STRONG RESULT FOR REAL ZEROS OF RANDOM POLYNOMIALS

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Let N_n be the number of real zeros of $\sum_{r=0}^{n} a_r X_r x^r = 0$ where X_r's are independent random variables identically distributed belonging to the domain of attraction of normal law; $a_0, a_1, a_2 \cdots a_n$ are nonzero real numbers such that $(k_n/t_n) = o(\log n)$ where $k_n = \max_{0 \le r \le n} |a_r|$ and $t_n = \min_{0 \le r \le n} |a_r|$. Further we suppose that the coefficients have zero means and $P\{X_r \ne 0\} > 0$. Then there exists a positive integer n_0 such that

$$P\{\sup_{n > n_0} (N_n/D_n) < \mu\} > 1 - \mu' \{\log ((k_{n_0}/t_{n_0}) \log \log n)/\log n_0\}^{1 - \varepsilon/2}$$

for $n > n_0$ and $1 > \varepsilon > 0$ where $D_n = (\log n/\log (k_n/t_n) \log \log n)^{(1-\varepsilon)/2}$.

1. Let N_n be the number of real roots of a random algebraic equation

$$\sum_{r=0}^{n} X_r x^r = 0 ,$$

where X_r's are independent, identically distributed random variables. The problem of finding the lower bound of N_n has been considered by various authors. Considering the coefficients as normally distributed or uniformly distributed in $[-1, 1]$, assuming the values $+1$ or -1 with equal probability Littlehood and Offord [8] have shown that $N_n > \mu \log n/\log \log n$ except for a set of measure at most $\mu'/\log n$, n being sufficiently large. Evans [4] has studied the strong version of Littlehood and Offord and has shown that in case of Gaussian distributed coefficients N_n is greater than $\mu \log n/\log \log n$ except for a set of measure at most μ' $(\log \log n_0/\log n_0)$ for $n > n_0$. The above result is strong in the following sense.

Theorem of Littlehood and Offord is of the form

$$P\{(N_n/D_n') < \mu\} \longrightarrow 1 \text{ as } n \longrightarrow \infty ,$$

where $D_n' = \log n/\log \log n$. But the theorem of Evans is of the form

$$P\{\sup_{n > n_0} (N_n/D_n') < \mu\} \longrightarrow 1 \text{ as } n_0 \longrightarrow \infty .$$

Considering the coefficients of $\sum_{r=0}^{n} a_r X_r x^r = 0$ as symmetric stable variables Samal and Mishra [13] have shown that

$$P\{(N_n/D_n^*) < \mu\} > 1 - \frac{\mu'}{\{\log (k_n/t_n) \log n\}(\log n)^{\alpha-1}} \text{ if } 1 \leq \alpha < 2$$

and

$$> 1 - \frac{\mu' \log (k_n/t_n) \log n}{\log n} \text{ if } \alpha = 2 \ ,$$

where $k_n = \max_{0 \leq r \leq n} |a_r|$, $t_n = \min_{0 \leq r \leq n} |a_r|$ and $D_n^* = (\log n/\log ((k_n/t_n) \log n))$. Samal and Mishra [13] have studied the strong version of the above theorem and have shown that $P\{\sup_{n > n_0} (N_n/D_n^*) < \mu\}$

$$> 1 - \frac{\mu}{\{\log (\log n_0/\log (k_{n_0}/t_{n_0}) \log n_0)\}^{\alpha-1}} \text{ where } \alpha > 1 \ .$$

Mishra and Nayak [9] have proved that

$$P\{(N_n/D_n^*) < \mu\} > 1 - \frac{\mu'}{\{\log ((k_n/t_n) \log n)\}(\log n)^{1-\varepsilon}}$$

for every positive $\varepsilon < 1$, when the coefficients belong to the domain of attraction of the normal law.

Object of this paper is to show that

$$P\{\sup_{n > n_0} (N_n/D_n) < \mu\}$$
$$> 1 - \mu' \left\{\frac{\log ((k_{n_0}/t_{n_0}) \log \log n_0)}{\log n_0}\right\}^{1/2}$$

for $0 < \varepsilon < 1$, when the coefficients belong to the domain of attraction of the normal law. Therefore it is a strong result of Mishra and Nayak.

Throughout this paper we shall denote μ's for positive constants which may assume different values in different occurences and $V(\cdot)$ for the variance of a random variable.

