ESTIMATES OF MEROMORPHIC FUNCTIONS AND SUMMABILITY THEOREMS

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The main goal of this paper is to prove the following theorem.

**Theorem 1.** Let $L$ be an unbounded operator in a Hilbert space $\mathcal{H}$, having a discrete spectrum $\{\lambda_j\}_{j\in\mathbb{Z}}$ where $B_R = \{\lambda: |\lambda| \leq R\}$, $P_{q,h} = \{\lambda: \Re\lambda \leq 0, |\lambda| > 1, |\Im\lambda| \leq h(\Re\lambda)^q, h > 0, -\infty < q < 1\}$, and for some $\gamma < \infty$, $L^{-1} \in \sigma_T$. Also let the estimate

$$
\| (I - L)^{-1} \| \leq C d^{-t}(\lambda, G), \lambda \in G
$$

hold outside the domain $G' = B_R \cup P_{q,2h}$, and for some $a > 0$, $p > 0$

$$
\sum_{|\lambda_j| \leq t} 1 = n(t) \leq dt^p
$$

provided $t$ is sufficiently large.

Then $L \in A(\alpha, \mathcal{H})$ for any $\alpha > \max 0, p - (1 - q)$.

Besides, if the numbers $a$ or $h$ can be chosen arbitrarily small and $p - (1 - q) > 0$, then $\alpha = p - (1 - q)$ is admissible.

**Introduction.** Let $L$ be an unbounded linear operator in a separable complex Hilbert space $\mathcal{H}$ with domain of definition $\mathcal{D}(L)$ which is dense in $\mathcal{H}$, having a discrete spectrum $\sigma(L)$. Let $\{e_j\}_{j=1}^\infty$ be a sequence consisting of bases in the root subspaces of $L$, where $e_j$ is a root vector corresponding to the eigenvalue $\lambda_j$. To each vector $x \in \mathcal{H}$ we associate its Fourier series $\sum (x, e_j^*)e_j$ with respect to this system (not necessarily convergent), where $\{e_j^*\}$ is a system which is biorthogonal to $\{e_j\}$.

We write $L \in A_\alpha(\alpha, \mathcal{M}, \mathcal{H})$ if for an arbitrary vector $x$ in $\mathcal{M}$, where $\mathcal{M}$ is some linear manifold in $\mathcal{H}$, the Fourier series $\sum (x, e_j^*)e_j$ is summable in $\mathcal{H}$ to $x$ by the Abel method of order $\alpha$ with parenthesis.

If we suppose that $L$ has no associated vectors and all its eigenvalues $\{\lambda_j\}$ lie in the sector $A_\theta = \{\lambda: |\arg\lambda| \leq \pi/2\theta, 1/2 \leq \theta < \infty\}$ then the Abel method of summability of order $\alpha(\alpha \leq \theta)$ consists in replacing the series $\sum (x, e_j^*)e_j$ by series

$$
(1) \quad u_x(t) = \sum_{j=1}^n e^{-i\xi_j t}(x, e_j^*)e_j;
$$

it is required that for any $t > 0$ after possible recombination of its terms and appropriate use of parenthesis (not depending on $x \in \mathcal{M}$, or $t > 0$) this series converges in $\mathcal{H}$ and its sum $u_x(t)$ converges...
to \( x \) in \( \mathcal{H} \) as \( t \to +0 \). The branch of the function \( \lambda^a \) in (1) is selected so that \( \lambda^a > 0 \) if \( \lambda > 0 \). In the general case, when there do exist associated vectors, the factors for the vectors \( e_j \) in the series (1) are defined by calculating the integral

\[
\frac{1}{2\pi i} \int e^{-it} (\lambda I - L)^{-1} x d\lambda
\]

along a contour which surrounds a corresponding eigenvalue (see [9], where the Abel method was first introduced).

By \( \sigma_\ast \) we denote the collection of all compact operators \( A \), for which \( \sum s_j(A) < \infty \), where \( s_j(A) \) are eigenvalues of operator \( (AA^*)^{1/2} \), and by \( \sigma_\infty \) the collection of all compact operators.

The following result combines those of many authors.

**Theorem.** Let \( L \) be an unbounded operator in a Hilbert space \( \mathcal{H} \) having a discrete spectrum \( \{\lambda_j\} \subseteq G = B_R \cup \Lambda_\theta \), where \( B_R = \{\lambda: |\lambda| \leq R\}, \Lambda_\theta = \{\lambda: \arg \lambda \leq \pi/2\theta\} \), and its inverse operator \( L^{-1} \in \sigma_r \) for some \( \gamma < \theta \). If the estimate

\[
\| (\lambda I - L)^{-1} \| \leq Cd^{-1}(\lambda, G), \ \lambda \in G
\]

holds outside the domain \( G \), where \( d(\lambda, G) \) is the distance between \( \lambda \) and \( G \), then

1. The system of root vectors of operator \( L \) is complete in the space \( \mathcal{H} \).
2. \( L \in \mathcal{A}(\alpha, \mathcal{H}) \), if \( \alpha \in (\gamma, \theta) \).

