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The linearly ordered net spaces are introduced in this paper. This concept is a generalization of both the sequential spaces and the linearly ordered base spaces. The fundamental properties of this class, including mapping properties, are presented. Various applications of a general nature are given as well as some applications to weak covering axioms. The class of lo-net spaces is characterized as the class of well-ordered net spaces.

I. Introduction. In this note the large class of linearly ordered net (lo-net) spaces is introduced, the fundamental properties of this class are presented and the study of applications is initiated. The lo-net spaces are very useful simultaneous generalizations of sequential spaces and linearly ordered base spaces. The applications of sequential spaces are numerous and well known. The linearly ordered base (lob) spaces were studied by Davis [6] and applications of a general nature were presented along with the beginnings of important applications to weak covering axioms. In [7] the applications of lob-spaces to various weak covering axioms were studied extensively.

Section 2 contains the definitions of the notions associated with the linearly ordered net spaces, as spaces with the weak topology generated by a class of subsets. The fundamental structural properties and some of the basic applications are presented in §3. In §4 some of the results of [6] and [7] are reexamined from the broadened view of lo-net spaces. The subtle differences between lo-net and very lo-net spaces are illustrated by examples. In particular, Lemma 2.3.1 of [7] is not true and a corrected version is presented along with its lo-net version. Finally the true nature of the lo-net spaces as a generalization of sequential spaces and lob-spaces is revealed in §5, where the class of lo-net spaces is characterized as the class of well-ordered net spaces.

II. Preliminaries. A linearly ordered net (lo-net) is a net whose directed set is linearly ordered. The collection of linearly ordered nets $\mathcal{L}_X$ in a space X determines a natural cover [9] of X. Many of the notions in this study are specific applications of concepts and properties introduced and developed by Stan Franklin [9]. A topological space X will be called a *linearly ordered net (lo-net) space* provided $H \subset X$ is closed if and only if for every convergent lo-net in H, say $x_\lambda \to x$, we have $x \in H$. That is X is a lo-not space

provided X has the weak topology determined by the collection $\mathcal{L}_X$. The class of lo-net spaces is very large including all sequential spaces and lob-spaces. (A space is a lob-space provided every point has a nhood base which is linearly ordered by set inclusion [6].) The lob-spaces and results of [6] are the primary motivation for this work. The terminology used by Arhangelskii, very k-space, is quite indicative of the property described and I use it as follows: X is a *very lo-net space* if for each $x \in \text{cl}(H)$ there exists a convergent lo-net (x_λ) in H such that $x_\lambda \to x$. For each $A \subset X$, let $\text{lo-cl}(A) = \{p\colon p$ is the limit of a convergent lo-net in $A\}$. Recall the essential subtle point that $\text{lo-cl}(A)$ is not necessarily closed, even in a lo-net space. For each $A \subset X$, let $A^0 = A$, $A^1 = \text{lo-cl}(A)$, for any fixed ordinal β where A^α is defined for each $\alpha < \beta$, if $\beta = \gamma + 1$, define $A^\beta = \text{lo-cl}(A^\gamma)$ and if β is a limit ordinal let $A^\beta = \cup\{A^\alpha\colon \alpha < \beta\}$. The *lo-net characteristic* of a space X, $\lambda(X)$, is the least ordinal α such that $A^\alpha = \text{cl}(A)$, for each $A \subset X$. X is a lo-net space if and only if $\lambda(X)$ exists, and in this case $\lambda(X) \leqq (t(X))^+$ where $t(X)$ is the tightness of X. X is a very lo-net space provided $\lambda(X) = 1$. The lo-net spaces are particularly cases of the Ω-net spaces studied by Jerry Vaughan [12], and in fact, they are the well-ordered net spaces (§5).

III. **Lo-net spaces.** In this section we present various properties of lo-net spaces and some fundamental applications of lo-net spaces.