2. In the sequel we shall need the following definition, and theorem due to Karamata, (cf. Ibragimov and Linnik [6] p. 394), for the proof of our main result.

DEFINITION. A function $H: R_+ \to R_+$ is called a slowly varying function if

(2.1) $$\lim_{x \to \infty} \frac{H(\gamma x)}{H(x)} = 1, (\gamma > 0) \ .$$

We have a few characterization of the slowly varying functions due to Karamata.

By writting $H(1/t) = h(t)$, we may define a slowly varying func-

tion h: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ with the property that

$$(2.2) \qquad \lim_{x \to 0} \frac{h(\gamma x)}{h(x)} = 1, \ (\gamma > 0) \ .$$

With this the Karamata theorem, (cf. Ibragimov and Linnik [6], p. 394), may be stated as follows.

THEOREM 1. *A slowly varying function h with the property (2.2) which is integrable on any finite interval may be represented in the form*

$$h(x) = c(x) \exp \left(-\int_a^x \frac{\bar{\varepsilon}(u)}{u} du \right) ,$$

where

$$\lim_{x \to 0} c(x) = c \neq 0, \quad \lim_{x \to 0} \bar{\varepsilon}(x) = 0 \ and \ a > 0 \ .$$

We establish the following formulae which will be necessary for the proof of the main theorem.

Let a sequence of independent and identically distributed random variables $\{X_r\}$ with mean zero belong to the domain of attraction of the normal law. Then their common characteristic function $\phi(t)$ is given by (cf. Ibragimov and Linnik [6], p. 91),

$$(2.3) \qquad \log \phi(t) = -\frac{t^2}{2} H(|t|^{-1})(1 + o(1)) \ ,$$

where $H(t)$ is a slowly varying function as $t \to \infty$ and is given by the formula

$$(2.4) \qquad H(x) = -\int_0^x u^2 d\psi(x) = \int_{-x}^x u^2 dG(u) \ ,$$

where $\Psi(x) = 1 - G(x) + G(x)$ and $G(x)$ is the common distribution function.

Also

$$(2.5) \qquad |\phi(t)| \sim \exp \left\{ -\frac{t^2}{2} H(|t|^{-1}) \right\} \ .$$

If we put $H(1/t) = L(t)$, then $L(t)$ is slowly varying as $t \to 0$. Then (2.3) and (2.5) will take the forms

$$\log \phi(t) = -\frac{t^2}{2} L(|t|)(1 + o(1))$$

and

$$|\phi(t)| \sim \exp\left\{-\frac{t^2}{2}L(|t|)\right\}$$

respectively. Since $L(|t|)$ is positive we can write the characteristic function ϕ as

(2.6) $$\phi(t) = \exp\left\{-\frac{t^2}{2}h(t)\right\}$$

where $h(t) = L(|t|)(1 + o(1))$ with the property

(2.7) $$h(t) = \operatorname{Re} h(t)(1 + o(1)) ,$$

as

$$\operatorname{Re} h(t) = L(|t|)\ (1 + o(1)) .$$

Now $h(t)$ is slowly varying as $t \to 0$, since for $\gamma > 0$,

$$\lim_{t \to 0}\frac{h(\gamma(t))}{h(t)} = \lim_{t \to 0}\frac{L(\gamma|t|)(1 + o(1))}{L(|t|)(1 + o(1))} = 1 .$$

Consider the function $h_1(t)$ determined by

$$h_1(t) = \begin{bmatrix} \operatorname{Re} h(t) \text{ if } V(X_r) = \infty , \\ \sigma^2 \text{ if } V(X_r) = \sigma^2 < \infty . \end{bmatrix}$$

Clearly $h_1(t)$ is slowly varying in a neighborhood of the origin. By (2.7),

(2.8) $h(t) = h_1(t)\ (1 + o(1))$, in both cases as $t \longrightarrow 0$.