M. V. Keldysh [6], [7] proved the first assertion in the case \( L = (I + V)H \), where \( H = H^* > 0 \), \( V \in \sigma_\infty \). Subsequently, the Keldysh method was generalized by many authors, in particular, in a similar form the first assertion was proved by S. Agmon [1], by I. C. Gohberg and M. G. Krein [3]. The second assertion is much stronger. In [9] V. B. Lidskii proved, that \( L \in \mathcal{A}(\alpha, D(L), \mathcal{H}) \), if \( \alpha \in (\gamma, \theta) \). Recently V. I. Macaev noticed that, indeed, the second assertion holds.\(^1\)

In many cases the spectrum of operator \( L \) lies asymptotically in an arbitrarily small sector \( \Lambda_\theta \), i.e., the number \( \theta \) may be chosen arbitrarily large. Such cases occur for some differential operators and are valid for operators which can be represented in the form \( L = (I + V)H \), where \( H > 0 \) and \( V \in \sigma_\infty \). In this situation the interval for \( \alpha \) is equal to \( (\gamma, \infty) \). For applications (see [9]) the most important case is when the order of summability \( \alpha = 1 \). In

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\(^1\) This note is reported in the appendix to the book [5]. The appendix is written by M. S. Agranovich.
this connection it is highly important to clear up the general conditions under which the interval for \( \alpha \) can be extended. Indeed, it can be extended if the spectrum of operator \( L \) lies asymptotically not only in an arbitrarily small sector but in some domain which is bounded by parabolas, lines, or hyperbolas.

The following theorem, which can be considered as the continuation of the previous theorem, formulates the exact result.

**Theorem 1.** Let \( L \) be an unbounded operator in a Hilbert space \( \mathcal{H} \), having a discrete spectrum \( \{ \lambda_j \} \subset G = B_r \cup P_{r,h} \), where \( B_r = \{ \lambda : |\lambda| \leq R \} \), \( P_{r,h} = \{ \lambda : \text{Re} \lambda \geq 0, |\lambda| > 1, |\text{Im} \lambda| \leq h (\text{Re} \lambda)^q, h > 0, -\infty < q < 1 \} \), and for some \( \gamma < \infty \), \( L^{-1} \in \mathcal{S} \). Also let the estimate
\[
\| (I - L)^{-1} \| \leq C d^{-\gamma}(\lambda, G), \quad \lambda \in G
\]
hold outside the domain \( G' = B_r \cup P_{r,2h} \), and for some \( a > 0, p > 0 \)
\[
\sum_{|\lambda_j| \leq t} 1 = n(t) \leq at^p
\]
provided \( t \) is sufficiently large.

Then \( L \in \mathcal{S}(\alpha, \mathcal{H}) \) for any \( \alpha > \max 0, p - (1 - q) \).

Besides, if the numbers \( a \) or \( h \) can be chosen arbitrarily small and \( p - (1 - q) > 0 \), then \( \alpha = p - (1 - q) \) is admissible.

Some results about extension of the interval for \( \alpha \) were obtained by V. B. Lidskii [10], by V. E. Katznelson and M. S. Agranovich (see [2]). All these results dealt with operators which can be represented in the form of a weakly perturbed self-adjoint positive operator, and the proofs of the appropriate statements used the specific properties of those operators.

Theorem 1 includes and generalizes these results. For its proof we use another more general method, where new estimates for meromorphic functions play the basic role. These estimates have independent significance; the following theorem formulates the relevant result.

**Theorem 2.** Let \( F(\lambda) \) be a meromorphic function of finite order \( \gamma \) in the sector \( \Lambda_\theta \), and its poles \( \{ \lambda_j \} \) lie in the domain \( P_{r,h} = \{ \lambda : \text{Re} \lambda > 0, |\lambda| > 1, |\text{Im} \lambda| < h (\text{Re} \lambda)^q, h > 0, -\infty < q < 1 \} \). Also let the estimate \( |F(\lambda)| \leq C \) hold on the boundary of the domain \( P_{r,2h} \) and for some \( a > 0, p > 0 \)
\[
\sum_{|\lambda_j| \leq t} 1 = n(t) \leq at^p
\]
provided \( t \) is sufficiently large.
Then there exists a sequence \( r_1 < r_2 < \cdots < r_k \to \infty \), such that the estimate

\[
|F(\lambda)| \leq C \exp(\sigma ah |\lambda|^{p-(1-\eta)})
\]

holds for all \( |\lambda| = r_h, \lambda \in P_{q,h} \), where the constants \( C, \sigma \) do not depend on \( \lambda, a, h \), if \( 0 < h < h_0 \) and \( h_0 \) is any fixed number.

In the case when the function \( F(\lambda) \) is meromorphic in the whole complex plane and has the finite order \( \gamma \), and when its poles \( \{\lambda_j\} \) are scattered in the sector \( A_\theta, \theta > \gamma, n(t) \leq at^\gamma \) and \( |F(\lambda)| \leq C \) on \( \partial A_\theta \) one can obtain, using the well-known theorem of Titchmarsh (see for example [1], p. 278), the following estimate

\[
|F(\lambda)| \leq C \exp|\lambda|^{p+\varepsilon}, \varepsilon > 0, |\lambda| = r_k, \lambda \in A_\theta,
\]

where \( r_1 < r_2 < \cdots < r_k \to \infty \). This estimate was used by V. B. Lidskii [9] for his summability theorem.

We note that in the case when \( \{\lambda_j\} \) are concentrated close to the real axis, namely, in some domain \( P_{q,h}, q < 1 \), Theorem 2 gives a much sharper estimate.

**Proof of Theorems 1 and 2.** In this section we will denote

1. \( A_\theta = \{\lambda: \arg |\lambda| \leq \pi/2\theta\} \),
2. \( P_{q,h} = \{\lambda: \Re \lambda > 0, |\lambda| > 1, |\Im \lambda| < h(\Re \lambda)^q, h > 0, -\infty < q < 1\} \),
3. \( P_{q,h}^+ = \{\lambda: \Re \lambda > 0, |\lambda| > 1, 0 < |\Im \lambda| < h(\Re \lambda)^q, h > 0, -\infty < q < 1\} \),
4. \( B(z, r) = \{\lambda: |\lambda - z| \leq r\}, B(0, r) = B_r \).

