ABSOLUTE $C^*$-EMBEDDING OF $F$-SPACES

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Let $\mathcal{U}$ be an open cover of a space $X$. We define $\mathcal{U}$ to be a $P$-cover if each element of $\mathcal{U}$ is a proper subset of $X$, $\mathcal{U}$ is closed under countable unions and for every $U \in \mathcal{U}$ there is a $V \in \mathcal{U}$ such that $U$ and $X \setminus V$ are completely separated. We prove an $F$-space $X$ is $C^*$-embedded in every $F$-space it is embedded in iff $X$ has no $P$-covers or $X$ is almost compact.

1. Introduction. In 1949, Hewitt [7] proved that a Tychonoff space is $C^*$-embedded in every Tychonoff space in which it is embedded iff $X$ is almost compact. C. E. Aull [1] has shown that a $P$-space $X$ is $C^*$-embedded in every $P$-space in which it is embedded iff $X$ is almost Lindelöf (given disjoint zero sets of $X$ at least one is Lindelöf). These two theorems are examples of absolute $C^*$-embedding theorems. In §3 of this paper we will provide the absolute $C^*$-embedding theorem for $F$-spaces. In §4 we obtain partial results concerning $C^*$-embeddings in basically disconnected spaces.

2. DEFINITIONS. All topological spaces will be assumed to be Tychonoff. The following theorem is useful when dealing with $F$-spaces and also provides a definition of $F$-spaces.

THEOREM 2.1 [6, 14.25]. The following are equivalent

(1) $X$ is an $F$-space.
(2) $\beta X$ is an $F$-space.
(3) disjoint cozero subsets of $X$ are completely separated.
(4) cozero subsets of $X$ are $C^*$-embedded.
(5) disjoint cozero subsets of $\beta X$ have disjoint closures.

$X$ is basically disconnected if the closure of every cozero set is clopen. $X$ is a $P$-space if every zero set of $X$ is open. The reader is referred to [6] for background on $P$-spaces, $F$-spaces and basically disconnected spaces. $X$ is weakly Lindelöf if every open cover of $X$ contains a countable subcollection whose union is dense in $X$ [2]. If $X$ is a subspace of $Y$ and $\mathcal{C}$ is a collection of subsets of $Y$, we define $\mathcal{C}|_X = \{C \cap X: C \in \mathcal{C}\}$.

The cardinality of a set $K$ is denoted by $|K|$ and the immediate successor of a cardinal $\alpha$ is denoted by $\alpha^+$. The cofinality of a non-
successor ordinal $\alpha$, denoted by $\text{cf}(\alpha)$, is the smallest cardinal $\kappa$ such that $\alpha = \sup \{\delta_\gamma : \gamma < \kappa\}$, where $\delta_\gamma < \alpha$. Our notation and terminology follows that of the Gillman-Jerison text [6].

3. Absolute $C^*$-embedding of $F$-spaces.

**Definition 3.1.** An open cover $\mathcal{G}$ of $X$ is called a $P$-cover if each $U \in \mathcal{G}$ is a proper subset of $X$, $\mathcal{G}$ is closed under countable unions and for each $U \in \mathcal{G}$ there is a $V$ in $\mathcal{G}$ such that $U$ and $X \setminus V$ are completely separated in $X$.

It is immediate from the definition that a weakly Lindelöf space has no $P$-covers. In this paper we will find similarities between weakly Lindelöf $F$-spaces and $F$-spaces without $P$-covers, but in §5 we will give an example of an $F$-space without $P$-covers and which is not weakly Lindelöf.

**Definition 3.2.** We will call $A \subset X$ a $P$-set of $X$ if $A$ is compact and any disjoint cozero set of $X$ is completely separated from $A$. If $A = \{p\}$ is a $P$-set, then $p$ (as usual) is called a $P$-point.

The following result motivates the use of the term "$P$-cover".

