Pacific Journal of Mathematics

ELEMENTARY PROOFS OF BERNDT'S RECIPROCITY LAWS

TOM M. (MIKE) APOSTOL AND THIENNU H. VU

Vol. 98, No. 1 March 1982

PACIFIC JOURNAL OF MATHEMATICS
Vol. 98, No. 1, 1982

ELEMENTARY PROOFS OF BERNDT'S RECIPROCITY LAWS

TOM M. APOSTOL AND THIENNU H. VU

Using analytic functional equations, Berndt derived three reciprocity laws connecting five arithmetical sums analogous to Dedekind sums. This paper gives elementary proofs of all three reciprocity laws and obtains them all from a common source, a polynomial reciprocity formula of L. Carlitz.

1. Introduction.

The classical Dedekind sums

$$s(h, k) = \sum_{r \bmod k} \left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right),$$

where h and k are integers, $k > 0$, $((x)) = x - [x] - 1/2$ if $x \neq$ integer, and $((x)) = 0$ for integer x, occur in the transformation formula for the logarithm of the Dedekind eta function

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}) . \qquad (\mathrm{Im}\,(\tau) > 0)$$

Dedekind's formula which describes the behavior of $\log \eta(\tau)$ under a unimodular substitution implies a reciprocity law relating $s(h, k)$ and $s(k, h)$ when $(h, k) = 1$. (See [1], Chapter 3.)

Berndt [2] derived transformation formulas for the logarithm of the theta function

$$\theta(\tau) = \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau})(1 + e^{(2n-1)\pi i \tau})^2$$

and related functions, and introduced five new arithmetical sums which are analogous to (but quite different from) the Dedekind sums, and showed that the analytic functional equations imply reciprocity laws for these sums. The sums in question are

$$S(h, k) = \sum_{r=1}^{k-1} (-1)^{r+1+[hr/k]} , \tag{1}$$

$$s_1(h, k) = \sum_{r=1}^{k-1} (-1)^{[hr/k]}\left(\left(\frac{r}{k}\right)\right) , \tag{2}$$

$$s_2(h, k) = \sum_{r=1}^{k-1} (-1)^{r}\left(\left(\frac{r}{k}\right)\right)\left(\left(\frac{hr}{k}\right)\right) , \tag{3}$$

$$s_3(h, k) = \sum_{r=1}^{k-1} (-1)^{r}\left(\left(\frac{hr}{k}\right)\right) , \tag{4}$$

and

$$s_4(h, k) = \sum_{r=1}^{k-1} (-1)^{[hr/k]} . \tag{5}$$

Berndt's reciprocity laws, which occur, respectively, as Theorems 4.2, 6.2, and 8.2 in [2], can be stated as follows:

THEOREM 1. *If h and k have opposite parity and $(h, k) = 1$, then*

$$S(h, k) + S(k, h) = 1 . \tag{6}$$

THEOREM 2. *If h is odd, k is even, and $(h, k) = 1$, then*

$$2s_2(h, k) - s_1(k, h) = \frac{1}{2}\left(\frac{1}{hk} + \frac{h}{k} - 1\right) . \tag{7}$$

THEOREM 3. *If k is odd and $(h, k) = 1$, then*

$$2s_3(h, k) - s_4(k, h) = 1 - \frac{h}{k} . \tag{8}$$

Since these theorems concern arithmetical sums, it seems desirable to have proofs independent of the theory of theta functions. An elementary proof of (6) has been given by Berndt, Evans and others [3]. This paper gives elementary proofs of all three reciprocity laws and, moreover, obtains them all from a common source, a polynomial reciprocity formula of L. Carlitz ([4], Eq. (5.11)) which states that

$$(u - 1) \sum_{r=1}^{k-1} u^{k-r-1} v^{[hr/k]} - (v - 1) \sum_{r=1}^{h-1} v^{h-r-1} u^{[kr/h]} = u^{k-1} - v^{h-1} . \tag{9}$$

Here h and k are coprime positive integers and u, v are arbitrary complex numbers.

