A CHARACTERIZATION OF $M$-IDEALS IN $B(l_p)$ FOR $1 < p < \infty$

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For $1 < p < \infty$ the only nontrivial $M$-ideal in $\mathcal{B}(\ell_p)$, the bounded linear operators on $\ell_p$, is $K(\ell_p)$, the ideal of compact operators on $\ell_p$.

1. Introduction. Certain theorems for $\mathcal{B}(H)$ (the bounded linear operators on $H$ a separable Hilbert space) are known to hold for $\mathcal{B}(\ell_p)$, $1 < p < \infty$. For example, it is well known that the only nontrivial closed two-sided ideal in $\mathcal{B}(\ell_p)$, $1 \leq p < \infty$ is $K(\ell_p)$, the compact linear operators on $\ell_p$. Hennefeld [4] has shown that $K(\ell_p)$ is an $M$-ideal in $\mathcal{B}(\ell_p)$ for $1 < p < \infty$. It is also known that $K(\ell_2)$ is the only nontrivial $M$-ideal in $\mathcal{B}(\ell_2)$. This follows from the fact that in a $B^*$-algebra, the $M$-ideals are precisely the closed two-sided ideals [5]. The purpose of this paper is to show that this result also generalizes to $\mathcal{B}(\ell_p)$, for $1 < p < \infty$. As this paper is largely based on the work of Smith and Ward [5] it is perhaps not surprising that a result of theirs, namely that every nontrivial $M$-ideal in $\mathcal{B}(\ell_p)$ for $1 < p < \infty$ contains $K(\ell_p)$, has a new proof.

2. Preliminaries. A closed subspace $L$ of a Banach space $X$ is said to be an $L$-ideal [$M$-summand] if there exists a closed subspace $L'$ such that $X = L \oplus L'$ and $\|\ell + \ell'\| = \|\ell\| + \|\ell'\|$ for every $\ell \in L$ and $\ell' \in L'$. A closed subspace $M$ of a Banach space $X$ is an $M$-ideal if $M^\perp$ is an $L$-ideal in $X^*$. Note that $M$-summands are $M$-ideals, but the latter is a more general concept. [For example, $K(\ell_p)$ is an $M$-ideal in $\mathcal{B}(\ell_p)$ but not an $M$-summand, as $K(\ell_p)$ is not complemented in $\mathcal{B}(\ell_p)$.] For basic properties of $M$-ideals, $L$-ideals and $M$-summands, refer to [1].

The state space $S$ of a banach algebra $A$ with identity $e$ is defined to be $\{\phi \in A^*: \phi(e) = \|\phi\| = 1\}$. An element $h \in A$ is hermitian if $\|e^{i\lambda}h\| = 1$ for all real $\lambda$. Equivalently [2] $h$ is hermitian if and only if $\{\phi(h): h \in S\} \subseteq \mathbb{R}$. $A^{**}$ when endowed with Arens multiplication [3] is a Banach algebra with identity $e$, and by the weak-star density of $A$ in $A^{**}$, $h \in A^{**}$ is hermitian if and only if $h$ is real valued on the state space of $A$.

In [5] it is shown that $M$-ideals in Banach algebras are necessarily subalgebras. Other results of this paper and [6] needed in the sequel are now summarized:

Let $M$ be an $M$-ideal in $\mathcal{B}(\ell_p)$, $1 < p < \infty$. Then clearly $M^{\perp\perp}$ is an $M$-summand in $\mathcal{B}(\ell_p)^{**}$; that is, $\mathcal{B}(\ell_p)^{**} = M^{\perp\perp} \oplus e_0 M^\prime$. Let
$P: B(\zeta_p)^{**} \to M^{1,1}$ be the associated $M$-projection. Let $I$ denote the identity in $B(\zeta_p)$, and let $P(I) = z$. Throughout this paper, the following arithmetical facts will be collectively referred to as (*):

$z = z^2$ is hermitian, and commutes with every other hermitian element of $B(\zeta_p)^{**}$. $zM^{1,1} \subseteq M^{1,1}$, $z^2 \subseteq M^2$, and $z^2 z = 0$. Likewise, $(e - z)M^{1,1} \subseteq M^{1,1}$, $(e - z)^2 \subseteq M^2$, and $(e - z)M^{1,1}(e - z) = 0$.