THEOREM 3.1.

$$\begin{array}{ccccc} \textit{first countable} & \Longrightarrow & \textit{Frechet} & \Longrightarrow & \textit{sequential} \\ \downarrow & & \downarrow & & \downarrow \\ \textit{lob-space} & \Longrightarrow & \textit{very lo-net space} & \Longrightarrow & \textit{lo-net space} \end{array}$$

The quotient space obtained by attaching the 0 of the $1/n$-sequence, S_1, to each countable ordinal in $[0, \omega_1]$ is a lo-net space which is not sequential and not very lo-net. Every lo-net closed set is sequentially closed, but sequentially closed does not imply lo-net closed as $[0, \omega_1)$ is sequentially closed in $[0, \omega]$ but not closed. However in countable sets we have the following.

THEOREM 3.2. *Every countable sequentially closed set is lo-net closed.*

Proof. Let H be a countable sequentially closed set, and let $\{p_\lambda\colon \lambda \in L\}$ be a lo-net in H where $p_\lambda \to p$. From Lemma 5.1, let W be a well-ordered cofinal subset of L. Since H is countable, if

$\mathrm{cf}(W) > \omega$, $\{p_\lambda: \lambda \in L\}$ must be constant on a cofinal subset of W. Since there is a constant subnet converging to p, $p \in H$. If $\mathrm{cf}(W) = \omega$, choose an increasing sequence $\{\lambda_n: n \in \omega\}$ which is cofinal in W. Then since $\{p_{\lambda_n}: n \in \omega\}$ is cofinal in $\{p_\lambda: \lambda \in L\}$, $p_{\lambda_n} \to p$. Since H is sequentially closed, $p \in H$. Thus H contains each of its lo-net limit points and is lo-net closed.

COROLLARY 3.3. *Every countable lo-net space is sequential and every countable very lo-net space if Frechet.*

COROLLARY 3.4. *If $\{x_n\}$ is a sequence in a lo-net space and $\{x_n: n \in N\}$ is not closed, then $\{x_n\}$ has a convergent subsequence.*

The mapping properties of lo-net spaces are as follows.

THEOREM 3.5. *X is a lo-net space (very lo-net space)* [lob-*space*] *if and only if X is the quotient (pseudo-open)* [*open*] *image of* alob-*space.*

THEOREM 3.6. *The property of being a lo-net (very lo-net)* [lob *space*] *is preserved under quotient (pseudo-open)* [*open*] *mappings.*

THEOREM 3.7. *The class of lo-net spaces is a coreflective subcategory of the category of all topological spaces.*

The next two theorems and Corollary 3.10 are generalizations of theorem 2.2 in [6] to arbitrary cardinals and lo-net spaces. An *m-lo-net space* is a lo-net space whose topology is generated by lo-nets with ranges of cardinality $\leqq m$. The *character of the point x* in a space X, $\chi(x, X)$ is the least cardinal α for which there is a nhood base at x of cardinality $\leqq \alpha$. The *character of X*, $\chi(X) = \sup\{\chi(x, X): x \in X\}$. The *pseudocharacter* of the point x in a space X, $\psi(x, X)$ is the least cardinal α such that $\{x\}$ is the intersection of $\leqq \alpha$ open sets.

THEOREM 3.8. *Let x be a non-isolated point in an* lob-*space X. $\chi(x, X) \leqq m$ if and only if there exist $M \subset X - \{x\}$ such that* card $(M) \leqq m$ *and* $x \in \mathrm{cl}(M)$.

Proof. We will prove only the sufficiency. Let $\{x\}$ be not open and let $M \subset X - \{x\}$ be such that $\mathrm{card}(M) \leqq m$ and $x \in \mathrm{cl}(M)$. Let $\mathcal{U}_x$ be a linearly ordered base at x. For each $p \in M$, let $U_p \in \mathcal{U}_x$ be such that $p \notin U_p$. If $\{U_p: p \in U\}$ is not cofinal in $\mathcal{U}_x$, then there is $U \in \mathcal{U}_x$ such that $U \subset U_p$ for each $p \in M$ and thus $U \cap M = \varnothing$. Thus $\{U_p: p \in M\}$ is cofinal and $\chi(x, X) \leqq m$ which completes the proof.

COROLLARY 3.9. *In a* lob*-space, the following are equivalent,*
(a) $\chi(X) \leqq m$
(b) *X is an m-lo-net space*
(c) $t(X) \leqq m$
(d) *If $\{x\}$ is not open, then there exists a set $M \subset X - \{x\}$ such that* card $(M) \leqq m$ *and* $x \in \text{cl}\,(M)$.