If the sequence \( \{\lambda_n\} \) lies in the upper half-plane \( (\Im \lambda_n > 0) \) and

\[
\sum_{n=1}^{\infty} \frac{\Im \lambda_n}{1 + |\lambda_n|^2} < \infty
\]

then the product

\[
B(\lambda) = \prod_{n=1}^{\infty} \frac{\lambda - \lambda_n}{\lambda_n - \lambda} \frac{1 + \lambda_n^2}{1 + \lambda^2}
\]

converges and is called Blaschke product for the sequence \( \{\lambda_n\} \).

We will start with the proof of the Theorem 2. The estimates of Blaschke product will play the main role in proving this theorem. First we will establish several lemmas.

**Lemma 1.** Given any number \( \varepsilon > 0 \) and complex numbers \( a_1, a_2, \cdots, a_N \), there is a system of circles in the complex plane, with the sum of the radii not greater than \( 2\varepsilon N \), such that for each
point $z$ lying outside these circles one has the inequalities

$$|z - a_k| \geq k\varepsilon, \quad n = 1, \ldots, N,$$

if the numbers $a_k$ have been enumerated in increasing order of $|z - a_k|$.

**Proof.** This lemma is essentially equivalent to H. Cartan's well-known theorem about estimating from below the modulus of a polynomial, and its proof can be obtained by following the proof of Cartan's theorem (see [8], Chap. 1, § 7).

Let $E$ be a set in the complex plane and suppose, that for any $r$ sufficiently large the set $E \cap B_r$ may be covered by a system of circles, such that the total sum of their radii is not greater than $\delta r$, where the number $\delta$ does not depend on $r$ and $0 < \delta < 1$. The number $\delta_0$ which is the minimum of such $\delta$, we will define as the linear density of the set $E$.

**Lemma 2.** Let $\{\lambda_k\}$ be a sequence in the complex plane, such that for all $t$ sufficiently large

$$\sum_{1 \leq k \leq t} 1 = n(t) \leq at.$$

Given any number $0 < \delta < 1$, there exists the set $E$ of linear density $\leq \delta$, such that for all $z \in D$ one has the inequalities

$$|z - \lambda_k| \geq \frac{\delta k}{4a}, \quad k = 1, 2, \ldots,$$

if the numbers $\lambda_k$ have been enumerated in order of increasing $|z - \lambda_k|$.

**Proof.** Let $\lambda_1, \ldots, \lambda_{k_0}$ be the points of the sequence $\{\lambda_k\}$ lying in the circle $B_{2r}$. From the inequality $n(t) \leq at$, ($t > t_0$) it follows that $k_0 \leq 2ar$. Fix any $\delta$, $0 < \delta < 1$. According to Lemma 1 (in this lemma we put $\varepsilon = \delta/4a$) there exists a set $E$, which consists of the circles with total sum of the radii $\leq 2\varepsilon k_0 \leq \delta r$, such that for all $z \in E$, one has the inequalities

$$|z - \lambda_k| \geq \frac{\delta k}{4a}, \quad k = 1, 2, \ldots, k_0,$$

if the numbers $\lambda_1, \ldots, \lambda_{k_0}$ have been enumerated in order of increasing $|z - \lambda_k|$.

By the inequality $n(t) \leq at$, we conclude $|\lambda_k| \geq k/a$ for all $\lambda_k \in$
$B_{2r}$. Consequently, if $|z| \leq r$, then

\[ |z - \lambda_k| \geq \frac{|\lambda_k|}{2} \geq \frac{k}{2^\alpha}, \quad k = k_0 + 1, \; k_0 + 2, \; \cdots. \tag{5} \]

Let $G_r = B_r \setminus E_r$, $G = \bigcup_{r > r_0} G_r$, $E = C \setminus G$, where by $C$ we denote the complex plane. According to (4), (5), the inequalities (3) hold, if $z \in G_r$ for all $r > r_0$, consequently, they hold for $z \in G$, i.e., for $z \in E$.

Evidently, $B_r \cap E \subset E_r$; therefore the set $B_r \cap E$ may be covered by a system of circles with sum of the radii $\leq \delta r$. Hence the linear density of the set $E$ is not greater than $\delta$. Thus Lemma 2 is proved.

**Lemma 3.** Let the sequence $\{\lambda_k\}$ lie in the domain $P_{\nu, h}^+$ and

\[ \sum_{|\lambda_k| \leq t} 1 = n(t) \leq at \]

provided $t$ is sufficiently large.

Given any number $0 < \delta < 1$ there exists a set $E$, such that its linear density $\leq \delta$ and for all $\lambda \in P_{\nu, h}^+ \setminus E$ one has the inequality

\[ |B(\lambda)| \geq \exp(-\sigma a h^{-1} |\lambda|^q), \quad \lambda \in P_{\nu, h}^+ \setminus E, \]

where the function $B(\lambda)$ is defined by (2), and the constant $\sigma$ does not depend on $\lambda, \alpha, h, \delta$, if $0 < h < h_0$ and $h_0$ is any fixed number.

**Proof.** Denote $\lambda = \mu + i\nu$, $\lambda_n = \mu_n + i\nu_n$. Then

\[ |B(\lambda)|^{-2} = \prod_{k=1}^{\infty} \left| \frac{\lambda - \lambda_k}{\lambda - \lambda_k} \right|^2 = \prod_{k=1}^{\infty} \left( 1 + \frac{4 \nu \nu_k}{|\lambda - \lambda_k|^2} \right). \]

Taking into account that $\nu \leq h\mu^q$, $\nu_k \leq h\mu_k^q$, $|\lambda - \lambda_k| \geq |\mu - \mu_k|$, $q < 1$, we obtain

\[ |B(\lambda)|^{-2} \leq \prod_{k=1}^{\infty} \left( 1 + \frac{4 h^q \mu^q \mu_k^q}{|\lambda - \lambda_k|^2} \right) \leq \prod_{k=1}^{\infty} \left( 1 + \frac{4 h^q \mu^q [(\mu_k - \mu) + \mu]^q}{|\lambda - \lambda_k|^2} \right) \]

\[ \leq \prod_{k=1}^{\infty} \left( 1 + \frac{4 h^q \mu^q}{|\lambda - \lambda_k|^2} \right) \prod_{k=1}^{\infty} \left( 1 + \frac{4 h^q \mu^q}{|\lambda - \lambda_k|^2} \right). \]