**Lemma 3.3.** There exists a $P$-set of $\beta X$ contained in $\beta X \setminus X$ iff $X$ has a $P$-cover.

**Proof.** Let $P$ be a $P$-set of $\beta X$ which is contained in $\beta X \setminus X$. Let $\mathcal{G} = \{C : C$ is a cozero subset of $\beta X$ and $C \cap P = \emptyset \}$. We will show that $\mathcal{G} |_X = \{C \cap X : C \in \mathcal{G}\}$ is a $P$-cover of $X$. It is immediate that $\mathcal{G} |_X$ is closed under countable unions. If $U \in \mathcal{G}$, then $P$ and $U$ are completely separated by the definition of a $P$-set. Hence there is a zero set $Z$ of $\beta X$ containing $P$ such that $U$ and $Z$ are completely separated in $\beta X$. Let $V = \beta X \setminus Z$; then $V \in \mathcal{G}$, and $U \cap X$ is completely separated from $Z \cap X = X \setminus V$. Also if $C \in \mathcal{G}$ then $\text{cl}_X (C \cap X) \cap P = \emptyset$ so $C \cap X$ is a proper subset of $X$. Therefore $\mathcal{G} |_X$ is a $P$-cover of $X$.

For the converse, assume $\mathcal{G}$ is a $P$-cover of $X$. Define $P$ to be $\cap \{\text{cl}_X (X \setminus C) : C \in \mathcal{G}\}$. We will show that $P$ is the required $P$-set. $P$ is compact and nonempty since $\mathcal{G}$ is closed under finite unions and therefore $\{\text{cl}_X (X \setminus C) : C \in \mathcal{G}\}$ has the finite intersection property. Also $P$ is contained in $\beta X \setminus X$ since $\mathcal{G}$ is a cover of $X$. Let $U$ be a cozero subset of $\beta X$ such that $U \cap P = \emptyset$. Then $U$ is Lindelöf and $\cap \{\text{cl}_X (X \setminus C) : C \in \mathcal{G}\} \cap U = \emptyset$, therefore there is a subset $\{C_n : n < \omega\}$ of $\mathcal{G}$ such that $\cap \{\text{cl}_X (X \setminus C_n) : n < \omega\} \cap U = \emptyset$. In parti-
LEMMA 3.4. Let $K$ be a compact $F$-space. If $P$ is a $P$-set of $K$ and $q$ is a point of $K$, then the quotient space formed by collapsing $P \cup \{q\}$ to a point is an $F$-space.

Proof. Let $Y$ be the quotient space and $f$ the quotient map. Since $P \cup \{q\}$ is compact, $Y$ is Tychonoff. All that remains to be shown is that disjoint cozero sets $C^0$ and $C^1$ of $Y$ can be completely separated. The cozero sets $f^{-1}(C^0)$ and $f^{-1}(C^1)$ of $K$ are disjoint, so $\text{cl}_K f^{-1}(C^0) \cap \text{cl}_K f^{-1}(C^1) = \emptyset$. We can assume w.l.o.g. that $q \notin \text{cl}_K f^{-1}(C^1)$. Since $q \notin \text{cl}_K f^{-1}(C^1)$ implies $q \notin f^{-1}(C^1)$, we have $(P \cup \{q\}) \cap f^{-1}(C^1) = \emptyset$, therefore the full preimage of $f(\text{cl}_K f^{-1}(C^1))$ is $\text{cl}_K f^{-1}(C^1)$. Thus $f(\text{cl}_K f^{-1}(C^0))$ and $f(\text{cl}_K f^{-1}(C^1))$ are disjoint compact sets of $Y$ which contain $C^0$ and $C^1$ respectively, so $C^0$ is completely separated from $C^1$ in $Y$. □

It is known that the property "weakly Lindelöf" is inherited by regular closed subspaces. Though a regular closed subspace of an $F$-space without $P$-covers may have a $P$-cover [3, pg. 70], we do have the following result.

LEMMA 3.5. If $C$ is a cozero set of an $F$-space $X$ and $X$ has no $P$-covers then $\text{cl}_X C$ has no $P$-covers.