If $S$ is the state space of $B(\zeta_p)$, then $S = 2^0 \cap F_2$ where $B(\zeta_p)^* = M_1^1 \varphi$, $x F_2$ and $1 F_2 = M \cap S$, and $F_2 = M \cap S$ (i.e., $\phi \in S \to$ there exist unique $\phi_1 \in F_1$, $\phi_2 \in F_2$, and $t \in [0, 1]$ such that $\phi = t\phi_1 + (1 - t)\phi_2$). If $z$ is regarded as a real valued affine function on $S$, then $z|_{F_1} = 0$ and $z|_{F_2} = 1$.

An important fact used in this paper which follows easily from the definition of the hermitian elements is that in $B(\zeta_p)$, any diagonal matrix with real entries is hermitian. [These are in fact precisely the hermitian elements of $B(\zeta_p)$ if $1 < p < \infty$, $p \neq 2$ [7].]

In § 3, a matrix $A \in B(\zeta_p)$ whose $i$th row $j$th column entry is $a_{ij}$ will be denoted $\sum_{i,j \geq 1} a_{ij} e_j \otimes e_i$, where $e_j \otimes e_i$ is the rank-one map that sends $e_j$ to $e_i$. $(e_i)_{i \geq 1}$ is the canonical basis for $\zeta_p$. Note that if $A \in B(\zeta_p)$, then $\|A(e_i)\| \leq \|A\|$ for every $i$. That is, every column of $A$ is an element of $\zeta_p$ whose norm does not exceed $\|A\|$. By considering the adjoint, we have that every row of $A$ is an element of $\zeta_p [1/p + 1/q = 1]$ whose norm is less than or equal to $\|A\|$. Clearly, $|a_{ij}| \leq \|A\|$ for every $i, j$, and if $A$ is a matrix with at most one nonzero entry in each row and column, [for example if $A$ is diagonal] then $\|A\|$ is the $\ell_\infty$-norm of the sequence of nonzero entries.

3. Results. Assume all notation in § 2, and assume $M \neq 0$. Recall that $I$ denotes the identity on $\zeta_p$, where throughout this section $1 < p < \infty$, $p \neq 2$.

**Lemma 1.** If $h$ is hermitian in $B(\zeta_p)$ and $h^2 = I$, then for every $m \in M$, $hm \in M$ and $mh \in M$.

**Proof.** Considering $h$ as canonically embedded in $B(\zeta_p)^{**}$, $h = h_1 + h_2$ where $h_1 \in M^{1,1}$, $h_2 \in M^2$, and $\|h\| = \max(\|h_1\|, \|h_2\|)$. Note that $h_1$ and $h_2$ are themselves hermitian elements of $B(\zeta_p)^{**}$, for if $f_1 \in F_1$ then $f_1(h_1) = 0$ and if $f_2 \in F_2$, $f_2(h_1) = f_2(h) \in R$. So for any $\phi \in S$, $\phi(h_1) \in R$, i.e., $h_1$ is hermitian. The same reasoning applied to $h_2$ shows that $h_2$ is also hermitian. $h^2 = I = h_1^2 + h_1 h_2 + h_2 h_1 + h_2^2$, however it is easy to see that $h_1 h_2 = 0 = h_2 h_1$, since by (*) we have that

$$h_1 h_2 = z h_1 h_2 + (e - z) h_1 h_2 = h_1 z h_2 + (e - z) h_1 (e - z) h_2 = 0 .$$

Similarly, $h_2 h_1 = 0$, hence $I = h_1^2 + h_2^2$. 
Now pick $m \in M$, and wlog assume $|m| = 1$. We'll show that $hm \in M$. [hm \in M is shown in similar fashion.] There exist $m_1 \in M^\perp$ and $m_2 \in M^*$ such that $hm = m_1 + m_2$. Claim: $zm_2 = 0 = m_2z$. To see this, note that $zhm = zm_1 + zm_2$ where [using (*)] $zhm = zhm \in M^\perp$ and $zm_1 \in M^\perp$. Hence $zm_2 \in M^\perp \cap M^*$ and so $zm_2 = 0$.

