a left $QI$-domain with the RLM condition is right $QI$ iff it is right Goldie is extended to left noetherian left $V$-rings.

**2. The restricted left minimum condition.** A module $M$ is said to satisfy the *restricted left minimum* condition, denoted RLM, provided $M/K$ is artinian for all $K \leqq_e M$. A ring $R$ is said to satisfy the RLM condition provided the left $R$-module $R$ satisfies the RLM condition. In this section, we investigate left noetherian left $V$-rings which satisfy the RLM condition.

LEMMA 2.1. *Let $R$ be a semiprime ring with Krull dimension. Then $R$ satisfies the RLM condition iff $K \dim R \leqq 1$.*

*Proof.* By Gordon and Robson [11; 6.1], $K \dim R = \sup \{K \dim R/I + 1 \mid I \leqq_e R\}$. If $R$ satisfies the RLM condition, then $K \dim R/I \leqq 0$ for all essential left ideals $I$. Thus, $K \dim R \leqq 1$. Conversely, if $K \dim R \leqq 1$, then $K \dim R/I \leqq 0$ for all $I \leqq_e R$. Hence, $R/I$ is artinian for all essential left ideals $I$.

The RLM condition has been shown by Faith [7] to be sufficient for a left $QI$ ring to be left hereditary. Michler and Villamayor [12] have shown that $K \dim R \leqq 1$ is sufficient for a left noetherian left $V$-ring $R$ to be left hereditary. Therefore, if all cyclic singular left $R$-modules are semisimple, then by 2.1, $K\dim R \leqq 1$ and $R$ is left hereditary. As the next result shows, $K\dim R \leqq 1$ and the RLM condition on $R$ are equivalent to all singular (cyclic) left $R$-modules semisimple.

THEOREM 2.2. *If $R$ is a left noetherian left $V$-ring, then the following are equivalent*:

(1) *$R$ satisfies the RLM condition.*

(2) $K\dim R \leqq 1$.

(3) *All singular left $R$-modules are semisimple.*

*Furthermore, if* (1)-(3) *hold, then $R$ is left hereditary.*

*Proof.* (1) implies (3). Let $M$ be a singular left $R$-module and let $0 \neq x \in M$. Then $Rx \cong R/I$ where $I \leqq_e R$ and hence, $Rx$ is artinian. Thus, $\operatorname{Soc} Rx \leqq_e Rx$. Since $R$ is a left noetherian left $V$-ring, $\operatorname{Soc} Rx$ is injective. Therefore, $\operatorname{Soc} Rx$ is a direct summand of $Rx$. This is impossible unless $\operatorname{Soc} Rx = Rx$. Therefore, every cyclic submodule of $M$ and hence, $M$ is semisimple.

(3) implies (1). Let $I$ be an essential left ideal of $R$. Since $R/I$ is singular, $R/I$ is finitely generated semisimple. It follows that $R/I$ is a finite direct sum of simple modules. Therefore, $R/I$ is artinian.

The equivalence of (1) and (2) follows from 2.1.

According to 2.2, if a left noetherian left $V$-ring $R$ satisfies the RLM condition, then $E(R)/R$ is a semisimple left $R$-module. In this case, $E(R)/R$ semisimple characterizes the RLM condition.

THEOREM 2.3. *Let $R$ be a left noetherian left $V$-ring. Then $R$ satisfies the RLM condition iff $E(R)/R$ is a semisimple left $R$-module.*

*Proof.* Suppose $E(R)/R$ is semisimple. Clearly, it suffices to show that every cyclic singular left $R$-module is semisimple. Let $I \leqq_e R$. Then there is a regular $c \in I$ and $Rc \leqq_e R$. The map $R \to Rc$ given by $r \to rc$ extends to an isomorphism $E(R) \to E(Rc)$. Since $E(R) = E(Rc)$, passing to the quotient yields an isomorphism $E(R)/R \cong E(R)/Rc$. Thus, $E(R)/Rc$ is semisimple. Now, $R/I \leqq E(R)/I \cong (E(R)/Rc)/(I/Rc)$. Therefore, $R/I$ is semisimple.