Proof. Assume $\text{cl}_X C$ has a $P$-cover. Then, by Lemma 3.3, there exists a $P$-set $P$ of $\beta(\text{cl}_X C)$ contained in $\beta(\text{cl}_X C) \setminus \text{cl}_X C$. $C$, and therefore $\text{cl}_X C$, are $C^*$-embedded in $X$, so $P \subset \beta(\text{cl}_X C) = \text{cl}_{\beta X} C \subset \beta X$. We will show that $P$ is a $P$-set of $\beta X$. Let $U$ be a cozero set of $\beta X$ such that $U \cap P = \emptyset$. Then $U \cap \text{cl}_{\beta X} C$ is a cozero set of $\text{cl}_{\beta X} C$ which misses $P$, hence $\text{cl}_{\beta X} (U \cap \text{cl}_{\beta X} C) \cap P = \emptyset$. Since $\text{cl}_{\beta X} (U \cap \text{cl}_{\beta X} C)$ and $P$ are disjoint compact sets of $\beta X$, there is a zero set $Z$ of $\beta X$ which contains $\text{cl}_{\beta X} (U \cap \text{cl}_{\beta X} C)$ and misses $P$. $(U \setminus Z) \cap X$ and $C$ are disjoint cozero sets of $X$, and have disjoint closures in $\beta X$. But now we have $Z \cup \text{cl}_{\beta X} [(U \setminus Z) \cap X]$ is a compact set containing $U$ which misses $P$, so $P \cap (\text{cl}_{\beta X} U) = \emptyset$. $X$ has a $P$-cover since $P$ is a $P$-set of $\beta X$ and $P \subset \text{cl}_{\beta X} C \setminus \text{cl}_X C \subset \beta X \setminus X$. □

THEOREM 3.6. Let $A$ and $B$ be subsets of an $F$-space $X$ such
that neither $A$ nor $B$ have $P$-covers and $\text{cl}_X A \cap B = A \cap \text{cl}_X B = \phi$. Then $A$ and $B$ are completely separated in $X$.

Proof. Let $K = \text{cl}_{\beta X}(A \cup B)$. The compact set $K$, as a $C^*$-embedded subset of an $F$-space, is an $F$-space [6, 14.26]. It will suffice to show $A$ and $B$ are completely separated in $K$.

Define $\mathcal{U} = \{U: U$ is a cozero set of $K$ and $U$ and $B$ are completely separated in $K\}$. Define $\mathcal{U}|_A = \{U \cap A: U \in \mathcal{U}\}$. By assumption, $A \cap \text{cl}_K B = \phi$, so $\mathcal{U}|_A$ is an open over of $A$. If $A \in \mathcal{U}|_A$, then $A$ and $B$ are completely separated so we assume $A \in \mathcal{U}|_A$ and we will arrive at a contradiction.

If $U \in \mathcal{U}$, then there exists a zero set $Z$ of $K$ containing $U$ and completely separated from $B$. Choose a cozero set $V$ containing $Z$ and completely separated from $B$. So we now have $U \subset Z \subset V \subset K \setminus B$ and $V \in \mathcal{U}$. Since $U$ is disjoint from the cozero set $K \setminus Z$, $U$ is completely separated from $K \setminus V \subset K \setminus Z$. Since $A$ has no $P$-covers, $\mathcal{U}|_A$ is not a $P$-cover, therefore there exist countably many cozero sets $\{U_i; i < \omega\} \subset \mathcal{U}$ such that $\cup\{U_i \cap A; i < \omega\} \in \mathcal{U}|_A$. Let $W = \cup\{U_i; i < \omega\}$.

Define $\mathcal{Y} = \{V: V$ is a cozero set of $K$ and $V \cap W = \phi\}$. $\mathcal{Y}|_B$ is a cover of $B$ since $B \cap \text{cl}_K A = \phi$ and $\text{cl}_K W \subset \text{cl}_K A$. If $U \in \mathcal{Y}$ then there exists a zero set $Z$ of $K$ containing $W$ and completely separated from $U$. If $V = K \setminus Z$ then $V \in \mathcal{Y}$. $U$ is completely separated from $K \setminus V = Z$, so $U \cap B$ is completely separated from $B \setminus (V \cap B)$. $\mathcal{Y}|_B$ is obviously closed under countable unions. But $\mathcal{Y}|_B$ cannot be a $P$-cover of $B$, so $B \in \mathcal{Y}|_B$, therefore there exists a cozero set $V$ of $K$ such that $B \subset V \in \mathcal{Y}$ and $V \cap W = \phi$. Therefore $B$ and $W$ are completely separated, which is a contradiction to $W \notin \mathcal{U}$.