THEOREM 3.10. *If X is a T_1* lob *space, $\chi(x, X) \leqq m$ if and only if $\psi(x, X) \leqq m$.*

Proof. The sufficiency is proven. Let $\{G_\alpha: \alpha \in A\}$ be a collection of open sets such that card $(A) \leqq m$ and $\{x\} = \cap\{G_\alpha: \alpha \in A\}$. Let $\mathcal{U}_x$ be a linearly ordered nhood base at x. For each $\alpha \in A$, choose $U_\alpha \in \mathcal{U}_x$ such that $U_\alpha \subset G_\alpha$. If $\{U_\alpha: \alpha \in A\}$ is not cofinal in $\mathcal{U}_x$, there is a $U_\alpha \in \mathcal{U}_x$ such that $U \subset U_\alpha \subset G_\alpha$, for each $\alpha \in A$. Thus $U = \{x\}$ and $\chi(x, X) = \omega \leqq m$. If $\{U_\alpha: \alpha \in A\}$ is cofinal, then it is a nhood base at x of cardinality $\leqq m$ and $\chi(x, X) \leqq m$ and this completes the proof.

The next theorem improves both Theorem 2.3 of [6] and Proposition 1.1 of [10], and supplies a different view of a similar theorem in [8].

THEOREM 3.11. *A T_1 lo-net space X is sequentially compact if and only if it is countably compact.*

Proof. We proof the sufficiency. Let $\{x_n\}$ be a sequence in X. If $\{x_n: n \in N\}$ is closed, then it is countable and compact. Hence $\{x_n: n \in N\}$ is first countable. If $\{x_n: n \in N\}$ is finite there is a convergent subsequence. If $\{x_n: n \in N\}$ is infinite there is a cluster point in $\{x_n: n \in N\}$ and a subsequence converges to it. If $\{x_n: n \in N\}$ is not closed, then it is not lo-net closed. Since countable sequentially closed sets are lo-net closed, by Theorem 3.2, $\{x_n: n \in N\}$ is not sequentially closed. Thus there is a sequence $\{x_{n_k}\}$ in $\{x_n\}$ that converges to some $p \notin \{x_n: n \in N\}$. Thus X is sequentially compact and this completes the proof.

COROLLARY 3.12. *If X_α is a T_1 countably compact lo-net space for each $\alpha \in \omega_1$, the $\Pi\{X_\alpha: \alpha \in \omega_1\}$ is countably compact.*

COROLLARY 3.13. *If X is a T_2 countably compact lo-net space and $\psi(X) \leqq c$, then card$(X) \leqq c$.*

IV. Further Applications. The introductory lemma in this section is the essential tool used in the applications of lo-nets to

various weak covering theorems. The published proof of Theorem 2.4 [6] was modified to be an argument using a linearly ordered base. However, the unpublished proof of this theorem in the preprint of [6] is particularly relevant to the lo-net argument. A variation of this construction is presented here as the proof of the following lemma. It is given here for completeness and because it shows clearly the essential interaction between lo-nets and closures of unions which are at the center of the applications that follow.

LEMMA 4.1. *If $(x_\lambda)_{\lambda\in L}$ is a lo-net in $\cup\mathscr{G}$ with $x_\lambda \to x$, then either there is some $G\in\mathscr{G}$ where $x\in \mathrm{cl}\,(G)$ or there is a subcollection $\mathscr{G}'\subset\mathscr{G}$ and a choice function y on $\mathscr{G}'$ such that $x\in \mathrm{cl}\,(\{y(G): G\in\mathscr{G}'\})$.*

Proof. Suppose $x\notin \mathrm{cl}\,(G)$ for each $G\in\mathscr{G}$. Well order $\mathscr{G}$. Let $\lambda_1\in L$ and let G_1 be the first set that contains x_{λ_1}. Suppose for each $\alpha<\beta$, λ_α and G_α are such that

(1) if $x_\lambda\in\bigcup_{\gamma<\alpha}G_\gamma$, then $\lambda<\lambda_\alpha$,

(2) $x_{\lambda_\alpha}\notin\bigcup_{\gamma<\alpha}G_\gamma$ and

(3) G_α is the first set such that $x_{\lambda_\alpha}\in G_\alpha$.

