

# Pacific Journal of Mathematics

## **STRONG RESULT FOR REAL ZEROS OF RANDOM POLYNOMIALS**

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## STRONG RESULT FOR REAL ZEROS OF RANDOM POLYNOMIALS

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Let  $N_n$  be the number of real zeros of  $\sum_{r=0}^n a_r X_r x^r = 0$  where  $X_r$ 's are independent random variables identically distributed belonging to the domain of attraction of normal law;  $a_0, a_1, a_2, \dots, a_n$  are nonzero real numbers such that  $(k_n/t_n) = o(\log n)$  where  $k_n = \max_{0 \leq r \leq n} |a_r|$  and  $t_n = \min_{0 \leq r \leq n} |a_r|$ . Further we suppose that the coefficients have zero means and  $P\{X_r \neq 0\} > 0$ . Then there exists a positive integer  $n_0$  such that

$$P\left\{\sup_{n > n_0} (N_n/D_n) < \mu\right\} > 1 - \mu' \left\{\log \left( (k_{n_0}/t_{n_0}) \log \log n \right) / \log n_0\right\}^{1-\varepsilon/2}$$

for  $n > n_0$  and  $1 > \varepsilon > 0$  where  $D_n = (\log n / \log(k_n/t_n) \log \log n)^{(1-\varepsilon)/2}$ .

1. Let  $N_n$  be the number of real roots of a random algebraic equation

$$\sum_{r=0}^n X_r x^r = 0,$$

where  $X_r$ 's are independent, identically distributed random variables. The problem of finding the lower bound of  $N_n$  has been considered by various authors. Considering the coefficients as normally distributed or uniformly distributed in  $[-1, 1]$ , assuming the values  $+1$  or  $-1$  with equal probability Littlewood and Offord [8] have shown that  $N_n > \mu \log n / \log \log n$  except for a set of measure at most  $\mu' / \log n$ ,  $n$  being sufficiently large. Evans [4] has studied the strong version of Littlewood and Offord and has shown that in case of Gaussian distributed coefficients  $N_n$  is greater than  $\mu \log n / \log \log n$  except for a set of measure at most  $\mu' (\log \log n_0 / \log n_0)$  for  $n > n_0$ . The above result is strong in the following sense.

Theorem of Littlewood and Offord is of the form

$$P\{(N_n/D'_n) < \mu\} \longrightarrow 1 \text{ as } n \longrightarrow \infty,$$

where  $D'_n = \log n / \log \log n$ . But the theorem of Evans is of the form

$$P\left\{\sup_{n > n_0} (N_n/D'_n) < \mu\right\} \longrightarrow 1 \text{ as } n_0 \longrightarrow \infty.$$

Considering the coefficients of  $\sum_{r=0}^n a_r X_r x^r = 0$  as symmetric stable variables Samal and Mishra [13] have shown that

$$P\{(N_n/D_n^*) < \mu\} > 1 - \frac{\mu'}{\{\log(k_n/t_n) \log n\}(\log n)^{\alpha-1}} \text{ if } 1 \leq \alpha < 2$$

and

$$> 1 - \frac{\mu' \log(k_n/t_n) \log n}{\log n} \text{ if } \alpha = 2,$$

where  $k_n = \max_{0 \leq r \leq n} |a_r|$ ,  $t_n = \min_{0 \leq r \leq n} |a_r|$  and  $D_n^* = (\log n / \log((k_n/t_n) \log n))$ . Samal and Mishra [13] have studied the strong version of the above theorem and have shown that  $P\{\sup_{n > n_0} (N_n/D_n^*) < \mu\}$

$$> 1 - \frac{\mu}{\{\log(\log n_0 / \log(k_{n_0}/t_{n_0}) \log n_0)\}^{\alpha-1}} \text{ where } \alpha > 1.$$

Mishra and Nayak [9] have proved that

$$P\{(N_n/D_n^*) < \mu\} > 1 - \frac{\mu'}{\{\log((k_n/t_n) \log n)\}(\log n)^{1-\varepsilon}}$$

for every positive  $\varepsilon < 1$ , when the coefficients belong to the domain of attraction of the normal law.

Object of this paper is to show that

$$P\{\sup_{n > n_0} (N_n/D_n) < \mu\} > 1 - \mu' \left\{ \frac{\log((k_{n_0}/t_{n_0}) \log \log n_0)}{\log n_0} \right\}^{1/2}$$

for  $0 < \varepsilon < 1$ , when the coefficients belong to the domain of attraction of the normal law. Therefore it is a strong result of Mishra and Nayak.

Throughout this paper we shall denote  $\mu$ 's for positive constants which may assume different values in different occurrences and  $V(\cdot)$  for the variance of a random variable.

2. In the sequel we shall need the following definition, and theorem due to Karamata, (cf. Ibragimov and Linnik [6] p. 394), for the proof of our main result.