Since expectation is zero, by virtue of (2.4), we have

$$\lim_{x \to \infty} H(x) = \int_{-\infty}^{\infty} u^2 dG(u) = \sigma^2 .$$

Therefore when variance is infinite, $\lim_{x \to \infty} H(x) = \infty$, so that $\lim_{t \to 0} L(t) = \infty$. Thus we have for infinite variance,

(2.9) $$\lim_{t \to 0} h_1(t) = \infty .$$

THEOREM 2. *Let*

$$f(x) = \sum_{r=0}^{n} a_r X_r x^r$$

be a polynomial of degree n, where X_r's are independent and identically distributed random variables which belong to the domain of attraction of the normal law, have zero means and $P\{X_r \neq 0\} > 0$. Let $a_0, a_1, a_2 \cdots a_n$ be nonzero real number such that $(k_n/t_n) = o\ (\log n)$

where $k_n = \max_{0 \leq r \leq n} |a_r|$ *and* $t_n = \min_{0 \leq r \leq n} |a_r|$. *Then there exists a positive* n_0 *such that the number of real roots of* $f(x) = 0$ *is at least* $\mu\{\log n/\log((k_n/t_n) \log \log n)\}^{1/2}$ *outside a set of measure at most* $\mu'\{\log((k_{n_0}/t_{n_0}) \log \log n_0)/\log n_0\}^{(1-\varepsilon)/2}$ *for* $n > n_0$ *and* $1 > \varepsilon > 0$.

3. Proof of the Theorem 2. Take constants A and D such that

$$(3.1) \qquad\qquad 0 < D < 1 \quad \text{and} \quad A > 1 .$$

Let

$$(3.2) \qquad\qquad \lambda_m = m \log \log n .$$

Let

$$(3.3) \qquad M_n = [d^2(\log \log n)^2(k_n/t_n)^2(\sqrt{2} + 1)^2(Ae/D)] + 1 ,$$

where b is a positive constant greater than one whose choice will be made later and $[x]$ denotes the greatest integer not exceeding x.

It follows from (3.3) that

$$(3.4) \qquad \mu_1\left(\frac{k_n}{t_n} \log \log n\right)^2 \leq M_n \leq \mu_2\left(\frac{k_n}{t_n} \log \log n\right)^2 .$$

We define

$$(3.5) \qquad\qquad \phi(x) = x^{[\log x] + x} .$$

Let k be the integer determined by

$$(3.6) \qquad \phi(8k + 7)M_n^{8k+7} \leq n < \phi(8k + 11)M_n^{8k+11}.$$

The first inequality of (3.5) gives

$$\log \phi(8k + 7) + (8k + 7) \log M_n \leq \log n ,$$

or

$$(8k + 7) \log M_n < \log n ,$$

which by help of (3.4) yields

$$k < \frac{\mu \log n}{\log\left(\dfrac{k_n}{t_n} \log \log n\right)} .$$

Again the right hand side inequality of (3.4) gives

$$\log n < \log \phi(8k + 11) + (8k + 11) \log M_n$$
$$= (\log(8k + 11) + 8k + 11) \log(8k + 11) + (8k + 11) \log M_n$$
$$< 2(8k + 11)^2 + (8k + 11) \log M_n < \mu_3 k^2 \log M_n ,$$

whence by (3.4), we have

$$\mu_0\Big(\frac{\log n}{\log (k_n/t_n \log \log n)}\Big)^{1/2} < k \,.$$

Therefore

$$(3.7) \quad \mu_0\Big(\frac{\log n}{\log (k_n/t_n \log \log n)}\Big)^{1/2} < k < \mu\frac{\log n}{\log (k_n/t_n \log \log n)} \,.$$

Since $(k_n/t_n) = o(\log n)$ by hypothesis, it follows from (3.7), that $k \to \infty$ as $n \to \infty$.

We have $f(x_m) = U_m + R_m$ at the points

$$(3.8) \qquad x_m = \Big(1 - \frac{1}{\phi(4m + 1)M_n^{4m}}\Big)^{1/2}$$

for $m = [k/2] + 1, [k/2] + 2 \cdots k$, where

$$U_m = \sum_1 a_r X_r x_m^r$$

and

$$R_m = (\sum_2 + \sum_3)a_r X_r x_m^r \,,$$

the index r ranging from $\phi(4m - 1)M_n^{4m-1} + 1$ to $\phi(4m + 3)M_n^{4m+3}$ in $\sum_1$, from 0 to $\phi(4m - 1)M_n^{4m-1}$ in $\sum_2$ and from $\phi(4m + 3)M_n^{4m+3} + 1$ to n in $\sum_3$. (We shall use the notations $\sum_1, \sum_2$ and $\sum_3$ to carry the above meaning throughout this paper.)