Of particular importance to the definition of the choice function is the following: if $\eta<\nu$, then $x_{\lambda_\nu}\notin\bigcup_{\gamma<\nu}G_\gamma$. Thus, $x_{\lambda_\nu}\notin G_\eta$ and $x_{\lambda_\nu}\in G_\nu$. Hence for $\eta\neq\nu$, $G_\eta=G_\nu$. If $\{\lambda_\alpha:\alpha<\beta\}$ is cofinal in L, then $x_{\lambda_\alpha}\to x$ and we are finished by letting $\mathscr{G}'=\{G_\alpha:\alpha<\beta\}$ and let $y(G_\alpha)=x_{\lambda_\alpha}$, for each $\alpha<\beta$. Then y is a choice function on $\mathscr{G}'$ such that $x\in \mathrm{cl}\,(\{y(G): G\in\mathscr{G}'\})$.

If $\{\lambda_\alpha:\alpha<\beta\}$ is not cofinal in L we continue as follows. If $\beta=\nu+1$, then by hypothesis G_ν is the least set such that $x_{\lambda_\nu}\in G_\nu$, $x_{\lambda_\nu}\notin\bigcup_{\gamma<\nu}G_\gamma$ and if $x_\lambda\in\bigcup_{\gamma<\nu}G_\gamma$ then $\lambda<\lambda_\nu$. Since $x\notin \mathrm{cl}\,(G_\nu)$, $\{\lambda: x_\lambda\in G_\nu\}$ is bounded. Let $\lambda_\beta\in L$ be such that $\lambda<\lambda_\beta$, for each $x_\lambda\in G_\nu$, and let G_β be the first set such that $x_{\lambda_\beta}\in G_\beta$. If $x_\lambda\in\bigcup_{\gamma<\beta}G_\gamma=(\bigcup_{\gamma<\nu}G_\gamma)\cup G_\nu$, then $\lambda<\lambda_\nu<\lambda_\beta$ or $x_\gamma\in G_\nu$ and $\lambda<\lambda_\beta$ and in either case $\lambda<\lambda_\beta$. Thus $x_{\lambda_\beta}\notin\bigcup_{\gamma<\beta}G_\gamma$. Otherwise, if β is a limit ordinal, let λ_β be any index such that $\lambda_\alpha<\lambda_\beta$ for each $\alpha<\beta$ and let G_β be the first set such that $\lambda_\alpha<\lambda_\beta$ for each $\alpha<\beta$ and let G_β be the first set such that $x_{\lambda_\beta}\in G_\beta$. If $x_\lambda\in\bigcup_{\alpha<\beta}G_\alpha$, then $x_\lambda\in G_{\alpha_0}$ for $\alpha_0<\beta$. Thus, $x_\lambda\in\bigcup\{G_\alpha:\alpha<\alpha_0+1\}$ and $\lambda<\lambda_{\alpha_0+1}<\lambda_\beta$. Also, if $x_{\lambda_\beta}\in G_\gamma$ for $\gamma<\beta$, then $\lambda_\beta<\lambda_\gamma<\lambda_\beta$ and this contradiction implies $x_{\lambda_\beta}\notin\bigcup_{\gamma<\beta}G_\gamma$. Thus appropriate λ_β and G_β are selected in either case. Since β is bounded by $\mathrm{card}\,(L)^+$, the induction continues until a cofinal case is reached. This completes the proof.

Theorem 2.4 [6] is the key theorem for the many applications of lob spaces to the various weak covering theorems in [6] and [7]. This important theorem is true for the weaker notion of very lo-net spaces and is restated here as a consequence of Lemma 4.1.