DEFINITION. A function  $H: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  is called a slowly varying function if

$$(2.1) \quad \lim_{x \rightarrow \infty} \frac{H(\gamma x)}{H(x)} = 1, (\gamma > 0).$$

We have a few characterizations of the slowly varying functions due to Karamata.

By writing  $H(1/t) = h(t)$ , we may define a slowly varying func-

tion  $h: \mathbf{R}_+ \rightarrow \mathbf{R}_+$  with the property that

$$(2.2) \quad \lim_{x \rightarrow 0} \frac{h(\gamma x)}{h(x)} = 1, \quad (\gamma > 0).$$

With this the Karamata theorem, (cf. Ibragimov and Linnik [6], p. 394), may be stated as follows.

**THEOREM 1.** *A slowly varying function  $h$  with the property (2.2) which is integrable on any finite interval may be represented in the form*

$$h(x) = c(x) \exp\left(-\int_a^x \frac{\bar{\varepsilon}(u)}{u} du\right),$$

where

$$\lim_{x \rightarrow 0} c(x) = c \neq 0, \quad \lim_{x \rightarrow 0} \bar{\varepsilon}(x) = 0 \text{ and } a > 0.$$

We establish the following formulae which will be necessary for the proof of the main theorem.

Let a sequence of independent and identically distributed random variables  $\{X_r\}$  with mean zero belong to the domain of attraction of the normal law. Then their common characteristic function  $\phi(t)$  is given by (cf. Ibragimov and Linnik [6], p. 91),

$$(2.3) \quad \log \phi(t) = -\frac{t^2}{2} H(|t|^{-1})(1 + o(1)),$$

where  $H(t)$  is a slowly varying function as  $t \rightarrow \infty$  and is given by the formula

$$(2.4) \quad H(x) = -\int_0^x u^2 d\psi(x) = \int_{-x}^x u^2 dG(u),$$

where  $\psi(x) = 1 - G(x) + G(x)$  and  $G(x)$  is the common distribution function.

Also

$$(2.5) \quad |\phi(t)| \sim \exp\left\{-\frac{t^2}{2} H(|t|^{-1})\right\}.$$

If we put  $H(1/t) = L(t)$ , then  $L(t)$  is slowly varying as  $t \rightarrow 0$ . Then (2.3) and (2.5) will take the forms

$$\log \phi(t) = -\frac{t^2}{2} L(|t|)(1 + o(1))$$

and

$$|\phi(t)| \sim \exp \left\{ -\frac{t^2}{2} L(|t|) \right\}$$

respectively. Since  $L(|t|)$  is positive we can write the characteristic function  $\phi$  as

$$(2.6) \quad \phi(t) = \exp \left\{ -\frac{t^2}{2} h(t) \right\}$$

where  $h(t) = L(|t|)(1 + o(1))$  with the property

$$(2.7) \quad h(t) = \operatorname{Re} h(t)(1 + o(1)),$$

as

$$\operatorname{Re} h(t) = L(|t|) (1 + o(1)).$$

Now  $h(t)$  is slowly varying as  $t \rightarrow 0$ , since for  $\gamma > 0$ ,

$$\lim_{t \rightarrow 0} \frac{h(\gamma t)}{h(t)} = \lim_{t \rightarrow 0} \frac{L(\gamma |t|)(1 + o(1))}{L(|t|)(1 + o(1))} = 1.$$

Consider the function  $h_1(t)$  determined by

$$h_1(t) = \begin{cases} \operatorname{Re} h(t) & \text{if } V(X_r) = \infty, \\ \sigma^2 & \text{if } V(X_r) = \sigma^2 < \infty. \end{cases}$$

Clearly  $h_1(t)$  is slowly varying in a neighborhood of the origin. By (2.7),

$$(2.8) \quad h(t) = h_1(t) (1 + o(1)), \text{ in both cases as } t \rightarrow 0.$$

Since expectation is zero, by virtue of (2.4), we have

$$\lim_{x \rightarrow \infty} H(x) = \int_{-\infty}^{\infty} u^2 dG(u) = \sigma^2.$$

Therefore when variance is infinite,  $\lim_{x \rightarrow \infty} H(x) = \infty$ , so that  $\lim_{t \rightarrow 0} L(t) = \infty$ . Thus we have for infinite variance,

$$(2.9) \quad \lim_{t \rightarrow 0} h_1(t) = \infty.$$

**THEOREM 2.** *Let*

$$f(x) = \sum_{r=0}^n a_r X_r x^r$$

*be a polynomial of degree  $n$ , where  $X_r$ 's are independent and identically distributed random variables which belong to the domain of attraction of the normal law, have zero means and  $P\{X_r \neq 0\} > 0$ . Let  $a_0, a_1, a_2 \cdots a_n$  be nonzero real number such that  $(k_n/t_n) = o(\log n)$*

where  $k_n = \max_{0 \leq r \leq n} |a_r|$  and  $t_n = \min_{0 \leq r \leq n} |a_r|$ . Then there exists a positive  $n_0$  such that the number of real roots of  $f(x) = 0$  is at least  $\mu \{ \log n / \log ((k_n/t_n) \log \log n) \}^{1/2}$  outside a set of measure at most  $\mu' \{ \log ((k_{n_0}/t_{n_0}) \log \log n_0) / \log n_0 \}^{(1-\varepsilon)/2}$  for  $n > n_0$  and  $1 > \varepsilon > 0$ .