According to Lemma 2, there exists a set $E$ such that its linear density $\leq \delta$ and for all $\lambda \in E$ one has the inequalities (3) after enumerating the sequence $\{\lambda_n\}$ properly. Hence, if $\lambda \in P_{\nu, h}^+ \setminus E$, then
\[ -2\ln |B(\lambda)| \leq \sum_{k=1}^{\infty} \ln \left( 1 + \frac{64a^2h^2\mu^2}{\delta^2x^2} \right) + \ln \left( 1 + \frac{4(4a)^2\mu^2}{\delta^2x^2} \right) < \int_{1/2}^{\infty} \left[ \ln \left( 1 + \frac{64a^2h^2\mu^2}{\delta^2x^2} \right) + \ln \left( 1 + \frac{4(4a)^2\mu^2}{\delta^2x^2} \right) \right] dx \]

\[ \leq x\ln \left( 1 + \frac{64a^2h^2\mu^2}{\delta^2x^2} \right) \bigg|_{1/2}^{\infty} + x\ln \left( 1 + \frac{4(4a)^2\mu^2}{\delta^2x^2} \right) \bigg|_{1/2}^{\infty} + \int_{1/2}^{\infty} \left[ \frac{128a^2h^2\mu^2}{\delta^2x^2} + \frac{4(2-q)(4a)^2\mu^2}{\delta^2x^2} \right] dx \]

\[ \leq \ln(1 + C_0a^2h^2\delta^{-2}\mu^2) + \ln(1 + C_0a^2\delta^{-2}\mu^2) + C_0ah\delta^{-1}\mu^2 \]

\[ + C_0ah^2/2\delta^{-1}\mu^2 \leq C_0ah\delta^{-1}\mu^2 \leq C_ah\delta^{-1}|\lambda|^\delta, \]

where the constant \( C \) does not depend on \( \lambda, a, h, \delta \), if \( 0 < h < h_0 \) and \( h_0 \) is any fixed number. We note that in (6) the following estimates were used:

\[ \int_{1/2}^{\infty} \frac{\omega dx}{\delta^2x^2 + \omega^2} = \omega \delta^{-1} \arctan \left( \frac{\pi}{2} \omega \delta^{-1} \right) ; \]

\[ \int_{1/2}^{\infty} \frac{\omega dx}{\delta^2x^2 + \omega^2} \leq \omega \delta^{-2} \left[ \int_{1/2}^{\infty} \frac{dx}{x^{2-q} + \omega \delta^{-2}} + \int_{1/2}^{\infty} \frac{dx}{x^{1/2-q}-1} \right] \]

\[ \leq \omega \delta^{-2} \left[ \int_{1/2}^{\infty} \frac{dx}{\omega \delta^{-2}} + \int_{1/2}^{\infty} \frac{dx}{x^{2-q}} \right] \]

\[ \leq \omega^{1/2-q} \delta^{-1} + \frac{1}{1-q} \omega^{1/2-q} \delta^{-1} = \frac{2(2-q)}{1-q} \omega^{1/2-q} \delta^{-1} . \]

The estimates (6) prove Lemma 3.

**Lemma 4.** Let the sequence \( \{\lambda_k\} \) lie in the domain \( P_{q,h} \) and

\[ \sum_{|\lambda_k| \leq t} 1 = n(t) \leq at^r . \]

Then there exists a holomorphic function \( A(\lambda) \) in the domain \( P_{q,h} \), such that

(a) \( A(\lambda_k) = 0 \) for all points of the sequence \( \{\lambda_k\} \), and \( \lambda_k \) is an \( s \)-multiple root of \( A(\lambda) \), if it is repeated in the sequence \( s \) times.

(b) \( |A(\lambda)| \leq 1, \) if \( \lambda \in P_{q,h} \).

(c) given \( \delta > 0 \) there exists a set \( E \), such that its linear density \( \leq \delta \) and for \( \lambda \in P_{q,h} \setminus E \) one has the estimate

\[ |A(\lambda)| \geq \exp \left( -\sigma ah\delta^{-\beta} |\lambda|^{p(1-q)} \right) , \]

where the constants \( \sigma, \beta > 0 \) do not depend on \( \lambda, a, h, \delta, \) if \( 0 < h < h_0 \) and \( h_0 \) is any fixed number.
Proof. Let us consider the function
\[ p(x) = [\lambda^{\frac{-q}{p}} + 3h(1 - q)i + \tau]^{\frac{1}{p' - q}}. \]
It is easy to verify, that provided \( \tau \) is sufficiently large, the function \( p(x) \) maps the domain \( P_{q,2h} \) inside the domain \( P_{q',k'} \), i.e., \( \rho(P_{q,2h}) \subset P_{q',k'} \), where \( q' = 1 - (1 - q)/p \), \( k' = 6ph \). Hence, the sequence \( \{\rho_n\} = \{\rho(\lambda_n)\} \) lies in the domain \( P_{q',k'} \), and furthermore, according to (7), we have
\[
\sum_{|z_n| \leq t} 1 = \sum_{|z_n| \leq t} 1 \leq 2at.
\]