To show $m_2z = 0$ is a little harder: $hmz = h_mz + h_2mz = m_1z + m_2z$ where $hmz \in M^\perp$ and $m_2z \in M^\perp$. If we knew that $hmz \in M^\perp$, then as before we'd have $m_2z \in M^\perp \cap M^* = 0$ and our claim would be established. So suppose $hmz \in M^\perp$. Then there exists some $f_1 \in S \cap M^\perp$ so that $f_1(h_mz) \neq 0$. [This happens as the state space spans $B(\mathcal{L})^*$ and hence $F_x$ spans $M^\perp$.] Choose $\theta \in \mathbb{R}$ so that $f_1(e^{i\theta}h_mz) = \delta > 0$. Then $e^{i\theta}mz \in M^\perp$ has norm at most one, $h_2 \in M^*$ has norm at most one, so $\|h_2(e^{i\theta}mz + h_2^2)\| \leq 1$. But $1 \geq f_1(e^{i\theta}h_2mz + h_2^2) = \delta + f_1(I) = \delta + 1$, a contradiction which proves the claim.

Now $(e - z)hm(e - z) = (e - z)m_1(e - z) + (e - z)m_2(e - z)$. But by (*) we have that $(e - z)hm(e - z) = (e - z)m_1(e - z) = 0 = (e - z)m_1(e - z)$, so $0 = (e - z)m_1(e - z) = m_2$, that is, $hm = m_1 \in M^\perp \cap B(\mathcal{L}) = M$. □

**Remark.** Although stated for $B(\mathcal{L})$, this lemma is true [by the same proof] for any $M$-ideal $M$ and norm-1 hermitian $h$ where $h^2 = I$.

**Corollary.** If $h$ is any diagonal matrix in $B(\mathcal{L})$, then $hM \subseteq M$ and $Mh \subseteq M$.

**Proof.** At this point we know that if $h$ is a diagonal matrix with only ±1’s on the diagonal, then $h^2 = I$ and so $hM \subseteq M$ and $Mh \subseteq M$. But by averaging two such hermitian elements, we have that if $h$ is any diagonal matrix with only 1’s or 0’s on the diagonal, then $hM \subseteq M$ and $Mh \subseteq M$. Hence the result holds for any finite valued diagonal matrix. But such matrices are dense in the diagonal elements of $B(\mathcal{L})$, and so as $M$ is closed, $hM \subseteq M$ and $Mh \subseteq M$ for any diagonal $h$. □