**3. Proof of the Theorem 2.** Take constants  $A$  and  $D$  such that

$$(3.1) \quad 0 < D < 1 \quad \text{and} \quad A > 1 .$$

Let

$$(3.2) \quad \lambda_m = m \log \log n .$$

Let

$$(3.3) \quad M_n = [d^2(\log \log n)^2(k_n/t_n)^2(\sqrt{2} + 1)^2(Ae/D)] + 1 ,$$

where  $b$  is a positive constant greater than one whose choice will be made later and  $[x]$  denotes the greatest integer not exceeding  $x$ .

It follows from (3.3) that

$$(3.4) \quad \mu_1 \left( \frac{k_n}{t_n} \log \log n \right)^2 \leq M_n \leq \mu_2 \left( \frac{k_n}{t_n} \log \log n \right)^2 .$$

We define

$$(3.5) \quad \phi(x) = x^{\lceil \log x \rceil + x} .$$

Let  $k$  be the integer determined by

$$(3.6) \quad \phi(8k + 7)M_n^{8k+7} \leq n < \phi(8k + 11)M_n^{8k+11} .$$

The first inequality of (3.5) gives

$$\log \phi(8k + 7) + (8k + 7) \log M_n \leq \log n ,$$

or

$$(8k + 7) \log M_n < \log n ,$$

which by help of (3.4) yields

$$k < \frac{\mu \log n}{\log \left( \frac{k_n}{t_n} \log \log n \right)} .$$

Again the right hand side inequality of (3.4) gives

$$\begin{aligned} \log n &< \log \phi(8k + 11) + (8k + 11) \log M_n \\ &= (\log (8k + 11) + 8k + 11) \log (8k + 11) + (8k + 11) \log M_n \\ &< 2(8k + 11)^2 + (8k + 11) \log M_n < \mu_3 k^2 \log M_n , \end{aligned}$$

whence by (3.4), we have

$$\mu_0 \left( \frac{\log n}{\log(k_n/t_n \log \log n)} \right)^{1/2} < k.$$

Therefore

$$(3.7) \quad \mu_0 \left( \frac{\log n}{\log(k_n/t_n \log \log n)} \right)^{1/2} < k < \mu \frac{\log n}{\log(k_n/t_n \log \log n)}.$$

Since  $(k_n/t_n) = o(\log n)$  by hypothesis, it follows from (3.7), that  $k \rightarrow \infty$  as  $n \rightarrow \infty$ .

We have  $f(x_m) = U_m + R_m$  at the points

$$(3.8) \quad x_m = \left( 1 - \frac{1}{\phi(4m+1)M_n^{4m}} \right)^{1/2}$$

for  $m = [k/2] + 1, [k/2] + 2 \dots k$ , where

$$U_m = \sum_1 a_r X_r x_m^r$$

and

$$R_m = \left( \sum_2 + \sum_3 \right) a_r X_r x_m^r,$$

the index  $r$  ranging from  $\phi(4m-1)M_n^{4m-1} + 1$  to  $\phi(4m+3)M_n^{4m+3}$  in  $\sum_1$ , from 0 to  $\phi(4m-1)M_n^{4m-1}$  in  $\sum_2$  and from  $\phi(4m+3)M_n^{4m+3} + 1$  to  $n$  in  $\sum_3$ . (We shall use the notations  $\sum_1, \sum_2$  and  $\sum_3$  to carry the above meaning throughout this paper.)

We have also

$$(3.9) \quad f(x_{2m}) = U_{2m} + R_{2m}, \quad f(x_{2m+1}) = U_{2m+1} + R_{2m+1}.$$

By (3.7), we have  $2k+1 < n$  for large  $n$ . Also the maximum index in  $U_{2m+1}$  for  $m = k$  is  $\phi(8k+7)M_n^{8k+7}$ , which by (3.6) is consistent with (3.9).

We define normalizing constants  $V_m$  starting from the relation

$$(3.10) \quad (1/V_m^2) \sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / V_m),$$

where  $\theta$  is a small positive number whose final choice will be dealt with later. Such normalizing constants  $V_m$  always exist when  $\theta$  is sufficiently small. (Cf. Ibragimov and Maslova [7], p. 232.)