We have also

$$(3.9) \qquad f(x_{2m}) = U_{2m} + R_{2m}, \ f(x_{2m+1}) = U_{2m+1} + R_{2m+1} \,.$$

By (3.7), we have $2k + 1 < n$ for large n. Also the maximum index in U_{2m+1} for $m = k$ is $\phi(8k + 7)M_n^{8k+7}$, which by (3.6) is consistent with (3.9).

We define normalizing constants V_m starting from the relation

$$(3.10) \qquad (1/V_m^2) \sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / V_m) \,,$$

where θ is a small positive number whose final choice will be dealt with later. Such normalizing constants V_m always exist when θ is sufficiently small. (Cf. Ibragimov and Maslova [7], p. 232.)

Now if $V(X_r) = \infty$, we have

$$V_m^2 = \sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / V_m) > \sum_1 a_r^2 x_m^{2r}$$

(by (2.9), since θ is small),

$$> t_n^2 \sum_{\phi(4m-1) M_n^{4m-1}+1}^{\phi(4m+1) M_n^{4m}} x_m^{2r}$$

$$> t_n^2 \{\phi(4m+1) M_n^{4m} - \phi(4m-1) M_n^{4m-1}\} \left\{ 1 - \frac{1}{\phi(4m+1) M_n^{4m}} \right\}^{\phi(4m+1) M_n^{4m}}$$

$$> t_n^2 \phi(4m+1) M_n^{4m}(D/Ae) .$$

Or

$$(3.11) \qquad M_n^{2m} < (Ae/D\,\phi(4m+1))^{1/2}(V_m/t_n) .$$

Again if $V(X_r) = \sigma^2 < \infty$, then

$$V_m^2 = \sigma^2 \sum_1 a_r^2 x_m^{2r}$$

$$> \sigma^2 \phi(4m+1) M_n^{4m}(D/Ae) .$$

Or

$$(3.12) \qquad M_n^{2m} < (Ae/D\,\phi(4m+1))^{1/2}(V_m/\sigma t_n) .$$

The following lemmas are necessary for the proof of the theorem.

LEMMA 1.

$$|\sum_2 a_r X_r x_m^r| < \lambda_m W_m ,$$

except for a set of measure at most $\mu/\lambda_m^{2-\varepsilon}$ *for* $\varepsilon > 0$, *where*

$$(3.13) \qquad W_m^2 = \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / W_m) .$$

Proof. The characteristic function of $(1/W_m) \sum_2 a_r X_r x_m^r$ is given by

$$\phi_m(t) = \exp\left(-\frac{t^2}{2} h_m(t) \right)$$

where

$$(3.14) \qquad h_m(t) = (1/W_m^2) \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / W_m) .$$

We have by Theorem 1 for $|t| < \theta$,

$$h_1(a_r x_m^r t / W_m)(h_1(a_r x_m^r \theta / W_m))^{-1} = \frac{L(|a_r x_m^r t / W_m|)(1 + o(1))}{L(|a_r x_m^r \theta / W_m|)(1 + o(1))}$$

$$= \frac{c(|a_r x_m^r t / W_m|)(1 + o(1))}{c(|a_r x_m^r \theta / W_m|)(1 + o(1))} \exp \left\{ \int_{|a_r x_m^r t / W_m|}^{|a_r x_m^r \theta / W_m|} \frac{\bar{\varepsilon}(u)}{u} \, du \right\} ,$$

where $\lim_{x\to 0} c(x) = c \neq 0$, $\lim_{x\to 0} \bar{\varepsilon}(x) = 0$. Again since $\lim_{u\to 0} \bar{\varepsilon}(u) = 0$,

there exists a positive t_0 such that for $|t| < \theta < t_0^{-1}$ and $\varepsilon > 0$, $|\bar{\varepsilon}(u)| < \varepsilon$. Thus we have

$$h_1(a_r x_m^r t / W_m) \leqq \left| \frac{t}{\theta} \right|^{-\varepsilon} h_1(a_r x_m^r \theta / W_m) .$$

Now

$$\operatorname{Re} h_m(t) = (1/W_m^2) \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r t / W_m)$$

$$\leqq |t/\theta|^{-\varepsilon} (1/W_m^2) \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / W_m) \leqq |t/\theta|^{-\varepsilon}$$

(by (3.13)) .