**Corollary.** $M \supseteq K(\mathcal{L})$.

**Proof.** By the previous corollary, if $E_{ij}$ denotes the elementary matrix with a 1 in the $i$th row and $j$th column and zeros elsewhere, then $E_{ij}ME_{ij} \subseteq M$ for every $i \geq 1$ and $j \geq 1$. As $M \neq 0$ there is an $A = \sum a_{ij}e_j \otimes e_i \in M$ such that for some $k$ and $r a_{kj} = 1$. Hence $E_{kr} = E_{kk}A_{1rr} \in M$. Claim: for every $p \geq 1$, $E_{p1} \in M$. If there is any $m = \sum m_{ij}e_j \otimes e_i \in M$ so that $m_{p1} \neq 0$, then $E_{p1} = (1/m_{p1})E_{pp}m_{11} \in M$. So if every $m = \sum m_{ij}e_j \otimes e_i \in M$ has the property that $m_{p1} = 0$, then the norm-1 functional $\rho \in B(\mathcal{L})^*$ defined by $\rho(\sum t_{ij}e_j \otimes e_i) = t_{11}$ is in $M^\perp$. Let $\rho \in B(\mathcal{L})^*$ be defined by $\rho(\sum t_{ij}e_j \otimes e_i) = t_{11}$. Then
Claim: $\|f_t\| = 1$. To see this, suppose that $f_t = \psi_1 + \tau/r^2$, where $\psi_1 \in M^1$, $\psi_2 \in M$. Then $\|f_t\| = \|\psi_1\| + \|\psi_2\|$, and $1 = \|\psi_1\| = \rho_1(E_kc) = \psi_1(E_kc) + \psi_2(E_kc) = \psi_2(E_kc)$, so $\|\psi_2\| = 1 \Rightarrow \|\psi_1\| = 0$. Hence $2 = \|\rho_1 + \rho_2\|$. Choose $T = \sum t_\ell e_j \otimes e_i \in B(\zeta_p)$ so that $\|T\| = 1$ and $|\rho_1(T) + \rho_2(T)| > 2^{1/2}$ where $1/p + 1/q = 1$. Then $2^{1/q} < |t_{\ell k}^1 + t_{\ell k}^2| \leq (\sum |t_{\ell k}|^p)^{1/p} \cdot 2^{1/q} \leq \|T(\zeta_p)\| \cdot 2^{1/q} \leq 2^{1/q}$, a contradiction implying that $E_{\ell k} \in M$. A similar argument shows that if $E_{ij} \in M$, then for every $k \geq 1$, $E_{ik} \in M$. Hence $M \supseteq \{E_{ij} : i, j \geq 1\}$ which is a basis for $K(\zeta_p)$, that is, $M \supseteq K(\zeta_p)$. 

Note that if $h$ is hermitian and $h \in M$ then $hB(\zeta_p)h \subseteq M$. This follows from the simple observation that if $h \in M$, then by (*), $(e - z)h = (e - z)h = (e - z)h = 0 = h(e - z)$, since $h$ is hermitian. So $zh = hz = h$, and for any $A \in B(\zeta_p)$, $hAh = hAzh \in M$. From this we see that if $I \in M$, then $M = B(\zeta_p)$.

**Lemma 2.** If $A = \sum a_{ij} e_j \otimes e_i \in M$ where $(a_{ii})_{i \geq 1} \in \zeta_p \epsilon_0$, then $M = B(\zeta_p)$.

**Proof.** Wlog there exists an infinite sequence of integers $f(1) < f(2) < \cdots$ so that $A = \sum e_{f(i)} \otimes e_{f(i)}$. The reduction to this case illustrates a typical use of Lemma 1 that occurs several times in this paper. This time it will be done in detail:

There exists a $\delta > 0$ and a sequence of positive integers $i_1 < i_2 < \cdots$ so that $\delta < |a_{i_k i_k}| \leq \|A\|$ for each $k$. As $hA \in M$ where $h = \sum_{k \geq 1} (1/|a_{i_k i_k}|) e_{i_k} \otimes e_{i_k}$ we may assume wlog that $a_{i_k i_k} = 1$ for every $k$. Choose a sequence of positive numbers $(\epsilon_i)_{i \geq 1}$ so that $\sum_{i \geq 1} \epsilon_i < \infty$. Let $f(1) = i_1$ and choose $\alpha > f(1)$ so that

$$\left(\sum_{j \geq \alpha_1} |a_{f(1) j}|^p\right)^{1/p} < \epsilon_1$$

and

$$\left(\sum_{i \geq \alpha_1} |a_{f(1) i}|^p\right)^{1/p} < \epsilon_2.$$

Choose a $k_2$ so that $i_{k_2} > \alpha_1$ and set $f(2) = i_{k_2}$. Now find $\alpha_2 > f(2)$ so that $\left(\sum_{j \geq \alpha_2} |a_{f(2) j}|^p\right)^{1/p} < \epsilon_3$ and $\left(\sum_{i \geq \alpha_2} |a_{f(2) i}|^p\right)^{1/p} < \epsilon_4$, etc. Fix $\epsilon > 0$. There is an $n$ such that $\sum_{i \geq n} \epsilon_i < \epsilon$. If $h = \sum h_{ij} e_j \otimes e_i$ where