Now if  $V(X_r) = \infty$ , we have

$$V_m^2 = \sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / V_m) > \sum_1 a_r^2 x_m^{2r}$$

(by (2.9), since  $\theta$  is small),

$$\begin{aligned}
 &> t_n^2 \sum_{\substack{\phi(4m+1)M_n^{4m} \\ \phi(4m-1)M_n^{4m-1}+1}} x_m^{2r} \\
 &> t_n^2 \{ \phi(4m+1)M_n^{4m} - \phi(4m-1)M_n^{4m-1} \} \left\{ 1 - \frac{1}{\phi(4m+1)M_n^{4m}} \right\}^{\phi(4m+1)M_n^{4m}} \\
 &> t_n^2 \phi(4m+1)M_n^{4m} (D/Ae) .
 \end{aligned}$$

Or

$$(3.11) \quad M_n^{2m} < (Ae/D \phi(4m+1))^{1/2} (V_m/t_n) .$$

Again if  $V(X_r) = \sigma^2 < \infty$ , then

$$\begin{aligned}
 V_m^2 &= \sigma^2 \sum_1 a_r^2 x_m^{2r} \\
 &> \sigma^2 \phi(4m+1)M_n^{4m} (D/Ae) .
 \end{aligned}$$

Or

$$(3.12) \quad M_n^{2m} < (Ae/D \phi(4m+1))^{1/2} (V_m/\sigma t_n) .$$

The following lemmas are necessary for the proof of the theorem.

LEMMA 1.

$$|\sum_2 a_r X_r x_m^r| < \lambda_m W_m ,$$

except for a set of measure at most  $\mu/\lambda_m^{2-\epsilon}$  for  $\epsilon > 0$ , where

$$(3.13) \quad W_m^2 = \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / W_m) .$$

*Proof.* The characteristic function of  $(1/W_m) \sum_2 a_r X_r x_m^r$  is given by

$$\phi_m(t) = \exp\left(-\frac{t^2}{2} h_m(t)\right)$$

where

$$(3.14) \quad h_m(t) = (1/W_m^2) \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / W_m) .$$

We have by Theorem 1 for  $|t| < \theta$ ,

$$\begin{aligned}
 h_1(a_r x_m^r t / W_m) (h_1(a_r x_m^r \theta / W_m))^{-1} &= \frac{L(|a_r x_m^r t / W_m|)(1 + o(1))}{L(|a_r x_m^r \theta / W_m|)(1 + o(1))} \\
 &= \frac{c(|a_r x_m^r t / W_m|)(1 + o(1))}{c(|a_r x_m^r \theta / W_m|)(1 + o(1))} \exp \left\{ \int_{|a_r x_m^r t / W_m|}^{|a_r x_m^r \theta / W_m|} \frac{\bar{\epsilon}(u)}{u} du \right\} ,
 \end{aligned}$$

where  $\lim_{x \rightarrow 0} c(x) = c \neq 0$ ,  $\lim_{x \rightarrow 0} \bar{\epsilon}(x) = 0$ . Again since  $\lim_{u \rightarrow 0} \bar{\epsilon}(u) = 0$ ,

there exists a positive  $t_0$  such that for  $|t| < \theta < t_0^{-1}$  and  $\varepsilon > 0$ ,  $|\bar{\varepsilon}(u)| < \varepsilon$ . Thus we have

$$h_1(a_r x_m^r t / W_m) \leq \left| \frac{t}{\theta} \right|^{-\varepsilon} h_1(a_r x_m^r \theta / W_m).$$

Now

$$\begin{aligned} \operatorname{Re} h_m(t) &= (1/W_m^2) \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r t / W_m) \\ &\leq |t/\theta|^{-\varepsilon} (1/W_m^2) \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / W_m) \leq |t/\theta|^{-\varepsilon} \\ &\text{(by (3.13))}. \end{aligned}$$

But by (2.7),  $h_m(t) = \operatorname{Re} h_m(t)(1 + o(1))$  as  $t \rightarrow 0$ . Therefore for  $|t| < t_0^{-1}$  and  $\varepsilon > 0$ , we have

$$|h_m(t)| < \mu_1 |t|^{-\varepsilon}.$$

Thus in a neighborhood of zero,

$$(3.15) \quad |\phi_m(t) - 1| = \left| \exp \left\{ -\frac{t^2}{2} h_m(t) \right\} - 1 \right| \leq \mu_1 |t|^{2-\varepsilon}.$$

By Gnedenko and Kolmogorov [5],

$$\begin{aligned} P\{|\sum_2 a_r X_r x_m^r| > \lambda_m W_m\} &< 2 - \left| (\lambda_m/2) \int_{-2/\lambda_m}^{2/\lambda_m} \phi_m(t) dt \right| \\ &\leq (\lambda_m/2) \int_{-2/\lambda_m}^{2/\lambda_m} |\phi_m(t) - 1| dt \leq \lambda_m \mu_1 \int_0^{2/\lambda_m} |t|^{2-\varepsilon} dt \quad \text{(by (3.15))}, \\ &\leq \mu/\lambda_m^{2-\varepsilon}. \end{aligned}$$

Hence the result.