The converse follows from 2.2.

For a two sided noetherian $V$-ring $R$, Michler and Villamayor [12] have demonstrated that $K\dim R \leqq 1$ and hereditary are equivalent. This result together with 2.3 allows us to characterize hereditary in terms of the left $R$-module $E(R)/R$. This is in contrast to Boyle and Goodearls result in [3] where $E(R)/R$ is required to be injective on both sides.

COROLLARY 2.4. *A noetherian $V$-ring $R$ is hereditary iff $E(R)/R$ is a semisimple left $R$-module.*

*Proof.* According to Michler and Villamayor [12; 4.4], hereditary is equivalent to $K\dim R \leqq 1$. The result follows from 2.2 and 2.3.

**3. $E(R)/R$ and critically nice rings.** A module $U$ is called *critical* provided $K\dim U/K < K\dim U$ for every $0 \neq K \leqq U$. Boyle [2] has shown that every finitely generated uniform left $R$-module over a left $QI$-ring is critical. Following Golan and Papp [8], we will call a ring over which every finitely generated uniform left $R$-module is *critically nice*. Since a hereditary noetherian $V$-ring is a $QI$-ring (Boyle [1; 5]), critically nice is necessary for a noetherian $V$-ring to be hereditary. Our purpose here will be to examine $E(R)/R$ when $R$ is critically nice and extend some of our previous results.

LEMMA 3.1. *If $R$ is a left noetherian ring, then the following are equivalent:*

(1) *$R$ is critically nice.*

(2) *If $A \neq 0$ is finitely generated, then every finitely generated submodule of $E(A)/A$ has Krull dimension strictly less than the Krull dimension of $A$.*

*Proof.* (1) implies (2). Let $A \neq 0$ be finitely generated and let $F \leqq E(A)/A$ be finitely generated. There are $U_1, \cdots, U_n$ uniform submodules of $A$ such that $U_1 \oplus \cdots \oplus U_n \leqq_e A$. Then $F$ is an epimorphic image of a finitely generated $F' \leqq E(U_1)/U_1 \oplus \cdots \oplus E(U_n)/U_n$. Since $F'$ is finitely generated, there are finitely generated $F_i/U_i \leqq E(U_i)/U_i$ such that $F' \leqq F_1/U_1 \oplus \cdots \oplus F_n/U_n$. Therefore, $K\dim F \leqq K\dim F' \leqq K\dim(F_1/U_1 \oplus \cdots \oplus F_n/U_n) = K\dim F_j/U_j$ for some $j$. Since $U_j \leqq A$, $K\dim U_j \leqq K\dim A$. Also, $F_j$ is critical. Thus, $K\dim F \leqq K\dim F_j/U_j < K\dim F_j = K\dim U_j \leqq K\dim A$.

(2) implies (1). Let $U \neq 0$ be finitely generated and uniform, and let $0 \neq K \leqq U$. Then $K\dim K \leqq K\dim U$. Since $U/K \leqq E(U)/K = E(K)/K$, $K\dim U/K < K\dim K \leqq K\dim U$.

A module $M$ is called *smooth* provided $K\dim F = K\dim H$ for all nonzero finitely generated submodules $F, H$ of $M$. According to 2.4 and 2.2, $R$ hereditary implies that $E(R)/R$ is smooth and injective. In case $R$ is critically nice, the following result shows that the reverse implication holds.

Note that by [7; 2, 3], we may freely use the hypothesis that our ring is a simple ring.