We now state and prove the main theorem of this paper.

**Theorem 3.7.** An $F$-space $X$ is $C^*$-embedded in every $F$-space it is embedded in iff $X$ has no $P$-covers or $X$ is almost compact.

Proof. Assume that $X$ is an $F$-space with no $P$-covers and $X$ is embedded in an $F$-space $Y$. It will suffice to show that disjoint cozero sets of $X$ are completely separated in $Y$. Let $C^0$ and $C^1$ be disjoint cozero sets of $X$. By Lemma 3.5, $\text{cl}_x C^0$ and $\text{cl}_x C^1$ have no $P$-covers. We note that $\text{cl}_Y (\text{cl}_X C^0) \cap \text{cl}_X C^1 = \phi$ and $\text{cl}_X C^0 \cap \text{cl}_Y (\text{cl}_X C^1) = \phi$, so by Theorem 3.6, they are completely separated in $Y$. 

For the converse assume $X$ is not almost compact and $X$ has a $P$-cover. By Lemma 3.3 there is a $P$-set $P$ of $\beta X \setminus X$. Choose a point $q \in \beta X \setminus X$ such that $|P \cup \{q\}| > 1$. Then by Lemma 3.4, the quotient space $\beta X / (P \cup \{q\})$ obtained by collapsing $P \cup \{q\}$ to a point is an $F$-space in which $X$ is densely embedded but not $C^*$-embedded.

The next corollary uses a construction similar to one given in [10, pg. 96]. We will show that for every space $X$ which is embedded in an $F$-space $Y$, there is an $F$-space $W$ in which

1. $X$ is embedded as a closed set and
2. $X$ is $C^*$-embedded in $W$ iff $X$ is $C^*$-embedded in $Y$.

**COROLLARY 3.8.** An $F$-space $X$ is $C^*$-embedded in every $F$-space if it is embedded in as a closed set iff $X$ has no $P$-covers or $X$ is almost compact.

**Proof.** Suppose $X$ is embedded in an $F$-space $Y$. Let $\lambda$ be the least ordinal of cardinality $|\beta Y|^+$. Define $A = (\lambda + 1)\{\alpha: cf(\alpha) = \omega\}$. Negrepontis [8] has shown that the product of a $P$-space with a compact $F$-space is an $F$-space. $A$ is a $P$-space, so $A \times \beta Y$ is an $F$-space. Let $W = (A \times \beta Y) / (\{\lambda\} \times \beta Y \setminus X)$. $W$ is a dense $C^*$-embedded subspace of $A \times \beta Y$ (see Example 5.1 or [10, pg. 96]), so $W$ is an $F$-space. $X$ is homeomorphic to the closed subspace $\{\lambda\} \times X$ of $W$. For every continuous real-valued function $f$ defined on $W$, there exists an $\alpha < \lambda$ such that for all $x \in X$, $f(\alpha, x) = f(\lambda, x)$. As a consequence, $\{\lambda\} \times X$ is $C^*$-embedded in $W$ iff $X$ is $C^*$-embedded in $Y$. This will show that Corollary 3.8 is equivalent to Theorem 3.7.

Note that if $X$ is $C$-embedded in $\beta X$ then $X$ is pseudocompact; and a pseudocompact space is $C$-embedded iff it is $C^*$-embedded. This, along with Theorem 3.7, proves the next corollary.

**COROLLARY 3.9.** An $F$-space $X$ is $C$-embedded in every $F$-space if it is embedded in iff $X$ is almost compact or $X$ is pseudocompact and has no $P$-covers.