Adopting the above procedure we can also prove the following lemma.

**LEMMA 2.**

$$|\sum_3 a_r X_r x_m^r| < \lambda_m Z_m,$$

except for a set of measure at most  $\mu/\lambda_m^{2-\varepsilon}$  where

$$Z_m^2 = \sum_2 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / Z_m).$$

Now we proceed to estimate  $R_m$ . By virtue of Lemma 1 and Lemma 2, we have

$$|R_m| < \lambda_m (W_m + Z_m),$$

for sufficiently large value of  $m$ .

Now if  $V(X_r) = \infty$ , we have

$$(3.16) \quad |R_m| < \lambda_m k_n d \{ (\sum_2 x_m^{2r})^{1/2} + (\sum_3 x_m^{2r})^{1/2} \},$$

where

$$d = \max_{0 \leq r \leq n} \{ (h_1(a_r x_m^r \theta / W_m))^{1/2}, (h_1(a_r x_m^r \theta / Z_m))^{1/2} \}.$$

We have

$$\begin{aligned} \frac{\phi(4m+3)}{\phi(4m+1)} &= \frac{(4m)^{\lceil \log(4m+3) \rceil + 4m+3} (1 + 3/4m)^{\lceil \log(4m+3) \rceil + 4m+3}}{(4m)^{\lceil \log(4m+1) \rceil + 4m+1} (1 + 1/4m)^{\lceil \log(4m+1) \rceil + 4m+1}} \\ &> (4m)^{\log(4m+3/4m+1)+2} = 16m^2(4m)^{\log(4m+3/4m+1)} > m^2. \end{aligned}$$

Therefore

$$(3.17) \quad \phi(4m+3) > m^2 \phi(4m+1)$$

and similarly

$$(3.18) \quad \phi(4m+1) > m^2 \phi(4m-1).$$

Now

$$(3.19) \quad \begin{aligned} \sum_2 x_m^{2r} &< 1 + \phi(4m-1) M_n^{4m-1} < 2\phi(4m-1) M_n^{4m-1} \\ &< (2/m^2) \phi(4m+1) M_n^{4m-1} \quad (\text{by (3.18)}), \end{aligned}$$

and

$$\begin{aligned} (\sum_3 x_m^{2r}) &< \left( \sum_{m^2 \phi(4m+1) M_n^{4m+1}} x_m^{2r} \right) \\ &(\text{since by (3.17), } m^2 \phi(4m+1) < m^2 \phi(4m+3)), \\ &= \phi(4m+1) M_n^{4m} \left\{ M_n - \frac{1}{\phi(4m+1) M_n^{4m}} \right\}^{m^2 \phi(4m+1) M_n^{4m+1}} \\ &< \phi(4m+1) M_n^{4m} e^{-m^2 M_n} < \phi(4m+1) M_n^{4m} (m^2 M_n)^{-1} \quad (\text{since } e^{-x} < x^{-1}), \\ (3.20) \quad &= (1/m^2) \phi(4m+1) M_n^{4m-1}. \end{aligned}$$

Hence by (3.19) and (3.20) we have from (3.16),

$$\begin{aligned} |R_m| &< d \lambda_m \frac{(\sqrt{2} + 1)}{m} \{ \phi(4m+1) \}^{1/2} (M_n^{2m} / M_n^{1/2}) \\ &< \frac{d(\sqrt{2} + 1)(Ae/D)^{1/2} (k_n/t_n) \log \log n V_m}{M_n^{1/2}} \end{aligned}$$

(by (3.2) and (3.11)),  $< V_m$  (by (3.3)).

Again if  $V(X_r) = \sigma^2 < \infty$ , then

$$|R_m| < \lambda_m \sigma \{ \sum_2 x_m^{2r} \}^{1/2} + \{ \sum_3 x_m^{2r} \}^{1/2} \\ < \frac{\log \log n (\sqrt{2} + 1) (D/Ae)^{1/2} (k_n/t_n) V_m}{M_n^{1/2}}$$

(by (3.2) and (3.12)),

$$< \frac{d(\sqrt{2} + 1)(k_n/t_n) \log \log n V_m}{M_n^{1/2}}. \quad (\text{since } d > 1.) < V_m.$$

Since  $k \rightarrow \infty$  as  $n \rightarrow \infty$ , it follows that when  $n$  is sufficiently large

$$|R_m| < V_m,$$

for  $m = [k/2] + 1, [k/2] + 2, \dots, k$ , except for a set of measure at most

$$(3.21) \quad (\mu/\lambda_m^{2-\epsilon}).$$

Thus we have  $|R_{2m}| < V_{2m}$  and  $|R_{2m+1}| < V_{2m+1}$  for  $m = m_0, m_0 + 1, \dots, k$ , where  $m_0 = [k/2] + 1$ .