But by (2.7), $h_m(t) = \operatorname{Re} h_m(t)(1 + o(1))$ as $t \to 0$. Therefore for $|t| < t_0^{-1}$ and $\varepsilon > 0$, we have

$$|h_m(t) < \mu_1 |t|^{-\varepsilon} .$$

Thus in a neighborhood of zero,

$$(3.15) \qquad |\phi_m(t) - 1| = \left| \exp\left\{ -\frac{t^2}{2} h_m(t) \right\} - 1 \right| \leqq \mu_1 |t|^{2-\varepsilon} .$$

By Gnedenko and Kolmogorov [5],

$$P\{|\sum_2 a_r X_r x_m^r| > \lambda_m W_m\} < 2 - \left| (\lambda_m/2) \int_{-2/\lambda_m}^{2/\lambda_m} \phi_m(t) dt \right|$$

$$\leqq (\lambda_m/2) \int_{-2/\lambda_m}^{2/\lambda_m} |\phi_m(t) - 1| dt \leqq \lambda_m \mu_1 \int_0^{2/\lambda_m} |t|^{2-\varepsilon} dt \quad \text{(by (3.15))} ,$$

$$\leqq \mu/\lambda_m^{2-\varepsilon} .$$

Hence the result.

Adopting the above procedure we can also prove the following lemma.

LEMMA 2.

$$|\sum_3 a_r X_r x_m^r| < \lambda_m Z_m ,$$

except for a set of measure at most $\mu/\lambda_m^{2-\varepsilon}$ *where*

$$Z_m^2 = \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / Z_m) .$$

Now we proceed to estimate R_m. By virtue of Lemma 1 and Lemma 2, we have

$$|R_m| < \lambda_m(W_m + Z_m) ,$$

for sufficiently large value of m.

Now if $V(X_r) = \infty$, we have

$$(3.16) \qquad |R_m| < \lambda_m k_n d\{(\textstyle\sum_2 x_m^{2r})^{1/2} + (\sum_3 x_m^{2r})^{1/2}\} \,,$$

where

$$d = \max_{0 \leq r \leq n} \{(h_1(a_r x_m^r \theta / W_m))^{1/2}, \ (h_1(a_r x_m^r \theta / Z_m))^{1/2}\} \,.$$

We have

$$\frac{\phi(4m + 3)}{\phi(4m + 1)} = \frac{(4m)^{[\log(4m+3)]+4m+3}(1 + 3/4m)^{[\log(4m+3)]+4m+3}}{(4m)^{[\log(4m+1)]+4m+1}(1 + 1/4m)^{[\log(4m+1)]+4m+1}}$$

$$> (4m)^{\log(4m+3/4m+1)+2} = 16m^2(4m)^{\log(4m+3/4m+1)} > m^2 \,.$$

Therefore

$$(3.17) \qquad \phi(4m + 3) > m^2\phi(4m + 1)$$

and similarly

$$(3.18) \qquad \phi(4m + 1) > m^2\phi(4m - 1) \,.$$

Now

$$(3.19) \qquad \begin{aligned}\textstyle\sum_2 x_m^{2r} &< 1 + \phi(4m - 1)M_n^{4m-1} < 2\phi(4m - 1)M_n^{4m-1}\\ &< (2/m^2)\phi(4m + 1)M_n^{4m-1} \ \text{(by (3.18))} \,,\end{aligned}$$

and

$$(\textstyle\sum_3 x_m^{2r}) < (\sum_{m^2\phi(4m+1)M_n^{4m+1}} x_m^{2r})$$

(since by (3.17), $m^2\phi(4m + 1) < m^2\phi(4m + 3)$),

$$= \phi(4m + 1)M_n^{4m}\Big\{M_n 1 - \frac{1}{\phi(4m + 1)M_n^{4m}}\Big\}^{m^2\phi(4m+1)M_n^{4m+1}}$$

$$< \phi(4m + 1)M_n^{4m}e^{-m^2M_n} < \phi(4m + 1)M_n^{4m}(m^2 M_n)^{-1} \ \text{(since } e^{-x} < x^{-1}) \,,$$

$$(3.20) \qquad = (1/m^2)\phi(4m + 1)M_n^{4m-1} \,.$$

Hence by (3.19) and (3.20) we have from (3.16),

$$|R_m| < d\lambda_m\frac{(\sqrt{2} + 1)}{m}\{\phi(4m + 1)\}^{1/2}(M_n^{2m}/M_n^{1/2})$$

$$< \frac{d(\sqrt{2} + 1)(Ae/D)^{1/2}(k_n/t_n)\log\log n V_m}{M_n^{1/2}}$$

(by (3.2) and (3.11)) , $< V_m$ (by (3.3)).