$$h_{ij} = \begin{cases} 1 & \text{if } i = j = f(k) \text{ for some } k \\ 0 & \text{otherwise} \end{cases}$$

and $K$ denotes the first $f(n)$ rows and columns of $hAh - \sum_{k \geq 1} e_{f(k)} \otimes e_{f(k)}$, then $K$ represents a compact operator on $\zeta_p$, and by choice of $K$, $\|hAh - \sum_{k \geq 1} e_{f(k)} \otimes e_{f(k)} - K\| < \epsilon$. As $\epsilon > 0$ is arbitrary and $hAh - K \in M$ we have that

$$\sum_k e_{f(k)} \otimes e_{f(k)} \in M.$$
If \( f(N) \) is finite, then there exists a compact \( K \) so that \( A + K = I \in M \to M = B(\ell_p) \). So assume \( f(N) \) is infinite and let \( g \) enumerate \( f(N) \).

**Claim.** \( B = \sum_i e_{g(i)} \otimes e_{f(i)} \in M \).

Note that proving this claim is sufficient to finish the lemma, since the same argument can be modified to show that

\[
C = \sum_i e_{f(i)} \otimes e_{g(i)} \in M, \text{ hence again } I = A + CB \in M.
\]

We first show that \( d(B, M) \) is zero or one.

Now if \( h = \sum_{i \in I} e_i \otimes e_i \) where \( I \) is any subset of positive integers, then \( d(h, M) \) is either zero or one for any \( M \)-ideal \( M \), for if there is a \( \delta > 0 \) and \( m \in M \) such that \( \| h - m \| = \delta \), then by the first corollary to Lemma 1, \( (h - m)^2 = h - (hm + mh - m^2) \to d(h, M) \leq \delta^2 \).

Let \( P \) be the permutation matrix which as an operator on \( \ell_p \) interchanges, for every \( i \), \( e_{f(i)} \) with \( e_{g(i)} \). Then \( AP = B \). It is easily checked that \( M_P = \{ mP : m \in M \} \) is an \( M \)-ideal isometric to \( M \). Indeed the isometry \( T: B(\ell_p) \to B(\ell_p) \) given by \( T(N) = NP \) induces an isometry [call it \( T \) again] on \( B(\ell_p)^* \) by \( \langle N, T\varphi \rangle = \langle NP, \varphi \rangle \). Then \( T(M) = M_P \), \( T(M^\perp) = M_P^\perp \) and \( B(\ell_p)^* = T(M^\perp) \oplus \ell_1 T(M) \). Therefore \( d(B, M) = d(A, M_P) = 1 \) or 0.

Now assuming that \( B \notin M \), there is a \( \varphi \in M^\perp \) so that \( \| \varphi \| = 1 = \varphi(B) \). Define \( \varphi_A \in B(\ell_p)^* \) by \( \varphi_A(N) = \varphi(NB) \). Then \( AB = B \to \varphi_A(A) = 1 = \| \varphi_A \| \). But then \( \varphi_A \in M \) since \( A \in M \). [This calculation occurs in the corollary above stating that \( M \supseteq K(\ell_p) \).] Thus \( \| \varphi_A + \varphi \| = 2 \). But there is an \( \varepsilon > 0 \) such that for any \( \text{norm-1} \; N \in B(\ell_p) \), we have

\[
|\varphi_A(N) + \varphi(N)| \leq \| \varphi \| \cdot \| N \| \cdot \| B + I \| < 2 - \varepsilon,
\]

a contradiction implying that \( B \in M \).