4. Absolute $C^*$-embedding in basically disconnected spaces.

Let $\mathcal{C}$ be a cover by cozero sets of a basically disconnected space $X$, and assume the union of every countable subcollection of $\mathcal{C}$ is not dense. The set of unions of every countable subset of the open cover $\{\text{cl}_X \cup \{C_n: n < \omega\}: \{C_n: n < \omega\} \subset \mathcal{C}\}$ is easily seen to be a $P$-cover of $X$. Therefore, for a basically disconnected space $X$, $X$ has a $P$-cover iff $X$ is not weakly Lindelöf. By Theorem 3.7 and this
remark we have the following corollary.

**Corollary 4.1.** A basically disconnected space $X$ is $C^*$-embedded in every $F$-space it is embedded in iff $X$ is weakly Lindelöf or $X$ is almost compact.

**Definition 4.2.** A space $X$ is almost weakly Lindelöf if given two disjoint cozero sets of $X$, at least one is weakly Lindelöf.

The next lemma is similar to Lemma 3.4.

**Lemma 4.3.** Let $K$ be a compact basically disconnected space. If $P$ is a $P$-set of $K$, then the quotient space formed by collapsing $P$ to a point is basically disconnected.

**Proof.** Let $Y$ be the quotient space and $f: K \to Y$ the quotient map. Since $P$ is compact, $Y$ is Tychonoff. Let $C$ be a cozero set of $Y$. $\text{cl}_K f^*(C)$ is open and $f$ is a quotient map so we will prove that $\text{cl}_Y C$ is open by showing $f^*(\text{cl}_Y C) = \text{cl}_K f^*(C)$. It is obvious that $\text{cl}_K f^*(C) \subset f^*(\text{cl}_Y C)$, so let $x \in K$ such that $f(x) \in \text{cl}_Y C = f(\text{cl}_K f^*(C))$. We wish to prove $x \in \text{cl}_K f^*(C)$. There is a $y \in \text{cl}_K f^*(C)$ such that $f(x) = f(y)$. If $x = y$, we are done so assume $x \neq y$. Then \{x, y\} $\subset P$. We now have $y \in P \cap \text{cl}_K f^*(C) \neq \emptyset$ and since $P$ is a $P$-set and $f^*(C)$ is a cozero set, $P \cap f^*(C) \neq \emptyset$. Therefore $x \in P \cap f^*(C) \subset \text{cl}_K f^*(C)$. \hfill $\Box$

We now prove the main result in this section.

**Theorem 4.4.** If a basically disconnected space $X$ is $C^*$-embedded is every basically disconnected space it is embedded in, then $X$ is almost weakly Lindelöf.

**Proof.** Let $X$ be a basically disconnected space which is not almost weakly Lindelöf. Let $C^0$ and $C^1$ be disjoint cozero subsets of $X$ neither of which is weakly Lindelöf. A cozero set of a weakly Lindelöf space is weakly Lindelöf [2, Lemma 1.2(c)], therefore $\text{cl}_x C^0$ and $\text{cl}_x C^1$ are not weakly Lindelöf, and since they are basically disconnected spaces, they both have $P$-covers. By the proof of Lemma 3.5 there are two disjoint $P$-sets, $P^0$ and $P^1$, of $\beta X$ contained in $\text{cl}_{\beta X} C^0 \text{cl}_x C^0$ and $\text{cl}_{\beta X} C^1 \text{cl}_x C^1$ respectively. Then $P^0 \cup P^1$ is a $P$-set and the quotient space obtained by collapsing $P^0 \cup P^1$ to a point is basically disconnected by Lemma 4.3. $X$ is a dense subspace of the quotient space, but it is not $C^*$-embedded since $|P^0 \cup P^1| > 1$. \hfill $\Box$
Unfortunately, an example in §5 will show that the property almost weakly Lindelöf is not a sufficient condition for C*-embedding. It remains an open question to characterize the basically disconnected spaces which are C*-embedded in every basically disconnected space in which they are embedded. But we do have the following theorem.