THEOREM 4.2. [Davis] *If X is a very lo-net space and U is a collection of subsets such that $p \in \mathrm{cl}(\cup \mathcal{G})$ then either there exists a $G \in \mathcal{G}$ such that $x \in \mathrm{cl}(G)$ or there exists $\mathcal{G}' \subset \mathcal{G}$ and a choice function y on $\mathcal{G}'$ such that $x \in \mathrm{cl}(\{y(G): G \in \mathcal{G}'\})$.*

Thus this class of pseudo-open images of lob-spaces suffices to insure the validity of many results found in [6] and [7]. The natural question is then if the weaker lo-net spaces suffice to prove Theorem 4.1. The counterexample is extremely simple and is a well known lo-net space.

EXAMPLE 4.3. *Theorem* 4.1 *is not true for lo-net spaces.*

Consider the space $S_2 = \{(0, 0)\} \cup \{(1/n, 0): n \in N\} \cup \{(1/n, 1/m): n, m \in N\}$ in [1] and [4]. Let $U_n = \{(1/n, 1/m): m \in N\}$. Then $(0, 0) \notin \mathrm{cl}(U_n)$ for each n and any choice function y on $\mathcal{U} = \{U_n: n \in N\}$ selects one point $y(U_n) = p_n$ in each U_n. But $(0, 0) \notin \mathrm{cl}(\{p_n: n \in N\})$.

A space X is called a *quasi k-space* if a subset $H \subset X$ is closed if and only if $F \cap C$ is closed in C for every countably compact subspace $C \subset X$. Theorem 4.2 has an interesting companion theorem; Lemma 2.3.1 of [7]. However, Lemma 2.3.1 is false as the previous example shows. S_2 is sequential and thus is a k-space. Hence S_2 is a quasi-k-space. Since $(0, 0) \in \mathrm{cl}(\cup \mathcal{U}) \setminus \cup \{\mathrm{cl}(U_n): n \in N\}$, $\{U_n: n \in N\}$ is not closure preserving. Any set formed by choosing a finite number of points in each U_n has no limit points. In particular, the choice function must be defined on the closures of the sets in some subcollection $\mathcal{U}'$. This is a subtle point relating to the weak topology induced by a class of sets and is extremely important here. A corrected version of this lemma follows.

THEOREM 4.4. *If X is a quasi k-space and $\mathcal{G}$ is a collection of subsets which is not closure preserving, then there is a subcollection $\mathcal{G}' \subset \mathcal{G}$ and a choice function y on $\{\mathrm{cl}(G): G \in \mathcal{G}\}$ such that $\{y(\mathrm{cl}(G)): G \in \mathcal{G}'\}$ has a cluster point.*

Proof. For some subcollection $\mathcal{G}^* \subset \mathcal{G}$, $\cup\{\mathrm{cl}(G): G \in \mathcal{G}^*\}$ is not closed. Then there exists a countably compact set C such that $C \cap (\cup\{\mathrm{cl}(G): G \in \mathcal{G}^*\})$ is not closed in C. Let $\mathcal{G}^*$ be well ordered, using an initial set of ordinals, and let ν be the least ordinal such that $C \cap (\cup\{\mathrm{cl}(G_\alpha): \alpha < \nu\})$ is not closed in C. Then ν is a limit ordinal and for each α in a cofinal subset of $[1, \nu)$, there is a point $x_\alpha \in (C \cap \mathrm{cl}(G_\alpha)) \setminus \cup\{\mathrm{cl}(G_\beta): \beta < \alpha\}$. Choose this $x_\alpha = y(\mathrm{cl}(G_\alpha))$ from each $\mathrm{cl}(G_\alpha)$ for this cofinal set of $[1, \nu)$ and let $\mathcal{G}'$ be the subcollec-

tion of $\mathcal{G}$ is lexed by this cofinal set. Thus, $\{y(\mathrm{cl}\,(G)): G \in \mathcal{G}'\}$ is an infinite subset of C and thus has a cluster point. This completes the proof.

Of additional interest is the following modified extension of theorem 4.2. This extension to the lo-net spaces is a parallel theorem to the previous theorem 4.4. This is parallel in the sense that this theorem also uses the weak topology induced by a collection of sets, namely the lo-nets, as theorem 4.4 used the countably compact subspaces. However, it is completely independent because Examples 3.7 and 3.8 of Davis [6] also show there is no subclass relationship between lo-net spaces and quasi k-spaces.