The measure of the exceptional set is at most

$$(3.22) \quad (\mu'/\lambda_{2m}^{2-\epsilon}) + (\mu'/\lambda_{2m+1}^{2-\epsilon}) < (\mu'/\lambda_m^{2-\epsilon}).$$

Again we proceed to estimate

$$P^* = P\{U_{2m} > V_{2m}, U_{2m+1} < -V_{2m+1}\} \cup \{U_{2m} < -V_{2m}, U_{2m+1} > V_{2m+1}\} \\ = P\{U_{2m} > V_{2m}\}P\{U_{2m+1} < -V_{2m+1}\} \\ + P\{U_{2m} < -V_{2m}\}P\{U_{2m+1} > V_{2m+1}\}.$$

Let  $G_m(x)$  and  $g_m(t)$  be the distribution function and the characteristic function of  $(U_m/V_m)$  respectively. Then

$$g_m(t) = \exp \left\{ \frac{t^2}{2} \frac{1}{V_m^2} \sum_1 a_r^2 x_m^{2r} h(a_r x_m^r t / V_m) \right\}.$$

Let

$$(3.23) \quad F(x) = \int_{-\infty}^x \exp(-u^2/2) du.$$

It follows from (3.11) and (3.12) that  $V_m \rightarrow \infty$  as  $m \rightarrow \infty$  and then  $(a_r x_m^r t / V_m) \rightarrow 0$ . Therefore when  $m \rightarrow \infty$  we have by (2.8),

$$h(a_r x_m^r t / V_m) = h_1(a_r x_m^r t / V_m) (1 + o(1))$$

and by Theorem 1, it can be shown that

$$h_1(a_r x_m^r t / V_m) = \|\theta/t\|^{o(1)} h_1(a_r x_m^r \theta / V_m) (1 + o(1))$$

and as such

$$\begin{aligned}
 g_m(t) &= \exp \left\{ -\frac{t^2}{2} \frac{1}{V_m^2} \sum_1 a_r^2 x_m^{2r} h_1(a_r x_m^r \theta / V_m) \left| \frac{\theta}{t} \right|^{o(1)} (1+o(1))(1+o(1)) \right\} \\
 &= \exp \left\{ \frac{|t|^{2-o(1)}}{2} \left| \frac{\theta}{t} \right|^{o(1)} (1+o(1)) \right\} \text{ (by (3.10)) .}
 \end{aligned}$$

Therefore as  $m \rightarrow \infty$ ,  $g_m(t) \rightarrow \exp(-t^2/2)$  uniformly in any bounded interval of  $t$ -values. Hence

$$(3.24) \quad \sup_x |G_m(x) - F(x)| = o(1) .$$

Then we have for  $\varepsilon > 0$ ,

$$(3.25) \quad |G_m(-1) - F(-1)| < \varepsilon$$

and

$$(3.26) \quad |G_{2m+1}(-1) - F(-1)| < \varepsilon .$$

By (3.25) and (3.26), we have

$$P\{U_{2m} < -V_{2m}\} > F(-1) - \varepsilon$$

and

$$P\{U_{2m+1} < -V_{2m+1}\} > F(-1) - \varepsilon .$$

In the similar way using (3.24) we can show that

$$P\{U_{2m} > V_{2m}\} > 1 - F(1) - \varepsilon$$

and

$$P\{U_{2m+1} > V_{2m+1}\} > 1 - F(1) - \varepsilon .$$

Therefore  $P^* > 2(F(1) - \varepsilon)(1 - F(1) - \varepsilon)$ . Thus  $P^*$  is greater than a quantity which tends to  $2F(-1)(1 - F(1))$  as  $m \rightarrow \infty$  with  $n$ . This limit being positive we conclude that

$$(3.27) \quad P^* > \delta > 0 \text{ for all large } m .$$

Now we define events  $E_m$  and  $F_m$  as follows:

$$\begin{aligned}
 E_m &= \{U_{2m} > V_{2m}, U_{2m+1} < -V_{2m+1}\} , \\
 F_m &= \{U_{2m} < -V_{2m}, U_{2m+1} > V_{2m+1}\} .
 \end{aligned}$$

By (3.27), we have

$$P\{E_m \cup F_m\} > \delta > 0 .$$

Let  $P\{E_m \cup F_m\} = \delta_m$ , so that  $\delta_m > \delta > 0$ .