**Theorem 4.5.** If a basically disconnected space $X$ is embedded as an open or dense subspace of a basically disconnected space $Y$, then $X$ is C*-embedded in $Y$ iff $X$ is almost weakly Lindelöf.

**Proof.** Assume $X$ is almost weakly Lindelöf and is embedded in a basically disconnected space $Y$. If $C^0$ and $C^1$ are disjoint cozero sets of $X$, then we can assume that one of them, say $C^0$, is weakly Lindelöf. Define $\mathcal{Y} = \{V: V$ is a cozero set of $Y$, $V \cap \operatorname{cl}_Y C^1 = \emptyset\}$. $\mathcal{Y}|_{\omega}$ is a cover of $C^0$, so there is a countable subcollection $\{V_n: n < \omega\}$ of $\mathcal{Y}$ such that, if $W = \bigcup \{V_n: n < \omega\}$, then $\operatorname{cl}_Y W \supset C^0$. But if $X$ is dense or open in $Y$, $\operatorname{cl}_Y W \cap C^1 = \emptyset$. $\operatorname{cl}_Y W$ is a clopen subset of $Y$ and it is easily seen $C^0$ is completely separated from $C^1$. The other part of the proof is provided by Theorem 4.4. □

5. Some further remarks and examples.

**Example 5.1.** We construct a non-weakly Lindelöf F-space which has no P-covers. Let $K = \beta(\omega \setminus \omega)$. Let $\lambda$ be the initial ordinal of cardinality $|K|^\dagger$. Define $D = (\lambda + 1)\setminus \{\alpha < \lambda: \operatorname{cf}(\alpha) = \omega\}$ where $\lambda + 1$ has the order topology. $D$ is a $P$-space and $K$ is a compact $F$-space, so $D \times K$ is an $F$-space [8]. Choose a non-clopen cozero set $C^0$ of $K$ [6, 6W], and let $B^0 = \operatorname{cl}_K C^0 \setminus C^0$. Our example will be $X = \beta(D \times K) \setminus (\{\lambda\} \times B^0)$. To show $X$ is an F-space we will first show that $Y = D \times K \setminus (\{\lambda\} \times K)$ is C*-embedded in $D \times K$. Let $f$ be a continuous real-valued function on $Y$. Modifying the arguments in [6, 9L] one has for every $k \in K$ an interval $[\alpha_k, \lambda]$ of $\lambda + 1$ such that $f$ is constant on $([\alpha_k, \lambda] \cap D) \times \{k\}$. Let $\beta = \sup \{\alpha_k: k \in K\}$. Since $\operatorname{cf}(\lambda) > |K|$, we have $\beta < \lambda$ and $[\beta + 1, \lambda] \cap D = \emptyset$ is a clopen neighborhood of $\lambda$ in $D$. Define $g: K \rightarrow \mathbb{R}$, where $\mathbb{R}$ is the real line, by declaring $g(k) = f(\beta, k)$. Obviously $g$ is continuous and for all $(\delta, k) \in (V \setminus \{\lambda\}) \times K$, $f(\delta, k) = g(k)$, so $f$ can be continuously extended to $V \times K$ and hence to $D \times K$. We now have $Y$ is a dense C*-embedded subspace of the $F$-space $D \times K$, so $Y$ is an $F$-space and $Y \subset X \subset \beta Y = \beta(D \times K)$, so $Y$ is also an $F$-space.

Choose a cozero set $C'$ of $\beta X = \beta(D \times K)$ such that $C' \cap (D \times K) = D \times C^0$. Then we have $C' \cap (\{\lambda\} \times B^0) = \emptyset$ and $\operatorname{cl}_X C' \supset (\{\lambda\} \times B^0) = \beta X \setminus X$, so there is no $P$-set of $\beta X$ contained in $\beta X \setminus X$. By Lemma
3.3, $X$ has no $P$-covers.