According to Lemma 3, given any number \( \delta_1 > 0 \), there exists a set \( E \), such that its linear density \( \leq \delta \), and for \( \lambda \in P_{q',k'} \setminus E \) one has the estimate
\[
|B(\rho)| \geq \exp \left(-\sigma ah\delta^{-1}|\rho|^\epsilon \right),
\]
where \( B(\rho) = II(\rho - \rho_0)(\rho - \bar{\rho})^{-1} \). Evidently, \( |B(\rho)| < 1 \) if \( \text{Im} \rho > 0 \). Taking into account that \( \rho^{-1}(P_{q',k'}) \supset P_{q,2h} \), we find that the function \( \Delta(\lambda) = B(\rho(\lambda)) \) is holomorphic in the domain \( P_{q,2h} \) and satisfies conditions (a) and (b). By virtue of (9), we have
\[
|\Delta(\lambda)| \geq \exp \left(-\sigma ah\delta^{-1}|\lambda|^{\frac{1}{p' - q}} \right)
\geq \exp \left(-\sigma ah\delta^{-1}|\lambda|^{\frac{1}{p' - q}} (1 + \sqrt{2}) \right),
\]
if \( \lambda \in \rho^{-1}(P_{q',k'}) \setminus \rho^{-1}(E) \supset P_{q,2h} \setminus \rho^{-1}(E) \). Thus, the proof of Lemma 4 will be complete if we show that the set \( \rho^{-1}(E) \) has the linear density \( \leq C\delta^{1/\beta} \), where the constants \( C > 0 \), \( \beta > 0 \) do not depend on \( a, h, \delta \). Then, supposing \( \delta = C\delta^{1/\beta} \), we will obtain the estimate (8).

It is sufficient to show that if the set \( W \) has linear density \( \leq \varepsilon < 1 \), then its image \( \xi(W) \) under the map \( \xi(\lambda) = \lambda^\epsilon \) has the linear density \( \leq C\varepsilon^{\kappa'} \), where \( \kappa' = \min(\kappa, 1) \) and the constant \( C \) does not depend on \( \varepsilon \).

For any \( r \) sufficiently large, there exists circles \( B(z_i, \varepsilon_i) \), such that they cover the set \( \mathcal{M} \cap B_r \) and \( \sum \varepsilon_i \leq \varepsilon r \). If \( \varepsilon_i \leq |z_i| \), then
\[
|(z_i + \varepsilon_i e^{i\theta})^\epsilon - z_i^\epsilon| \leq |z_i|^\epsilon |(1 + \varepsilon_i e^{i\theta}z_i^{-1})^\epsilon - 1| \leq C_i \varepsilon_i |z_i|^{\epsilon - 1}
\]
by virtue of the simple inequality \( |(1 + z)^\epsilon - 1| \leq C_i(z)|z| \), which holds for \( |z| \leq 1 \). If \( |z_i| < \varepsilon_i \), then \( B(z_i, \varepsilon_i) \subset B(0, 2\varepsilon_i) \subset B(0, 2r\varepsilon) \). Taking into account (10), we find that the set \( \xi(W) \cap B(0, r^\epsilon) \) may be covered by circles with the sum of the radii not greater than \( (2r\varepsilon)^\epsilon + C_i \sum \varepsilon_i |z_i|^{\epsilon - 1} \leq (2r\varepsilon)^\epsilon + C_i\varepsilon^{\epsilon - 1} \sum \varepsilon_i \leq C_i\varepsilon^{\epsilon} (\varepsilon + 1) \).

This means that the set \( \xi(W) \) has linear density \( \leq 2C_2\varepsilon^{\kappa'} \), where \( \kappa' = \min(\kappa, 1) \). Hence Lemma 4 is established.
LEMMA 5. Let the sequence \( \{\lambda_k\} \) lie in the domain \( P_{q,h} \) and

\[
\sum_{|\lambda_k| \leq t} 1 = n(t) \leq at^p,
\]
provided \( t \) is sufficiently large.

Then outside the domain \( P_{q,2h} \) one has the estimate

\[
|V(\lambda)| \geq C \exp(-|\lambda|^{p+\varepsilon}), \varepsilon > 0, \lambda \in C \setminus P_{q,2h},
\]
where the function \( V(\lambda) \) is the canonical product for the sequence \( \{\lambda_k\} \).

\[
V(\lambda) = \prod_{k=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right) \exp\left(\frac{\lambda}{\lambda_k} + \cdots + \frac{\lambda^p}{\nu \lambda_k^p}\right), \nu = [p].
\]

Proof. If \( \pm \lambda \in P_{q,2h}, \lambda \in P_{q,h}, \) then there exist the constants \( C_1, C_2 \) depending only on \( h, q \), such that

\[
|\lambda - \lambda_k| \geq C_1 |\text{Im} \lambda| \geq C_1 C_2 |\lambda|^q.
\]

If \( -\lambda \in P_{q,2h} \), then the estimate (14) holds for \( q = 1 \), and consequently (14) holds for all \( \lambda \in P_{q,2h} \).

It follows from condition (11), that \( |\lambda_{k+1}| \geq a^{-1/p} k^{1/p} \). Taking into account that

\[
\sum_{k=1}^{\kappa} k^\varepsilon \leq \begin{cases} 2N^{\varepsilon+1} & \text{if } \kappa \neq -1 \\ 2\ln N, & \text{if } \kappa = -1 \end{cases}
\]

we obtain the estimate

\[
\sum_{|\lambda_k| \leq 2|\lambda|} \ln \left| \exp\left(\frac{\lambda}{\lambda_k} + \cdots + \frac{\lambda^p}{\nu \lambda_k^p}\right) \right| \geq -\sum_{k=1}^{n(\lambda)} \left| \frac{\lambda}{\lambda_k} \right| + \cdots
\]

\[
+ \frac{1}{\nu} \left| \frac{\lambda}{\lambda_k} \right|^\nu \geq -C_3 \sum_{k=1}^{n(\lambda)} |\lambda|^q k^{-p/|\lambda|}
\]

\[
\geq -2C_3 \sum_{j=1}^{n(\lambda)} |\lambda|^q n(2|\lambda|)^{1-\varepsilon} \ln n(2|\lambda|)
\]

\[
\geq -C_4 \sum_{k=1}^{n(\lambda)} |\lambda|^q |\lambda|^{-\varepsilon} \ln |\lambda| = -\nu C_4 |\lambda|^p \ln |\lambda|.
\]