THEOREM 3.2. *A simple noetherian $V$-ring $R$ is hereditary iff $R$ is critically nice and $E(R)/R$ is smooth and injective.*

*Proof.* Sufficiency. According to 2.4, it suffices to show that $E(R)/R$ is semisimple. Consequently, it suffices to show that every cyclic submodule of $E(R)/R$ is injective. Let $0 \neq C \leqq E(R)/R$ be cyclic. Then $C \cong R/I$ where $I \leqq_e R$. As in the proof of 2.3, $E(R)/R \cong E(R)/Rc$ where $c \in I$ is regular. Thus, $E(R)/Rc = E(R/Rc) \oplus E'$. Also, $E(R)/R \cong (E(R)/Rc)/(R/Rc) \cong E(R/Rc)/(R/Rc) \oplus E'$. Thus, if $0 \neq F \leqq E(R/Rc)/(R/Rc)$ is finitely generated, then $K\dim F < K\dim R/Rc$ by 3.1. However, $F$ imbeds in $E(R)/R \cong E(R)/Rc$ and hence, $K\dim F = K\dim R/Rc$ which is a contradiction. It follows that $E(R/Rc) = R/Rc$. Thus, $R/Rc = E(I/Rc) \oplus E''$. Now, $R/I \cong (R/Rc)/(I/Rc) \cong E(I/Rc)/(I/Rc) \oplus E''$. Thus, if $0 \neq K \leqq E(I/Rc)/(I/Rc)$ is finitely generated, then $K\dim K < K\dim I/Rc$ by 3.1. However, since $R/I$ imbeds in $E(R)/R \cong E(R)/Rc$ and $K$ imbeds in $R/I$, $K\dim R/I = K\dim K = K\dim R/Rc = K\dim I/Rc$ which is a contradiction. It follows that $E(I/Rc) = I/Rc$. Therefore, $R/I \cong E''$ is injective.

Necessity follows from the remark prior to 3.1 and by 2.4.

Since a $QI$-ring is critically nice by Boyle [2], we immediately obtain the following corollary.

COROLLARY 3.3. *A $QI$-ring $R$ is hereditary iff $E(R)/R$ is smooth and injective.*

4. **Left-right symmetry.** In this section, we examine the question of symmetry for left noetherian left $V$-rings which satisfy the RLM condition. We determine that right Goldie is equivalent to the ring being a right noetherian right $V$-ring. As a corollary to this result, we obtain a theorem of Faith's.

LEMMA 4.1. *Let $R$ be a simple right Goldie ring. Then $R$ is right noetherian iff $R$ satisfies the ascending chain condition on finitely generated essential right ideals.*

*Proof.* The forward implication is trivial. For the reverse implication, let $U \neq 0$ be a uniform right ideal of $R$. By [10; 1.2], there is a $1-1$ map $R \to U^n$ where $U^n$ is a direct sum of $n$ copies of $U$ for some $n$. Thus, if every submodule of $U$ is finitely generated, then $U$ and hence $R$ is right noetherian. Therefore, it suffices to show that every uniform right ideal of $R$ is finitely generated. If not, then there is a uniform right ideal $U$ with an infinite ascending chain $K_1 < K_2 < \cdots < U$ where each $K_i$ is finitely generated. Since $R$ is right Goldie, there are finitely generated right ideals $U_1, \cdots, U_m$ such that $U \oplus U_1 \oplus \cdots \oplus U_m \leqq_e R$. Thus, if $F_i = K_i \oplus U_1 \oplus \cdots \oplus U_m$, then $F_i \leqq_e R$ for all $i$ and $F_1 < F_2 < \cdots < R$ is an infinite ascending chain which is a contradiction. Therefore, every uniform right ideal of $R$ is finitely generated.

THEOREM 4.2. *Let $R$ be a simple left noetherian left $V$-ring which satisfies the RLM condition. Then $R$ is a right noetherian right $V$-ring iff $R$ is right Goldie.*

*Proof.* Sufficiency. By 2.2, $R$ is left hereditary. If $R$ is right noetherian, then by Small [13], $R$ is right hereditary, and by Boyle and Goodearl [3; 2], $R$ is a right $V$-ring. Thus, it suffices to show that $R$ is right noetherian.