Again if $V(X_r) = \sigma^2 < \infty$, then

$$|R_m| < \lambda_m \sigma \{ (\sum_2 x_m^{2r})^{1/2} + (\sum_3 x_m^{2r})^{1/2} \}$$

$$< \frac{\log \log n (\sqrt{2} + 1)(D/Ae)^{1/2}(k_n/t_n) V_m}{M_n^{1/2}}$$

(by (3.2) and (3.12)),

$$< \frac{d(\sqrt{2} + 1)(k_n/t_n) \log \log n V_m}{M_n^{1/2}} . \quad \text{(since } d > 1.) \quad < V_m .$$

Since $k \to \infty$ as $n \to \infty$, it follows that when n is sufficiently large

$$|R_m| < V_m ,$$

for $m = [k/2] + 1, [k/2] + 2, \cdots, k$, except for a set of measure at most

(3.21) $(\mu/\lambda_m^{2-\varepsilon}) .$

Thus we have $|R_{2m}| < V_{2m}$ and $|R_{2m+1}| < V_{2m+1}$ for $m = m_0$, $m_0 + 1, \cdots, k$, where $m_0 = [k/2] + 1$.

The measure of the exceptional set is at most

(3.22) $(\mu'/\lambda_{2m}^{2-\varepsilon}) + (\mu'/\lambda_{2m+1}^{2-\varepsilon}) < (\mu'/\lambda_m^{2-\varepsilon}) .$

Again we proceed to estimate

$$P^* = P\{U_{2m} > V_{2m}, U_{2m+1} < -V_{2m+1}\} \cup \{U_{2m} < -V_{2m}, U_{2m+1} > V_{2m+1}\}$$
$$= P\{U_{2m} > V_{2m}\}P\{U_{2m+1} < -V_{2m+1}\}$$
$$+ P\{U_{2m} < -V_{2m}\}P\{U_{2m+1} > V_{2m+1}\} .$$

Let $G_m(x)$ and $g_m(t)$ be the distribution function and the characteristic function of (U_m/V_m) respectively. Then

$$g_m(t) = \exp \left\{ \frac{t^2}{2} \frac{1}{V_m^2} \sum_1 a_r^2 x_m^{2r} h(a_r x_m^r t/V_m) \right\} .$$

Let

(3.23) $$F(x) = \int_{-\infty}^{x} \exp(-u^2/2) du .$$

It follows from (3.11) and (3.12) that $V_m \to \infty$ as $m \to \infty$ and then $(a_r x_m^r t/V_m) \to 0$. Therefore when $m \to \infty$ we have by (2.8),

$$h(a_r x_m^r t/V_m) = h_1(a_r x_m^r t/V_m) (1 + o(1))$$

and by Theorem 1, it can be shown that

$$h_1(a_r x_m^r t/V_m) = \|\theta/t\|^{o(1)} h_1(a_r x_m^r \theta/V_m) (1 + o(1))$$

and as such

$$g_m(t) = \exp\left\{ -\frac{t^2}{2}\frac{1}{V_m^2}\sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta/V_m)\left|\frac{\theta}{t}\right|^{o(1)}(1+o(1))(1+o(1))\right\}$$

$$= \exp\left\{\frac{|t|^{2-o(1)}}{2}\left|\frac{\theta}{t}\right|^{o(1)}(1+o(1))\right\} \text{ (by (3.10))}.$$

Therefore as $m \to \infty$, $g_m(t) \to \exp(-t^2/2)$ uniformly in any bounded interval of t-values. Hence