Further, using (14), we have

\[
\sum_{|\lambda_k| \leq 2|\lambda|} \ln \left| 1 - \frac{\lambda}{\lambda_k} \right| \geq \sum_{|\lambda_k| \leq 2|\lambda|} \ln C_1 C_2 |\lambda|^q
\]

\[
\geq n(2|\lambda|) \ln \frac{1}{2} C_1 C_2 |\lambda|^{-1} \geq -C_5 |\lambda|^p \ln |\lambda|.
\]

Since
we finally get the estimate

\[
\prod_{|\lambda_k| \geq |z|} \left| \left(1 - \frac{\lambda}{\lambda_k}\right) \exp \left(\frac{\lambda}{\lambda_k} + \cdots + \frac{\lambda^r}{r!}\right) \right| \\
\geq \prod_{|\lambda_k| \geq |z|} \exp \left(-\sum_{s=\nu+1}^{\infty} \left| \frac{\lambda}{\lambda_k} \right|^s \right) \\
\geq \prod_{|\lambda_k| \geq |z|} \exp \left(-2 \left| \frac{\lambda}{\lambda_k} \right|^{|s+1|} \right) \\
= \exp \left(-2 \sum_{|\lambda_k| \geq |z|} \left| \frac{\lambda}{\lambda_k} \right|^{|s+1|} \right) \\
\geq \exp \left(-\sum_{|\lambda_k| \geq |z|} \left| \frac{\lambda}{\lambda_k} \right|^{|s+2|/2} \right) \\
\geq C_0 \exp \left(-|\lambda|^{p+\varepsilon} \right),
\]

if \( \varepsilon > 0 \) is chosen, such that \( \rho + \varepsilon/2 \leq \nu + 1 \), and \( |\lambda| \) is sufficiently large. The estimates (15)-(17) prove Lemma 5.

Remark. According to Titchmarsh's theorem, there exists a sequence \( r_1 < r_2 < \cdots < r_k \to \infty \), such that the estimate (12) holds not only for \( \lambda \in \mathbb{C}\setminus P_{q,h} \), but also for \( |\lambda| = r_k \).

Lemma 6.\(^2\) Let the function \( f(\lambda) \) be holomorphic in the sector \( A_0 = \{\lambda: \arg \lambda < \pi/2, \theta \geq 1/2\} \), let \( f \) have no zeros inside this sector and let the order of its growth be \( \rho < \infty \). Then for any given \( \delta > 0 \) inside the sector \( A_{\theta+\delta} \) one has the estimate

\[
|f(\lambda)| \geq \exp \left(-\sigma |\lambda|^{\max(\rho+\delta,\theta)} \right), \quad \lambda \in A_{\theta+\delta}, \quad |\lambda| > 1,
\]

where the constant \( \sigma \) does not depend on \( \lambda \).

Proof. If the function \( \psi(z) \) is holomorphic in the circle \( |z| \leq 1 \) and has no zeros in this circle, then its modulus satisfies the inequality ([8], Chap. 1, § 6):

\[
\ln |\psi(z)| \geq - \frac{c |z|}{1-|z|}, \quad |z| < 1,
\]

where \( c = 2^{-1} |0| \ln \max_{|z| \leq 1} |\psi(z)| \). The function \( z(\mu) = (\mu-1)(\mu+1)^{-1} \)

\(^2\) A similar lemma (unpublished) was obtained by another method by G. V. Radzievskii.
maps the right half-plane into the unit circle, and as \( |\mu| \to \infty \) we have asymptotically \( (\mu = |\mu| e^{i\phi}) \)

\[
\frac{|\mu - 1|}{|\mu + 1|} = \frac{\sqrt{|\mu|^2 + 1 - 2|\mu| \cos \phi}}{\sqrt{|\mu|^2 + 1 + 2|\mu| \cos \phi} - \sqrt{|\mu|^2 + 1 - 2|\mu| \cos \phi}}
\]

\[= \frac{|\mu| - \cos \phi + 0(|\mu|^{-1})}{2 \cos \phi + 0(|\mu|^{-1})}.
\]

Hence, the function \( g(\mu) \), which is holomorphic, bounded and has no zeros in the right half-plane, satisfies the inequality

\[
\ln |g(\mu)| \geq \frac{-C|\mu|}{2 \cos (\text{arg} \mu)}, \quad \text{if } \mu \in A_{\theta + \delta}, \ |\mu| > 1.
\]

Suppose that \( \theta > \rho \). Then \( \theta - \tau > \rho \) for some \( \tau > 0 \). In this case the function \( f(\lambda) \exp (-\lambda^{\theta-\tau}) \) is holomorphic and bounded in the sector \( A_\theta \). If \( \lambda = \mu^{1/\theta} \), then the function \( g(\mu) = f(\mu^{1/\theta}) \exp (-\mu^{\theta-\tau}) \) satisfies (19) and one has the estimate

\[
\ln |f(\lambda)| \geq -\sigma|\lambda|^\theta, \ \lambda \in A_{\theta + \delta}, \ |\lambda| > 1.
\]

In case \( \theta \leq \rho \) we consider the function \( f_\phi(\lambda) = f(\lambda) \exp (-e^{i\phi/2}\lambda^{\rho+\delta}) \), which is holomorphic and bounded in the sector \( A_{\rho+\delta} = \{ \lambda: -\pi/2(1/\rho + 2\delta) + \phi < \text{arg} \lambda < \pi/2(1/(\rho + 2\delta) - \phi) \} \subset A_\theta \), if \( 1/(\rho + 2\delta) - 1/\theta < \phi < 1/\theta - 1/(\rho + 2\delta) \). All sectors \( A_{\rho+\delta} \) cover the sector \( A_\theta \), when \( \phi \) changes in the indicated limits. Therefore, it is sufficient to show that the function \( f(\lambda) \) satisfies (18) in every sector \( A_{\rho+\delta} \).