THEOREM 4.5. *If X is a lo-net space and $\mathcal{G}$ is a collection of subsets of X which is not closure preserving, then there is a subcollection $\mathcal{G}' \subset \mathcal{G}$ and a choice function y on $\{\mathrm{cl}\,(G): G \in \mathcal{G}'\}$ such that $\{y(\mathrm{cl}\,(G)): G \in \mathcal{G}'\}$ has a cluster point.*

Proof. For a subcollection $\mathcal{G}^* \subset \mathcal{G}$, $\cup\{\mathrm{cl}\,(G): G \in \mathcal{G}^*\}$ is not closed. Since X is a lo-net space, there exists a lo-net (x_λ) in $\cup\{\mathrm{cl}\,(G): G \in \mathcal{G}^*\}$ converging to a point x which is not in $\cup\{\mathrm{cl}\,(G): G \in \mathcal{G}^*\}$. Thus, $x \notin \mathrm{cl}\,(G)$, for each $G \in \mathcal{G}^*$ and by the selection process in Lemma 4.1 and [6, Th. 2.4] there exists $\mathcal{G}' \subset \mathcal{G}^*$ and a choice function y on $\{\mathrm{cl}\,(G): G \in \mathcal{G}'\}$ such that $x \in \mathrm{cl}\,(\{y(\mathrm{cl}\,(G)): G \in \mathcal{G}\})$. Thus $\{y(\mathrm{cl}\,(G)): G \in \mathcal{G}'\}$ has a cluster point and this completes the proof.

The following definitions are due to Briggs [5]. A collection of subsets of a space X, $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ is a *ppc-collection* (*$\aleph$-ppc-collection*) provided: if $B \subset A$ is infinite (uncountable) and if p_β and $q_\beta \in H_\beta$ for each $\beta \in B$ and $\alpha \neq \beta$ implies $p_\alpha \neq p_\beta$ and $q_\alpha \neq q_\beta$, then $Q = \{q_\beta: \beta \in B\}$ has a cluster point whenever $P = \{p_\beta: \beta \in B\}$ has a cluster point. A space is *preparacompact* (*$\aleph$-preparacompact*) if every open cover has a ppc-refinement ($\aleph$-pp-refinement).

Since theorems 4.4 and 4.5 are true for the collection of closures, we add further that $\mathcal{H}$ will be called a *strong* ppc-*collection* (*strong* $\aleph$-ppc-*collection*) if the p_β and q_β can be chosen from the closures of the sets H_β. Also, a space will be *strongly preparacompact* (*strongly $\aleph$-preparacompact*) if every open cover has a strong ppc-refinement (strong $\aleph$-ppc-refinement). These notions were independently introduced and studied by Nitta [11]. Observe in Example 4.2 the collection $\{U_n: n \in N\}$ is a ppc-collection which is not a strong ppc-collection.

THEOREM 4.6. [Davis [6]] *Let X be a very lo-net space and let $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ be an* $\aleph$-ppc *collection of open subsets of X. If there*

exists a discrete collection $\{D_\beta: \beta \in B\}$ *of nonempty subsets of* X *such that* $D_\beta \subset G_\beta$ *for* $\beta \in B \subset A$, *then* $\{G_\beta: \beta \in B\}$ *is either countable or closure preserving.*

Theorem 4.6 is not true for lo-net spaces as the following example shows.

EXAMPLE 4.7. Let T be the set of countable nonlimit ordinals. For each $\beta \in T$, let $S_1(\beta)$ be a copy of the convergent $1/n$-sequence with limit point 0_β. Let X be the quotient space formed by attaching the limit point 0_β of $S_1(\beta)$ to β in the ordinal space $[0, \omega_1]$, for each $\beta \in T$. X is a lo-net space. If $G_\beta = S_1(\beta) - \{0_\beta\}$, for each $\beta \in T$, $\mathcal{G} = \{G_\beta: \beta \in T\}$ is an $\aleph$-ppc-collection. If D_β consists of any finite subset of G_β for each $\beta \in T$, then $\{D_\beta: \beta \in T\}$ is discrete. However, $\{G_\beta: \beta \in T\}$ is neither countable nor closure preserving.