In [4] Carlitz gives an elementary proof of (9). We give a different elementary proof involving lattice points in a triangle and then use (9) to deduce Theorems 1, 2 and 3. We also show that in the cases not covered by Berndt's theorems the sums in question vanish. Thus we have the following companion theorems.

THEOREM 1a. *If both h and k are odd and $(h, k) = 1$, then*

$$S(h, k) = S(k, h) = 0 . \tag{10}$$

THEOREM 2a. *If k is odd and $(h, k) = 1$, then*

$$s_2(h, k) = s_1(k, h) = 0 . \tag{11}$$

THEOREM 3a. *If k is even and $(h, k) = 1$, then*

$$s_3(h, k) = s_4(k, h) = 0 \,. \tag{12}$$

2. Proof of Carlitz's reciprocity formula (9). We have

$$\begin{aligned} u^{k-1} - v^{h-1} &= (u^{k-1} - 1) - (v^{h-1} - 1) \\ &= (u - 1)\sum_{r=1}^{k-1} u^{k-1-r} - (v - 1)\sum_{r=1}^{h-1} v^{h-1-r} \,. \end{aligned}$$

This identity reduces to (9) if, and only if, we have

$$(u - 1)\sum_{r=1}^{k-1} u^{k-r-1}(1 - v^{[hr/k]}) = (v - 1)\sum_{r=1}^{h-1} v^{h-r-1}(1 - u^{[kr/h]}) \,. \tag{13}$$

Now if $hr/k \geqq 1$ we have

$$1 - v^{[hr/k]} = (1 - v)\sum_{n=0}^{[hr/k]-1} v^n$$

and there is a corresponding formula for $1 - u^{[kr/h]}$ if $kr/h \geqq 1$. Hence (13) is equivalent to the identity

$$\sum_{\substack{r=1 \\ hr/k \geqq 1}}^{k-1} \sum_{n=0}^{[hr/k]-1} u^{k-1-r} v^n = \sum_{\substack{s=1 \\ ks/h \geqq 1}}^{h-1} \sum_{m=0}^{[ks/h]-1} v^{h-1-s} u^m \,. \tag{14}$$

Because of symmetry in h and k, we can assume that $h < k$ so the condition $ks/h \geqq 1$ is automatically satisfied. Let L denote the left member of (14). In the sum over r introduce a new index of summation, $m = k - 1 - r$. Then

$$\left[\frac{hr}{k}\right] = \left[\frac{h(k - 1 - m)}{k}\right] = h - 1 - \left[\frac{h(1 + m)}{k}\right]$$

and we get

$$L = \sum_{m=0}^{k-2} \sum_{n=0}^{h-1-[h(1+m)/k]} u^m v^n \,.$$

Now replace the index n by $s = h - 1 - n$. This gives

$$L = \sum_{m=0}^{k-2} \sum_{s=[h(1+m)/k]}^{h-1} u^m v^{h-1-s} \,.$$

This double sum is extended over the lattice points (m, s) in the xy-plane which lie inside or on the boundary of the right triangle bounded by the lines

$$x = 0 \,, \qquad y = h - 1 \,, \quad \text{and} \quad y = h(1 + x)/k \,.$$

Interchanging the order of summation we find

$$L = \sum_{s=1}^{h-1} \sum_{m=0}^{[ks/h]-1} u^m v^{h-1-s} ,$$

which proves (14), and hence (9).

3. Proof of Theorems 1 and 1a. Taking $u = v = -1$ in (9) and dividing by $2(-1)^{k+1}$ we obtain

$$\sum_{r=1}^{k-1} (-1)^{r+1+[hr/k]} + (-1)^{h-k+1} \sum_{r=1}^{h-1} (-1)^{r+1+[kr/h]} = \frac{1-(-1)^{k+h}}{2} .$$

If h and k have opposite parity this implies Berndt's Theorem 1, and if h and k have the same parity (both odd since $(h, k) = 1$), we obtain

$$S(h, k) - S(k, h) = 0 .$$

But if h and k are both odd we have

$$\begin{aligned} S(h, k) &= \sum_{r=1}^{k-1} (-1)^{(k-r)+1+[h(k-r)/k]} \\ &= (-1)^k \sum_{r=1}^{k-1} (-1)^{-r+1+h-1-[hr/k]} = (-1)^{k+h-1} S(h, k) = -S(h, k) , \end{aligned}$$

so $S(h, k) = 0$ and hence also $S(k, h) = 0$.