We now show $X$ is not weakly Lindelöf. Let $\mathcal{U} = \{C: C$ is a cozero set of $\beta X$, $C \cap (\{x\} \times B^0) = \emptyset\}$. $\mathcal{U}$ is an open cover of $X$. If $C \in \mathcal{U}$ choose a continuous function $f: \beta X \to \mathbb{R}$ such that $C = \text{coz}(f)$. There is a clopen neighborhood $V$ of $\lambda$ in $D$ and a continuous function $g: K \to \mathbb{R}$ such that $f(\delta, k) = g(k)$ for all $(\delta, k) \in (V \setminus \{\lambda\}) \times K$. Since nonempty zero sets of $K$ have nonempty interior [5], $N = \text{int}_K g^-(0)$ is not empty. Thus $(V \setminus \{\lambda\}) \times N$ is an open set of the dense subspace $(D \setminus \{\lambda\}) \times K$ of $X$ and it is disjoint from $C$, so $C$ cannot be dense in $X$. Since a countable union of elements of $\mathcal{U}$ is again an element of $\mathcal{U}$ we have shown no union of a countable subcollection of $\mathcal{U}$ is dense in $X$.

**Example 5.2.** The next example shows that an almost weakly Lindelöf basically disconnected space need not be $C^*$-embedded in every basically disconnected space it is embedded in.

Let $A$ be $(\omega_2 + 1) \setminus \{\alpha: \text{cf}(\alpha) = \omega\}$ where $\omega_2 + 1$ has the order topology. The space $A$ is basically disconnected, in fact a $P$-space [6, 9L]. Let $X$ be the free union of $A \setminus \{\omega_2\}$ with the countable discrete space $\omega$. This space is almost weakly Lindelöf (see [9]) but we will construct a basically disconnected space in which it is embedded but not $C^*$-embedded. The product $Y = A \times \beta \omega$ is a basically disconnected space [8, Theorem 6.3]. Let $q$ be any point of $\beta \omega \setminus \omega$. The subspace $(A \setminus \{\omega_2\}) \times \{q\} \cup (\{\omega_2\} \times \omega)$ of $Y$ is homeomorphic to $X$. The closures in $Y$ of the sets $(A \setminus \{\omega_2\}) \times \{q\}$ and $\{\omega_2\} \times \omega$ have the point $(\omega_2, q)$ in common, so this copy of $X$ is not $C^*$-embedded in $Y$.

Example 5.2 suggests a proof for the following theorem.

**Theorem 5.3.** A $P$-space $X$ is $C^*$-embedded in every basically disconnected space it is embedded in iff $X$ is Lindelöf.

**Proof.** Suppose $X$ is a $P$-space which is not Lindelöf. Then $X$ is infinite and therefore not pseudocompact [6, 4K.2]. This also means that $X$ is not almost compact. Zero sets of $X$ are clopen so let $A$ and $B$ be complementary clopen subsets of $X$ neither of which is compact. As $X$ is not Lindelöf we can assume that $A$ is not Lindelöf. A non-Lindelöf $P$-space also fails to be weakly Lindelöf and if a basically disconnected space is not weakly Lindelöf, it has a $P$-cover. Therefore there is a $P$-set $P$ of $\beta A$ contained in $\beta A \setminus A$. If we let $Y = A \cup \{P\}$ be the quotient space of $A \cup P$ obtained by collapsing $P$ to a point then $Y$ is also a $P$-space. Since $B$ is not
compact we can choose \( q \in \beta B \setminus B \). The space \( Y \times \beta B \) is basically disconnected \([8, \text{Theorem 6.3}]\) and \( (A \times \{ q \}) \cup (\{ P \} \times B) \) is homeomorphic to \( X \) but it is not \( C^* \)-embedded in \( Y \times \beta B \). The converse follows from Corollary 4.1.

Recall that \( X \) is an \textit{extremally disconnected} space if the closure of every open set of \( X \) is open. The class of extremally disconnected spaces is contained in the class of basically disconnected spaces, and though the absolute \( C^* \)-embedding theorem for basically disconnected spaces is not known, the first author has proven,

\textbf{Theorem 5.4.} \([4]\) An extremally disconnected space \( X \) is \( C^* \)-embedded in every extremally disconnected space it is embedded in iff \( X \) is weakly Lindelöf or almost compact.

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