The function \( \mu(\lambda) = (e^{i\phi/2}\lambda)^{\rho+\delta} \) maps the sector \( A_{\rho+\delta} \) into the right half-plane. Then the function \( g(\mu) = f_\phi(e^{-i\phi/2}\mu^{1/\rho+\delta}) \) is holomorphic and bounded in the right half-plane, and hence its modulus satisfies (19). From this inequality we obtain

\[
\ln f(\lambda) \geq -\sigma|\lambda|^\rho + \delta, \ \lambda \in A_{\rho+\delta}, \ |\lambda| > 1.
\]

The inequalities (20), (21) prove Lemma 6.

Now let us go on to the proof of Theorem 2.

**Proof of Theorem 2.** Let \( F(\lambda) = F_1(\lambda)(F_2(\lambda))^{-1} \), where \( F_1(\lambda), F_2(\lambda) \) are holomorphic functions of finite order \( \leq \gamma \) in the sector \( A_\theta \), and \( \{\nu_n\} \) are the zeros of the function \( F_2(\lambda) \). According to Lemma 4, there exists a function \( D(\lambda) \), which satisfies the condition (a)-(c) of this lemma. By \( V(\lambda) \) we denote the canonical product (13) for the sequence \( \{\nu_n\} \). Let us consider the function
where
\[ G(\lambda) = F(\lambda)J(\lambda) = \frac{F(\lambda)}{\phi(\lambda)} \psi(\lambda), \]

The function \( G(\lambda) \) is holomorphic in the domain \( P_{q,t^2} \); we want to show that it has growth of finite order in that domain. Let \( D_h = P_{q,t^2} \cap B_{r_k} \) where \( r_k \) are the numbers that were mentioned in the remark after Lemma 5. According to Lemma 5 and Titchmarsh's theorem, we have

\[ |\phi(\lambda) \exp(-\lambda')|_{\partial D_h} \leq C, \quad |\psi(\lambda) \exp(-\lambda')|_{\partial D_h} \leq C, \]

if \( \gamma' = \max(\gamma, \theta) \), and the constant \( C \) does not depend on \( k \). By virtue of the maximum principle, we have the functions \( \phi(\lambda) \exp(-\lambda') \) and \( \psi(\lambda) \exp(-\lambda') \) are bounded in the domain \( P_{q,t^2} \), i.e., the functions \( \phi(\lambda), \psi(\lambda) \) have finite order \( \leq \gamma' \) in the domain \( P_{q,t^2} \). By inequality (12) we find that the function \( \phi(\lambda) \) has order \( \leq \gamma' \) in the domain \( A_\delta \). As soon as \( \phi(\lambda) \) has no zeros in the sector \( A_\delta \), we conclude from Lemma 6 that the function \( \phi^{-1}(\lambda) \) has order \( \leq \gamma' \) in sector \( A_{\theta+\delta}, \delta > 0 \). Hence, the function \( G(\lambda) \) has order \( \leq \gamma' \) in domain \( P_{q,t^2} \).

For \( \lambda \in \partial P_{q,t^2} \) we have the inequality \( |G(\lambda)| \leq C \), insofar as

\[ |F(\lambda)|_{\partial P_{q,t^2}} \leq C, \quad |J(\lambda)|_{\partial P_{q,t^2}} \leq 1. \]

Using the Phragmen-Lindelöf principle (see, for example [8], chap. 1, §14), we have \( |G(\lambda)| \leq C \) for all \( \lambda \in P_{q,t^2} \). Hence, according to Lemma 6,

\[ |F(\lambda)| \leq C |J(\lambda)|^{-1} \leq C \exp(-\sigma a h^{-\beta} |\lambda|^{p-(1-q)}), \]

if \( \lambda \in P_{q,t^2} \setminus E \), where the set \( E \) has linear density \( \leq \delta \); the constants \( C, \sigma, \beta \) do not depend on \( \lambda, a, h, \delta \), and one can choose \( \delta \) arbitrarily small.

Obviously, if the set \( E \) has linear density \( < 1/2 \), then there exist numbers \( 0 < r_1 < r_2 < 1 \cdots < r_h \to \infty \), such that the circles \( |\lambda| = r_h \) do not intersect the set \( E \). Then, the assertion of theorem 2 follows from the estimate (22).

Essentially, Theorem 1 may be considered as a corollary of the Theorem 2.

**Proof of Theorem 1.** Fix the number \( \alpha \), such that \( p-(1-q) < \alpha \). All eigenvalues of the operator \( L \), except for a finite number, lie in the sector \( A_{\alpha+\varepsilon}, \varepsilon > 0 \). Without loss of generality, we suppose
that there are no eigenvalues of the operator $L$ outside the sector $\Lambda_{a+\varepsilon}$.

Let $x \in \mathcal{D}$. Consider the integral

$$u_x(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{-i\beta t} (\lambda I - L)^{-1} x d\lambda, \quad t > 0,$$

where the contour $\Gamma$ is the boundary of the sector $\Lambda_{a+\varepsilon}$. Since $\| (\lambda I - L)^{-1} \| \leq C|\lambda|^{-1}$ for $\lambda \in \Gamma$, the function $u_x(t)$ is correctly defined for all $t > 0$. For the proof of the theorem we have to show that

(a) the function $u_x(t)$ can be represented in the form (1) and the series (1) converges in $\mathcal{D}$ after some rearrangement of parentheses not depending on $t$ and $x$.