Let  $y_m$  be the random variable such that it takes value 1 on  $E_m \cup F_m$  and 0 elsewhere. In otherwords,

$$y_m = \begin{cases} 1 & \text{with probability } \delta_m, \\ 0 & \text{with probability } 1 - \delta_m. \end{cases}$$

The  $y_m$ 's are thus independent random variables with  $E(y_m) = 0$  and  $V(y_m) = \delta_m - \delta_m^2 < 1$ . We write

$$z_m = \begin{cases} 0 & \text{if } |R_{2m}| < V_{2m} \text{ and } |R_{2m+1}| < V_{2m+1} \\ 1 & \text{otherwise.} \end{cases}$$

Moreover, we have  $f(x_{2m}) = U_{2m} + R_{2m}$  and  $f(x_{2m+1}) = U_{2m+1} + R_{2m+1}$ . Let  $\alpha_m = y_m - y_m z_m$ . Now  $\alpha_m = 1$  only if  $y_m = 1$  and  $z_m = 0$ , which implies the occurrence of one of the events

- (i)  $U_{2m} > V_{2m}, |R_{2m}| < V_{2m};$   
 $U_{2m+1} < -V_{2m+1}, |R_{2m+1}| < V_{2m+1},$
- (ii)  $U_{2m} < -V_{2m}, |R_{2m}| < V_{2m};$   
 $U_{2m+1} > V_{2m+1}, |R_{2m+1}| < V_{2m+1}.$

It is obvious that (i) implies  $f(x_{2m}) > 0$  and  $f(x_{2m+1}) < 0$ , and (ii) implies that  $f(x_{2m}) < 0$  and  $f(x_{2m+1}) > 0$ . Thus if  $\alpha_m = 1$ , there is a root of the polynomial in the interval  $(x_{2m}, x_{2m+1})$ . Hence the number of roots in  $(x_{2m_0}, x_{2k+1})$  must exceed  $\sum_{m=m_0}^k \alpha_m$ .

We appeal to the strong law of large numbers in the following form. The technique has been earlier used by Evans [4], Samal and Mishra [12] and [13].

Let  $y_1, y_2, \dots$ , be a sequence of independent random variables identically distributed with  $V(y_i) < 1$  for all  $i$ , then for each  $\varepsilon > 0$ ,

$$(3.28) \quad P\left\{ \sup_{k \geq k_0} \left| \frac{1}{k} \sum_{i=1}^k (y_i - E(y_i)) \right| > \varepsilon \right\} < B/\varepsilon^2 k_0,$$

where  $B$  is a positive constant.

In the present case,

$$(3.29) \quad \left| \sum_{m=m_0}^k (\alpha_m - E(y_m)) \right| \leq \left| \sum_{m=m_0}^k (y_m - E(y_m)) \right| + \left| \sum_{m=m_0}^k y_m z_m \right| \\ \leq \left| \sum_{m=m_0}^k (y_m - E(y_m)) \right| + \left| \sum_{m=m_0}^k z_m \right| \text{ (since } y_m \leq 1 \text{)}.$$

Since  $E(z_m) = 1 \cdot P\{z_m = 1\} < P\{|R_m| > V_m\}$  we have from (3.21),

$$(3.30) \quad E(z_m) < \mu/\lambda_m^{2-\varepsilon}.$$

Now we have

$$P\left\{ \sum_{m=m_0}^k z_m \geq (k - m_0 + 1)\varepsilon_1 \right\} < \mu/\lambda_{m_0}^{2-\varepsilon}.$$

Hence outside an exceptional set of measure at most

$$\sum_{(k-m_0+1) \geq k_0} (\mu/\lambda_{m_0}^{2-\varepsilon}),$$

we have

$$\sup_{(k-m_0+1) \geq k_0} (1/(k - m_0 + 1)) \sum_{m=m_0}^k z_m < \varepsilon_1;$$

and therefore,

$$\begin{aligned} & \sup_{(k-m_0+1) \geq k_0} (1/(k - m_0 + 1)) \left| \sum_{m=m_0}^k (\alpha_m - E(y_m)) \right| \\ & \leq \sup_{(k-m_0+1) \geq k_0} (1/(k - m_0 + 1)) \left| \sum_{m=m_0}^k (y_m - E(y_m)) \right| + \varepsilon_1. \end{aligned}$$

Now by using strong law of large numbers,

$$\begin{aligned} P \left\{ \sup_{(k-m_0+1) \geq k_0} \left| (1/(k - m_0 + 1)) \sum_{m=m_0}^k (\alpha_m - E(y_m)) \right| > \varepsilon \right\} \\ < B/(\varepsilon - \varepsilon_1)^2 k_0 = \mu/k_0. \end{aligned}$$