(3.24) $$\sup_x |G_m(x) - F(x)| = o(1).$$

Then we have for $\varepsilon > 0$,

(3.25) $$|G_m(-1) - F(-1)| < \varepsilon$$

and

(3.26) $$|G_{2m+1}(-1) - F(-1)| < \varepsilon.$$

By (3.25) and (3.26), we have

$$P\{U_{2m} < -V_{2m}\} > F(-1) - \varepsilon$$

and

$$P\{U_{2m+1} < -V_{2m+1}\} > F(-1) - \varepsilon.$$

In the similar way using (3.24) we can show that

$$P\{U_{2m} > V_{2m}\} > 1 - F(1) - \varepsilon$$

and

$$P\{U_{2m+1} > V_{2m+1}\} > 1 - F(1) - \varepsilon.$$

Therefore $P^* > 2(F(1) - \varepsilon)(1 - F(1) - \varepsilon)$. Thus P^* is greater than a quantity which tends to $2F(-1)(1 - F(1))$ as $m \to \infty$ with n. This limit being positive we conclude that

(3.27) $$P^* > \delta > 0 \text{ for all large } m.$$

Now we define events E_m and F_m as follows:

$$E_m = \{U_{2m} > V_{2m},\ U_{2m+1} < -V_{2m+1}\},$$
$$F_m = \{U_{2m} < -V_{2m},\ U_{2m+1} > V_{2m+1}\}.$$

By (3.27), we have

$$P\{E_m \cup F_m\} > \delta > 0.$$

Let $P\{E_m \cup F_m\} = \delta_m$, so that $\delta_m > \delta > 0$.

Let y_m be the random variable such that it takes value 1 on $E_m \cup F_m$ and 0 elsewhere. In otherwords,

$$y_m = \begin{bmatrix} 1 \text{ with probability } \delta_m, \\ 0 \text{ with probability } 1 - \delta_m . \end{bmatrix}$$

The y_m's are thus independent random variables with $E(y_m) = 0$ and $V(y_m) = \delta_m - \delta_m^2 < 1$. We write

$$z_m = \begin{bmatrix} 0 \text{ if } |R_{2m}| < V_{2m} \text{ and } |R_{2m+1}| < V_{2m+1} \\ 1 \text{ otherwise} . \end{bmatrix}$$

Moreover, we have $f(x_{2m}) = U_{2m} + R_{2m}$ and $f(x_{2m+1}) = U_{2m+1} + R_{2m+1}$. Let $\alpha_m = y_m - y_m z_m$. Now $\alpha_m = 1$ only if $y_m = 1$ and $z_m = 0$, which implies the occurrence of one of the events

(i) $U_{2m} > V_{2m}, \ |R_{2m}| < V_{2m}$;

             $U_{2m+1} < -V_{2m+1}, \ |R_{2m+1}| < V_{2m+1}$,

(ii) $U_{2m} < -V_{2m}, \ |R_{2m}| < V_{2m}$;

             $U_{2m+1} > V_{2m+1}, \ |R_{2m+1}| < V_{2m+1}$.

It is obvious that (i) implies $f(x_{2m}) > 0$ and $f(x_{2m+1}) < 0$, and (ii) implies that $f(x_{2m}) < 0$ and $f(x_{2m+1}) > 0$. Thus if $\alpha_m = 1$, there is a root of the polynomial in the interval (x_{2m}, x_{2m+1}). Hence the number of roots in (x_{2m_0}, x_{2k+1}) must exceed $\sum_{m=m_0}^{k} \alpha_m$.

We appeal to the strong law of large numbers in the following form. The technique has been earlier used by Evans [4], Samal and Mishra [12] and [13].

Let $y_1, y_2, \cdots$, be a sequence of independent random variables identically distributed with $V(y_i) < 1$ for all i, then for each $\varepsilon > 0$,

$$(3.28) \qquad P\left\{ \sup_{k \geq k_0} \left| \frac{1}{k} \sum_{i=1}^{k} (y_i - E(y_i)) \right| > \varepsilon \right\} < B/\varepsilon^2 k_0 ,$$

where B is a positive constant.