Theorem 4.6 has the following valid variation for lo-net spaces.

THEOREM 4.8. *Let* X *be a lo-net space (or a quasi k-space) and let* $\mathcal{G} = \{G_\alpha: \alpha \in A\}$ *be a strongly* $\aleph$-ppc-*collection of subsets of* X. *If there exists a discrete collection* $\{D_\beta: \beta \in B\}$ *of nonempty subsets of* X *such that* $D_\beta \subset G_\beta$, *for each* $\beta \in B \subset A$, *then* $\{G_\beta: \beta \in B\}$ *is either countable or closure preserving.*

Proof. Let B be uncontable and suppose $\{G_\beta: \beta \in B\}$ is not closure preserving. By theorem 4.5 (or theorem 4.4) there is a subset $B_1 \subset B$ and a choice function y on $\{\mathrm{cl}\,(G_\beta): \beta \in B_1\}$ such that $\{y(\mathrm{cl}\,(G_\beta)): \beta \in B_1\}$ has a cluster point. Let $B_2 \subset B_1$ be such that for distinct elements $\alpha, \beta \in B_2$, $y(\mathrm{cl}\,(G_\alpha)) \neq y(\mathrm{cl}\,(G_\beta))$. For each $\beta \in B$, choose any $q_\beta \in D_\beta$. For $\beta \in B_2$, let $p_\beta = y(\mathrm{cl}\,(G_\beta))$ and for $\beta \in B - B_2$, let $p_\beta = q_\beta$. Then $\{q_\beta: \beta \in B\}$ is a closed discrete set, but $\{p_\beta: \beta \in B\}$ has a cluster point. This contradicts the fact that $\mathcal{G}$ is a strong $\aleph$-ppc-collection. This proves the theorem.

Accordingly, the following variation of [6, Th. 3.3] is a valid characterization of paracompactness in either lo-net spaces or quasi-k-spaces.

THEOREM 4.9. *If* X *is a regular lo-net space (or quasi k-space), then* X *is paracompact if and only if* X *is irreducible and strongly* $\aleph$-*preparacompact.*

Also, if H is an $\aleph$-ppc collection and $\mathrm{cl}\,(G_\alpha) \subset H_\alpha$ for each $\alpha \in A$, then $\{\mathrm{cl}\,(G_\alpha): \alpha \in A\}$ is an $\aleph$-ppc-collection. Thus Lemma 3.4.2 [7] which uses quasi-k-spaces can also be stated for lo-net spaces and

its validity follows from Theorem 4.8.

THEOREM 4.10. *Let X be a lo-net space and let $\mathcal{H} = \{H_\alpha: \alpha \in A\}$ be an $\aleph$-ppc-collection such that for the collection $\{G_\alpha: \alpha \in A\}$, $\mathrm{cl}\,(G_\alpha) \subset H_\alpha$ for each $\alpha \in A$. If there exists a discrete collection $\{D_\beta: \beta \in B \subset A\}$ of nonempty subsets such that $D_\beta \subset G_\beta$, for each $\beta \in B$, then $\{G_\beta: \beta \in B\}$ is either countable or closure preserving.*

With ppc in place of $\aleph$-ppc the word countable may be omitted from the conclusion.

Sheldon Davis has been gracious enough to carefully study the preprint of this paper and has supplied the following results, which extend various theorems in [7]. The proofs are easily excessible as modifications of those in [7] and are omitted here.

THEOREM 4.11. *If X is a normal (regular) preparacompact (or $\aleph$-preparacompact) lo-net space, then X is collectionwise normal (Hausdorff).*

COROLLARY 4.12. *If $\mathcal{P}$ is property such that $\mathcal{P}$ plus collectionwise normality implies paracompact, then every normal, preparacompact lo-net space which satisfies $\mathcal{P}$ is paracompact.*

COROLLARY 4.13. *If X is a normal lo-net space, then X is paracompact if and only if X is preparacompact and θ-refinable.*

COROLLARY 4.14. *If X is a normal lo-net space and $X = \cup\{F_n: n \in \omega\}$ where F_n is closed and isocompact for each $n \in \omega$, then X is isocompact* (See Th. 2.6 of [7].)