Let $0 \neq I_1 \leqq I_2 \leqq \cdots \leqq R$ where each $I_i$ is a finitely generated essential right ideal of $R$. Since $R$ is right Goldie, there is a regular $c \in I_1$ and hence, $I_0 = cR$ is essential in $R$. For every $i$, let $I_i^* = \mathrm{Hom}_R(I_i, R)$ and let $g_i\colon I_i^* \to I_{i-1}^*$ be given by $g_i(f) = f \mid I_{i-1}$. Since each $I_i$ is essential in $R$ and $R$ is nonsingular, $g_i$ is $1-1$ for all $i$. Define a $1-1$ map $h_i\colon I_i^* \to R$ for all $i$ by $h_i(f) = f(c)$. Let $J_i = h_i(I_i^*)$ for all $i$. It is easily verified that each $J_i$ is a left ideal and that since each $g_i$ is $1-1$, $R \geqq J_0 \geqq J_1 \geqq \cdots$. Also, since each $I_i$ contains the inclusion map, $c \in J_i$ for all $i$. Thus, since $R/Rc$ is artinian, there is an $n$ for which $J_n = J_{n+k}$ for all $k$. It is well known that this forces $I_n = I_{n+k}$ for all $k$. By 4.1, $R$ is right noetherian.

Necessity is trivial.

COROLLARY 4.3 [6; 22]. *Let $R$ be a left $QI$-domain which satisfies the RLM condition. If $R$ is right Goldie, then $R$ is a right $QI$-ring.*

*Proof.* By 4.2, $R$ is a right noetherian right $V$-ring. By Small [13], $R$ is right hereditary. According to Boyle [1; 5], $R$ is a right $QI$-ring.

## REFERENCES

1. A. K. Boyle, *Hereditary QI-rings*, Trans. A.M.S., **192** (1974), 115–120.
2. ———, *Injectives containing no proper quasi-injective submodules*, Comm. in Alg., **4**(8), (1976), 775–785.

3. A. K. Boyle and K. R. Goodearl, *Rings over which certain modules are injective*, Pacific. J. Math., **58** No. 1, (1975), 43-53.
4. J. H. Cozzens, *Homological properties of the ring of differential polynomials*, Bull. A.M.S., **76** (1970), 75-79.
5. J. H. Cozzens and J. Johnson, *Some applications of differential algebra to ring theory*, Proc. A.M.S., **31** No. 2, (1972), 354-356.
6. C. Faith, *When are proper cyclics injective*? Pacific. J. Math., **45** No. 1, (1973), 97-111.
7. ———, *On hereditary rings and Boyles conjecture*, Archiv Der Mathematik., **27** No. 2, (1976), 113-119.
8. J. S. Golan and Z. Papp, *Cocritically nice rings and Boyles conjecture*, Preprint.
9. K. R. Goodearl, *Ring Theory*, Marcel Dekker, 1976.
10. K. R. Goodearl, D. Handelman, and J. Lawrence, *Strongly Prime and Completely Torsion Free Rings*, Carleton University Mathematical Series, No. 109, June 1974.
11. R. Gordon and J. C. Robson, *Krull Dimension*, Memoirs A.M.S., No. 133, 1973.
12. G. O. Michler and O. E. Villamayor, *On rings whose simple modules are injective*, J. Algebra, **25** (1973), 185-201.
13. L. W. Small, *Semihereditary rings*, Proc. Nat. Acad. Sci. U.S.A., **55** (1966), 25-27.

Received October 1, 1980 and in revised form October 26, 1981. This paper represents a portion of the author's Ph. D. thesis written under the direction of Professor Ann K. Boyle at the University of Wisconsin-Milwaukee.

ST NORBERT COLLEGE
DE PERE, WI 54115

The Supporting Institutions listed above contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its content or policies,

Mathematical parers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd unmbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.


The *Pacific Journal of Mathematics* is issued monthly as of January 1966, Regular subscription rate: $114.00 a year (6 Vol., 12 issues). Special rate: $57.00 a year to individual members of supporting institution.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address shoud be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

# Pacific Journal of Mathematics

Vol. 103, No. 2 April, 1982