In the present case,

$$(3.29) \quad \begin{aligned} \left| \sum_{m=m_0}^{k} (\alpha_m - E(y_m)) \right| &\leq \left| \sum_{m=m_0}^{k} (y_m - E(y_m)) \right| + \left| \sum_{m=m_0}^{k} y_m z_m \right| \\ &\leq \left| \sum_{m=m_0}^{k} (y_m - E(y_m)) \right| + \left| \sum_{m=m_0}^{k} z_m \right| \text{ (since } y_m \leq 1) . \end{aligned}$$

Since $E(z_m) = 1 \cdot P\{z_m = 1\} < P\{|R_m| > V_m\}$ we have from (3.21),

$$(3.30) \qquad E(z_m) < \mu/\lambda_m^{2-\varepsilon} .$$

Now we have

$$P\left\{ \sum_{m=m_0}^{k} z_m \geq (k - m_0 + 1)\varepsilon_1 \right\} < \mu/\lambda_{m_0}^{2-\varepsilon} .$$

Hence outside an exceptional set of measure at most

$$\sum_{(k-m_0+1)\geq k_0} (\mu/\lambda_{m_0}^{2-\varepsilon}) \,,$$

we have

$$\sup_{(k-m_0+1)\geq k_0} (1/(k-m_0+1)) \sum_{m=m_0}^{k} z_m < \varepsilon_1 \,;$$

and therefore,

$$\sup_{(k-m_0+1)k\geq k_0} (1/(k-m_0+1)) \left| \sum_{m=m_0}^{k} (\alpha_m - E(y_m)) \right|$$

$$\leq \sup_{(k-m_0+1)\geq k_0} (1/(k-m_0+1)) \left| \sum_{m=m_0}^{k} (y_m - E(y_m)) \right| + \varepsilon_1 \,.$$

Now by using strong law of large numbers,

$$P\left\{ \sup_{(k-m_0+1)\geq k_0} \left| (1/(k-m_0+1)) \sum_{m=m_0}^{k} (\alpha_m - E(y_m)) \right| > \varepsilon \right\}$$
$$< B/(\varepsilon - \varepsilon_1)^2 k_0 = \mu/k_0 \,.$$

By (3.1)

$$\lambda_{m_0}^{2-\varepsilon} = (m_0 \log\log n)^{2-\varepsilon} \,.$$

For large n, $m_0 \log\log n > m_0$, and therefore

$$\sum (\mu/\lambda_{m_0}^{2-\varepsilon}) < \sum (\mu/m_0^{2-\varepsilon}) \,.$$

Hence outside a set S_{k_0}, where

(3.31) $$P(S_{k_0}) < \mu/k_0 + \sum_{(k-m_0+1)\geq k_0} (\mu/m_0^{2-\varepsilon}) \,,$$

we have

(3.32) $$(1/(k-m_0+1)) \left| \sum_{m=m_0}^{k} (\alpha_m - E(y_m)) \right| < \varepsilon \,.$$

Also

$$E(y_m) = \delta_m > \delta \,.$$

Therefore,

$$N_n > \sum_{m=m_0}^{k} \alpha_m > \sum_{m=m_0}^{k} \delta - (k-m_0+1)\varepsilon > (k-[k/2])$$
$$> \mu(\log n/\log((k_n/t_n)\log\log n))^{1/2} \text{ (by (3.7))} \,,$$

for all k such that $k - m_0 + 1 > k_0$, or in otherwords for all $n > n_0$. Now

$$P(S_{k_0}) < (\mu/k_0) + \mu \sum_{k\geq(2k_0-1)} (1/m_0)^{2-\varepsilon}$$
$$= \frac{\mu}{k_0} + \mu \left\{ \frac{1}{k_0^{2-\varepsilon}} + 2\left(\frac{1}{k_0^{2-\varepsilon}+1} + \frac{1}{k_0^{2-\varepsilon}+2} + \cdots \right) \right\}$$
$$< (\mu/k_0) + 2\mu \sum_{k\geq k_0} (1/k^{2-\varepsilon}) \,.$$

It can be easily shown that for $0 < \varepsilon < 1$,

$$\sum_{k \geq k_0} (1/k^{2-\varepsilon}) < (1/(1-\varepsilon)k_0^{1-\varepsilon}) \, .$$

Hence

$$P(S_{k_0}) < (\mu/k_0) + (1/(1-\varepsilon)k_0^{1-\varepsilon}) < \mu_1/k_0^{1-\varepsilon}$$

(since by hypothesis $0 < \varepsilon < 1$, $k_0 > k_0^{(1-\varepsilon)/2}$),

$$< \mu'\{\log((k_{n_0}/t_{n_0}) \log \log n_0)/\log n_0\}^{1-\varepsilon} \quad \text{(by (3.7))} \, .$$

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