By (3.1)

$$\lambda_{m_0}^{2-\varepsilon} = (m_0 \log \log n)^{2-\varepsilon}.$$

For large  $n$ ,  $m_0 \log \log n > m_0$ , and therefore

$$\sum (\mu/\lambda_{m_0}^{2-\varepsilon}) < \sum (\mu/m_0^{2-\varepsilon}).$$

Hence outside a set  $S_{k_0}$ , where

$$(3.31) \quad P(S_{k_0}) < \mu/k_0 + \sum_{(k-m_0+1) \geq k_0} (\mu/m_0^{2-\varepsilon}),$$

we have

$$(3.32) \quad (1/(k - m_0 + 1)) \left| \sum_{m=m_0}^k (\alpha_m - E(y_m)) \right| < \varepsilon.$$

Also

$$E(y_m) = \delta_m > \delta.$$

Therefore,

$$\begin{aligned} N_n & > \sum_{m=m_0}^k \alpha_m > \sum_{m=m_0}^k \delta - (k - m_0 + 1)\varepsilon > (k - [k/2]) \\ & > \mu(\log n/\log((k_n/t_n) \log \log n))^{1/2} \text{ (by (3.7))}, \end{aligned}$$

for all  $k$  such that  $k - m_0 + 1 > k_0$ , or in otherwords for all  $n > n_0$ .

Now

$$\begin{aligned} P(S_{k_0}) & < (\mu/k_0) + \mu \sum_{k \geq (2k_0-1)} (1/m_0)^{2-\varepsilon} \\ & = \frac{\mu}{k_0} + \mu \left\{ \frac{1}{k_0^{2-\varepsilon}} + 2 \left( \frac{1}{k_0^{2-\varepsilon} + 1} + \frac{1}{k_0^{2-\varepsilon} + 2} + \dots \right) \right\} \\ & < (\mu/k_0) + 2\mu \sum_{k \geq k_0} (1/k^{2-\varepsilon}). \end{aligned}$$

It can be easily shown that for  $0 < \varepsilon < 1$ ,

$$\sum_{k \geq k_0} (1/k^{2-\varepsilon}) < (1/(1-\varepsilon))k_0^{1-\varepsilon}.$$

Hence

$$P(S_{k_0}) < (\mu/k_0) + (1/(1-\varepsilon))k_0^{1-\varepsilon} < \mu_1/k_0^{1-\varepsilon}$$

(since by hypothesis  $0 < \varepsilon < 1$ ,  $k_0 > k_0^{(1-\varepsilon)/2}$ ),

$$< \mu' \{ \log((k_{n_0}/t_{n_0}) \log \log n_0) / \log n_0 \}^{1-\varepsilon} \text{ (by (3.7))}.$$

The authors are thankful to the referee for his valuable comments.

#### REFERENCES

1. J. E. A. Dunnage, *The number of real zeros of a class of random algebraic polynomials*, Proc. London Math. Soc., (3) **18** (1968), 439-460.
2. ———, *The number of real zeros of a class of random algebraic polynomials II*, Quart. J. Math. Oxford, (2), **21** (1970), 309-319.
3. ———, *The number of real zeros of a class of random algebraic polynomials III*, J. London Math. Soc., (2) **6** (1972), 143-152.
4. E. A. Evans, *On the number of real roots of a random algebraic equation*, Proc. London Math. Soc., **15** (1965), 731-749.
5. B. V. Gnedenko and A. N. Kolmogorov, *Limit Distribution of Sums of Independent Random Variables*, Addison Wesley, Inc., 1954.
6. I. A. Ibragimov and Yu. V. Linnik, *Independent and Stationary Sequence of Random Variables*, Wolters-Noordhoff Publishing Groningen, 1972.
7. I. A. Ibragimov and N. B. Maslova, *On the expected number of real zeros of random algebraic polynomial 1. Coefficients with zero means*, (Translated by B. Seckler), Theor. Probability Application, XVI, 2 (1971), 228-248.
8. J. E. Littlewood and A. C. Offord, *On the number of real roots of a random algebraic equation II*, Proc. Cambridge Phil. Soc., **35** (1939), 133-148.
9. M. N. Mishra and N. N. Nayak, *On the lower bound of the number of real roots of a random algebraic equation*, accepted in the Journal of I. M. S. for publication.
10. G. Samal, *On the number of real roots of a random algebraic equation*, Proc. Cambridge Phils. Soc., **58** (Part 3) (1962), 433-442.
11. G. Samal and M. N. Mishra, *On the lower bound of the number of real roots of a random algebraic equation with infinite variance*, Proc. Amer. Math. Soc., **33** (1972), 523-528.
12. ———, *On the lower bound of the number of real roots of a random algebraic equation with infinite variance II*, Proc. Amer. Math. Soc., **36** (1972), 557-563.
13. ———, *On the lower bound of the number of real roots of a random algebraic equation with infinite variance III*, Proc. Amer. Math. Soc., **39** (1973), 184-189.

Received August 19, 1980 and in revised form September 16, 1981.

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Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).  
8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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