4. Proof of Theorems 3 and 3a. We differentiate each member of (9) with respect to v and then put $v = 1$ to obtain

$$\sum_{r=1}^{k-1} (u^{k-r} - u^{k-r-1})\left[\frac{hr}{k}\right] - \sum_{r=1}^{h-1} u^{[kr/h]} = 1 - h . \tag{15}$$

When $u = -1$ this becomes

$$2(-1)^k \sum_{r=1}^{k-1} (-1)^r\left[\frac{hr}{k}\right] - s_4(k, h) = 1 - h . \tag{16}$$

But $[hr/k] = hr/k - 1/2 - ((hr/k))$ so

$$\begin{aligned} \sum_{r=1}^{k-1} (-1)^r\left[\frac{hr}{k}\right] &= \frac{h}{k}\sum_{r=1}^{k-1} (-1)^r r - \frac{1}{2}\sum_{r=1}^{k-1} (-1)^r - s_3(h, k) \\ &= (-1)^{k-1}\frac{h}{k}\left[\frac{k}{2}\right] + \frac{(-1)^k + 1}{4} - s_3(h, k) \end{aligned} \tag{17}$$

since

$$\sum_{r=1}^{k-1} (-1)^r r = \left[\frac{k}{2}\right](-1)^{k-1} \quad \text{and} \quad \sum_{r=1}^{k-1} (-1)^r = -\frac{(-1)^k + 1}{2} .$$

Using (17) in (16) we obtain

$$2(-1)^{k-1}s_3(h,k) - s_4(k,h) = 1 - h + \frac{2h}{k}\left[\frac{k}{2}\right] - \frac{1+(-1)^k}{2}.$$

When k is odd this gives Berndt's Theorem 3, and when k is even it gives

$$-2s_3(h,k) - s_4(k,h) = 0 . \tag{18}$$

But when k is even it is easy to see that $s_4(k,h) = 0$ because

$$\begin{aligned} s_4(k,h) &= \sum_{r=1}^{h-1}(-1)^{[kr/h]} = \sum_{r=1}^{h-1}(-1)^{[k(h-r)/h]} = \sum_{r=1}^{h-1}(-1)^{k-1-[kr/h]} \\ &= -\sum_{r=1}^{h-1}(-1)^{[kr/h]} = -s_4(k,h) . \end{aligned}$$

Therefore $s_4(k,h) = 0$ and (18) shows that $s_3(h,k) = 0$ when k is even.

5. **Proof of Theorems 2 and 2a.** Start with Equation (15) and rewrite it as follows:

$$(u-1)\sum_{r=1}^{k-1}u^{k-r-1}\left[\frac{hr}{k}\right] - \sum_{r=1}^{h-1}u^{[kr/h]} = 1 - h .$$

Replace r by $k-r$ in the first sum and note that $[h(k-r)/k] = h - 1 - [hr/k]$ to obtain

$$(u-1)(h-1)\sum_{r=1}^{k-1}u^{r-1} - (u-1)\sum_{r=1}^{k-1}u^{r-1}\left[\frac{hr}{k}\right] - \sum_{r=1}^{h-1}u^{[kr/h]} = 1 - h ,$$

or

$$(u-1)\sum_{r=1}^{k-1}u^{r-1}\left[\frac{hr}{k}\right] + \sum_{r=1}^{h-1}u^{[kr/h]} = u^{k-1}(h-1) . \tag{19}$$

Differentiate with respect to u and multiply by u to obtain

$$\begin{aligned} (u-1)\sum_{r=1}^{k-1}ru^{r-1}\left[\frac{hr}{k}\right] + \sum_{r=1}^{k-1}u^{r-1}\left[\frac{hr}{k}\right] + \sum_{r=1}^{h-1}\left[\frac{kr}{h}\right]u^{[kr/h]} \\ = (k-1)u^{k-1}(h-1) . \end{aligned}$$