(b) $\lim_{t \to +0} u_x(t) = x$.

It follows from $L \in \sigma$, that $(\lambda I - L)^{-1}$ is a meromorphic function of order $\leq \gamma$ (see, for example, [5]). According to Theorem 2, there exists a sequence $r_1 < r_2 < \cdots < r_k \to \infty$, such that for $|\lambda| = r_k$ one has the estimate

$$\| (\lambda I - L)^{-1} \| \geq \exp (-\sigma a h |\lambda|^{p-1-q}),$$

where the constant $\sigma$ does not depend on $a, h$, $0 < h < h_0$.

Let $K_n = \{ \lambda: r_n \leq |\lambda| \leq r_{n+1} \}$, $\mathcal{D}_n = \Lambda_{a+\varepsilon} \cap K_n$. It follows from estimate (23), that for any $t > 0$ ($\alpha > p - (1 - q)$)

$$u_x(t) = \sum_{n=1}^{\infty} \int_{\mathcal{D}_n} e^{-i\beta t} (I\lambda - L)^{-1} x d\lambda,$$

and the series converges in $\mathcal{D}$.

Calculating the integrals in (24), we obtain the assertion (a). We note also that in the case when either $a$, or $h$ can be chosen arbitrarily small and $p - (1 - q) > 0$, the assertion (a) is valid for $\alpha = p - (1 - q)$.

As was mentioned before, under assumption $\| (I\lambda - L)^{-1} \| \leq Cd^{-1}(\lambda, A_{a+\varepsilon})$, the assertion (b) was proved by V. B. Lidskii [9] for any $x \in \mathcal{D}(L)$, and by V. I. Macaev (see [5]) for any $x \in \mathcal{D}$.

We note only that (b) is valid for $x$, which can be represented as a finite linear combination of eigenvectors of the operator $L$, and all such $x$ are closed in $\mathcal{D}$. Consequently, the assertion (b) is valid for all $x$, if $\| u_x(t) \| \leq C \| x \|$, where the constant $C$ does not depend on $t$, $0 < t \leq 1$. But this fact may be proved also by using the ideas from a theorem of E. Hille about the generation of holomorphic semi-groups (see [4], § 12.8).

As was shown in [9], the summability theorems have an important role in solving some nonstationary differential equations. The applications of Theorem 1 to such problems will be considered by
the author in a subsequent paper.

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Alberto Alesina and Leonede De Michele, A dichotomy for a class of positive definite functions .............................................. 251
Kahtan Alzubaidy, Rank\textsubscript{2} \(p\)-groups, \(p > 3\), and Chern classes ................................. 259
James Arney and Edward A. Bender, Random mappings with constraints on coalescence and number of origins ...................... 269
Bruce C. Berndt, An arithmetic Poisson formula .............................................................. 295
Julius Rubin Blum and J. I. Reich, Pointwise ergodic theorems in l.c.a. groups ........ 301
Jonathan Borwein, A note on \(\varepsilon\)-subgradients and maximal monotonicity .................. 307
Andrew Michael Brunner, Edward James Mayland, Jr. and Jonathan Simon, Knot groups in \(S^4\) with nontrivial homology .................................................. 315
Luis A. Caffarelli, Avner Friedman and Alessandro Torelli, The two-obstacle problem for the biharmonic operator .................. 325
Aleksander Calka, On local isometries of finitely compact metric spaces .................. 337
William S. Cohn, Carleson measures for functions orthogonal to invariant subspaces ............................................................................ 347
Roger Fenn and Denis Karmen Sjerve, Duality and cohomology for one-relator groups .................................................................................. 365
Gen Hua Shi, On the least number of fixed points for infinite complexes ................. 377
George Golightly, Shadow and inverse-shadow inner products for a class of linear transformations ......................................................... 389
Joachim Georg Hartung, An extension of Sion’s minimax theorem with an application to a method for constrained games .................. 401
Vikram Jha and Michael Joseph Kallaher, On the Lorimer-Rahilly and Johnson-Walker translation planes ................................................. 409
Kenneth Richard Johnson, Unitary analogs of generalized Ramanujan sums .... 429
Peter Dexter Johnson, Jr. and R. N. Mohapatra, Best possible results in a class of inequalities .................................................................................. 433
Dieter Jungnickel and Sharad S. Sane, On extensions of nets ........................................ 437
Johan Henricus Bernardus Kemperman and Morris Skibinsky, On the characterization of an interesting property of the arcsin distribution ................................. 457
Karl Andrew Kosler, On hereditary rings and Noetherian \(V\)-rings .................................. 467
William A. Lampe, Congruence lattices of algebras of fixed similarity type. II ........ 475
M. N. Mishra, N. N. Nayak and Swadeenananda Pattanayak, Strong result for real zeros of random polynomials ........................................ 509
Sidney Allen Morris and Peter Robert Nickolas, Locally invariant topologies on free groups ................................................................................. 523
Richard Cole Penney, A Fourier transform theorem on nilmanifolds and nil-theta functions ................................................................................. 539
Andrei Shkalikov, Estimates of meromorphic functions and summability theorems .................................................................................. 569
László Székelyhidi, Note on exponential polynomials ................................................... 583
William Thomas Watkins, Homeomorphic classification of certain inverse limit spaces with open bonding maps ............................. 589
David G. Wright, Countable decompositions of \(E^n\) ......................................................... 603
Takayuki Kawada, Correction to: “Sample functions of Pólya processes” ............. 611
Z. A. Chanturia, Errata: “On the absolute convergence of Fourier series of the classes \(H^\omega \cap V[u]\)” ......................................................... 611