COROLLARY 4.15. *Isoparacompactness is F_σ-hereditary in normal lo-net spaces.*

THEOREM 4.16. *If Y is a preparacompact lo-net space and $f: X \to Y$ is closed, continuous, finite-to-one, then X is paracompact.* (Proof of Th. 2.5 [7] suffices.)

COROLLARY 4.17. *If Y is a lo-net space, $f: X \to Y$ is closed continuous, finite to one and X is isocompact, then Y is isocompact.*

COROLLARY 4.18. *If $X = \cup\{X_\alpha: \alpha \in A\}$ is a lo-net space where $\{X_\alpha: \alpha \in A\}$ is a closed, locally finite collection and X_α is isoparacompact for each $\alpha \in A$, then X is isoparacompact.*

THEOREM 4.19. *Every preparacompact lo-net space is* ppc-*expandable.* (See Th. 3.2. [7].)

THEOREM 4.20. *If X is a normal* ppc-*expandable lo-net space, then any locally finite collection of closed sets has a* ppc-*expansion.*

THEOREM 4.21. *If X is a normal (regular) discretely* ppc-*expandable lo-net space, then X is collectionwise normal (Hausdorff).*

V. **Well-ordered net spaces.** In this section the true nature of lo-net spaces as a generalization of sequential spaces and lob-spaces, is revealed in the characterization, Theorem 5.2. The following fact is essential.

LEMMA 5.1. *For each linearly ordered set, there is a well-ordered cofinal subset.*

This lemma implies that the lob-spaces of Davis [6] are precisely the spaces which have well-ordered local bases, wob-spaces (well ordered by reverse inclusion). The quotient spaces of wob-spaces would be characterized as those spaces which have the weak topology generated by the collection of well-ordered nets. Thus, I define a space X to be a *well-ordered net space* (wo-net space) provided X has the weak topology generated by the natural cover of well-ordered nets in X. Since lob-space $\equiv$ wob space, the classes of quotient spaces are identical.

THEOREM 5.2. *A space is a lo-net space if and only if it is a* wo-*net space.*

The properties of lo-net spaces presented in §3 should be re-examined as wo-net space properties. The class of test spaces for the wo-net spaces (thus lo-net spaces has been determined in [3] as the test spaces for sequential spaces were determined in [2].

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Pacific Journal of Mathematics
Vol. 98, No. 1 March, 1982

Humberto Raul Alagia, Cartan subalgebras of Banach-Lie algebras of operators 1
Tom M. (Mike) Apostol and Thiennu H. Vu, Elementary proofs of Berndt's reciprocity laws 17
James Robert Boone, A note on linearly ordered net spaces 25
Miriam Cohen, A Morita context related to finite automorphism groups of rings 37
Willibald Doeringer, Exceptional values of differential polynomials 55
Alan Stewart Dow and Ortwin Joachim Martin Forster, Absolute C^*-embedding of F-spaces 63
Patrick Hudson Flinn, A characterization of M-ideals in $B(l_p)$ for $1 < p < \infty$ 73
Jack Emile Girolo, Approximating compact sets in normed linear spaces 81
Antonio Granata, A geometric characterization of nth order convex functions 91
Kenneth Richard Johnson, A reciprocity law for Ramanujan sums 99
Grigori Abramovich Kolesnik, On the order of $\zeta(\frac{1}{2}+it)$ and $\Delta(R)$ 107
Daniel Joseph Madden and William Yslas Vélez, Polynomials that represent quadratic residues at primitive roots 123
Ernest A. Michael, On maps related to σ-locally finite and σ-discrete collections of sets 139
Jean-Pierre Rosay, Un exemple d'ouvert borné de $\mathbf{C}^3$ "taut" mais non hyperbolique complet 153
Roger Sherwood Schlafly, Universal connections: the local problem 157
Russel A. Smucker, Quasidiagonal weighted shifts 173
Eduardo Daniel Sontag, Remarks on piecewise-linear algebra 183
Jan Søreng, Symmetric shift registers. II 203
H. M. (Hari Mohan) Srivastava, Some biorthogonal polynomials suggested by the Laguerre polynomials 235