Now multiply by $(u-1)$ and use (19) to obtain

$$\begin{aligned} (u-1)^2\sum_{r=1}^{k-1}ru^{r-1}\left[\frac{hr}{k}\right] - \sum_{r=1}^{h-1}u^{[kr/h]} + (u-1)\sum_{r=1}^{h-1}\left[\frac{kr}{h}\right]u^{[kr/h]} \\ = u^{k-1}(h-1)\{(k-1)(u-1) - 1\} . \end{aligned}$$

When $u = -1$ this gives us

$$\begin{aligned} 4\sum_{r=1}^{k-1}(-1)^{r-1}r\left[\frac{hr}{k}\right] - s_4(k,h) - 2\sum_{r=1}^{h-1}(-1)^{[kr/h]}\left[\frac{kr}{h}\right] \\ = (-1)^k(h-1)(2k-1) . \end{aligned} \tag{20}$$

If k is even, $s_4(k, h) = 0$ by Theorem 3a, and (20) becomes

$$(21)\qquad 4\sum_{r=1}^{k-1}(-1)^{r-1}r\left[\frac{hr}{k}\right] - 2\sum_{r=1}^{h-1}(-1)^{[kr/h]}\left[\frac{kr}{h}\right] = (h-1)(2k-1)\ .$$

Theorem 2 now follows at once from (21) and the following lemma:

LEMMA. *If k is even and h is odd, $(h, k) = 1$, then we have*

$$(22)\qquad -2\sum_{r=1}^{h-1}(-1)^{[kr/h]}\left[\frac{kr}{h}\right] = 1 - \frac{1}{h} - 2ks_1(k, h)\ ,$$

and

$$(23)\qquad 4\sum_{r=1}^{k-1}(-1)^{r-1}r\left[\frac{hr}{k}\right] = 4ks_2(h, k) + 2hk - 2h - k\ .$$

To prove (22) we evaluate the sum

$$t(h, k) = 2\sum_{r=1}^{h-1}(-1)^{[kr/h]}\left(\left(\frac{kr}{h}\right)\right)$$

in two ways. On the one hand we have (since $s_4(k, h) = 0$ if k is even)

$$\begin{aligned} t(h, k) &= 2\sum_{r=1}^{h-1}(-1)^{[kr/h]}\left(\frac{kr}{h} - \left[\frac{kr}{h}\right] - \frac{1}{2}\right) \\ &= -2\sum_{r=1}^{h-1}(-1)^{[kr/h]}\left[\frac{kr}{h}\right] + 2k\sum_{r=1}^{h-1}(-1)^{[kr/h]}\left(\left(\frac{r}{h}\right)\right) \\ &= -2\sum_{r=1}^{h-1}(-1)^{[kr/h]}\left[\frac{kr}{h}\right] + 2ks_1(k, h)\ . \end{aligned}$$

On the other hand we have, since k is even,

$$t(h, k) = 2\sum_{r\not\equiv 0 \pmod h}(-1)^{[kr/h]}\left(\left(\frac{kr}{h}\right)\right)\ .$$

Write $kr = qh + \rho$, where $q = [kr/h]$ and $0 < \rho < h$. Since k is even and h is odd we have $qh + \rho \equiv q + \rho \equiv 0 \pmod 2$ so $(-1)^\rho = (-1)^q$. Hence

$$t(h, k) = 2\sum_{\rho=1}^{h-1}(-1)^\rho\left(\left(\frac{\rho}{h}\right)\right) = 2\sum_{\rho=1}^{h-1}(-1)^\rho\left(\frac{\rho}{h} - \frac{1}{2}\right) = \frac{h-1}{h} = 1 - \frac{1}{h}\ .$$

Equating the two expressions for $t(h, k)$ we obtain (22).

To prove (23) we write

$$4ks_2(h, k) = 4k\sum_{r=1}^{k-1}(-1)^r\left(\frac{r}{k} - \frac{1}{2}\right)\left(\frac{hr}{k} - \left[\frac{hr}{k}\right] - \frac{1}{2}\right)$$

$$= -4\sum_{r=1}^{k-1}(-1)^r r\left[\frac{hr}{k}\right] + 2k\sum_{r=1}^{k-1}(-1)^r\left[\frac{hr}{k}\right] + \frac{4h}{k}\sum_{r=1}^{k-1}(-1)^r r^2$$
$$-2(h+1)\sum_{r=1}^{k-1}(-1)^r r + k\sum_{r=1}^{k-1}(-1)^r .$$

Now if k is even we have

$$\sum_{r=1}^{k-1}(-1)^r r^2 = -\frac{k(k-1)}{2}, \qquad \sum_{r=1}^{k-1}(-1)^r r = -\frac{k}{2},$$
$$\text{and} \quad \sum_{r=1}^{k-1}(-1)^r = -1 ,$$

so

$$(24) \qquad 4ks_2(h, k) = -4\sum_{r=1}^{k-1}(-1)^r r\left[\frac{hr}{k}\right] + 2k\sum_{r=1}^{k-1}(-1)^r\left[\frac{hr}{k}\right] + 2h - hk .$$

Let S denote the second sum on the right. Replacing r by $k - r$ we find (since k is even),

$$S = \sum_{r=1}^{k-1}(-1)^{k-r}\left(h - 1 - \left[\frac{hr}{k}\right]\right) = (h-1)\sum_{r=1}^{k-1}(-1)^r - S ,$$

so $2S = 1 - h$ and $2kS = k - hk$. Therefore (24) reduces to (23). This completes the proof of the lemma and also of Theorem 2.

Finally, to prove Theorem 2a we replace the index r by $k - r$ in (3) to obtain

$$s_2(h, k) = \sum_{r=1}^{k-1}(-1)^{k-r}\left(\left(\frac{k-r}{k}\right)\right)\left(\left(\frac{h(k-r)}{k}\right)\right) = (-1)^k s_2(h, k) .$$

Therefore $s_2(h, k) = 0$ if k is odd and $(h, k) = 1$. A similar argument shows that $s_1(h, k) = 0$ if h is odd and $(h, k) = 1$. This implies Theorem 2a.

References

1. Tom M. Apostol, *Modular Functions and Dirichlet Series in Number Theory. Graduate texts in Mathematics,* 41. Springer-Verlag, New York, 1976.
2. Bruce C. Berndt, *Analytic Eisenstein series, theta-functions, and series relations in the spirit of Ramanujan,* J. für die reine und angewandte Math. 303/304 (1978), 332–365. MR 80b, #10035.
3. Bruce C. Berndt and Ronald J. Evans, Problem E2758, American Math. Monthly, v. 86(1979), p. 128, and v. 87(1980), p. 404.
4. Leonard Carlitz, *Some polynomials associated with Dedekind sums,* Acta Math. Acad. Sci. Hungaricae, 26(1975), 311–319. MR 52, #5540.

Received January 21, 1981.

California Institute of Technology
Pasadena, CA 91125

PACIFIC JOURNAL OF MATHEMATICS

Mathematical papers intended for publication in the *Pacific Journal of Mathematics* should be in typed form or offset-reproduced, (not dittoed), double spaced with large margins. Please do not use built up fractions in the text of the manuscript. However, you may use them in the displayed equations. Underline Greek letters in red, German in green, and script in blue. The first paragraph or two must be capable of being used separately as a synopsis of the entire paper. Please propose a heading for the odd numbered pages of less than 35 characters. Manuscripts, in triplicate, may be sent to any one of the editors. Please classify according to the scheme of Math. Reviews, Index to Vol. **39**. Supply name and address of author to whom proofs should be sent. All other communications should be addressed to the managing editor, or Elaine Barth, University of California, Los Angeles, California, 90024.

50 reprints to each author are provided free for each article, only if page charges have been substantially paid. Additional copies may be obtained at cost in multiples of 50.

The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$102.00 a year (6 Vols., 12 issues). Special rate: \$51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address shoud be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.). 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

Pacific Journal of Mathematics

Vol. 98, No. 1 March, 1982