**Lemma 3.** If \( B = \sum b_{ij} e_j \otimes e_i \in M \) where \( B \) contains a sequence of entries \( (b_{ij})_{k \geq 1} \in \ell_\infty \setminus \ell_0 \), then \( M = B(\ell_p) \).

**Proof.** As in the proof of Lemma 2, we may assume wlog that there exist infinite sequences \( f(1) < f(2) < \cdots \) and \( g(1) < g(2) < \cdots \) such that \( f(i) \neq g(j) \) for all \( i \) and \( j \), and so that \( \sum_i e_{g(i)} \otimes e_{f(i)} \in M \). Call this matrix \( B \), and let \( A = \sum_i e_{g(i)} \otimes e_{f(i)} \). If \( P \) and \( M_P \) are as in Lemma 2, then \( 0 = d(B, M) = d(A, M_P) \to [\text{by Lemma 2}] \; M_P = B(\ell_p) \to M = B(\ell_p) \).

If \( T = \sum t_{ij} e_j \otimes e_i \in M \) and \( T \) is not compact, then it is not necessarily the case that there is a subsequence of entries \( (t_{ij})_{k \geq 1} \in \ell_\infty \setminus \ell_0 \). But what is true [and will be shown in the proof of the next
theorem] is that $T$ has infinitely many square blocks each of whose norm is larger than some fixed $\varepsilon > 0$. So what essentially remains to be done is to generalize preceding arguments from 1 by 1 blocks to square blocks of arbitrary dimension.

**Theorem.** Suppose $T = \sum t_{ij} e_j^* \otimes e_i$ is not compact. Then $T \in M \rightarrow M = B(\ell_p)$.

**Proof.** wlog $\|T\| = 1$. The argument of Lemma 2 modifies to show that wlog $T$ is a direct sum of diagonal square blocks $\mathcal{T}_i$ where $\|\mathcal{T}_i\| = 1$. Although this is well known, it is included for the sake of completeness. We can do this in more generality as follows:

Suppose $T = \sum t_{ij} e_j^* \otimes e_i \in B(X)$ where $X$ is a reflexive space with 1 unconditional basis $(e_i)_{i \geq 1}$ [so $(e_i^*)_{i \geq 1}$ is a basis for $X^*$]. Suppose $T$ is in an $M$-ideal $M \subseteq B(X)$. Since $T$ is not compact, there is a $\delta > 0$ and a sequence $(z_i)_{i \geq 1} \subseteq X$ such that $\|z_i\| = 1$ and $\|T(z_i)\| > 2\delta$ for every $i$, and $z_i \rightarrow 0$ in the weak topology. Let $x_i = z_i$ where $x_i = \sum_{k \geq 1} x_{ik} e_k$. Then there exist $p_i \geq 1$ and $p_i' \geq 1$ so that $\|T(\sum_{k=1}^{p_i} x_{ik} e_k)\| > \delta$, and if $T(\sum_{k=1}^{p_i'} y_{ik} e_k) = \sum_{k \geq 1} y_{ik} e_k$, then also $\|\sum_{k=1}^{p_i'} y_{ik} e_k\| > \delta$. Define $m_1 = 0$, let $n_i = \max\{p_i, p_i'\}$ and let $\mathcal{T}_i = \sum_{j=1, j \neq m_i}^{n_i} t_{ij} e_j^* \otimes e_i$. Then $\delta < \|\mathcal{T}_i\| \leq 1$. Choose a sequence $(e_i)_{i \geq 1}$ of positive numbers so that $\sum_{i \geq 1} \varepsilon_i < \infty$. Now $\sum_{i=1}^{\infty} \sum_{j=1}^{n_i} t_{ij} e_j^* \otimes e_i$ represents a compact operator [its adjoint is finite rank] and so there exists $\beta_i > n_i$ such that $\|\sum_{i=1}^{\infty} \sum_{j=1}^{n_i} t_{ij} e_j^* \otimes e_i\| < \varepsilon_i$ [if $(P_n)_{n \geq 1}$ are the natural basis projections defined by $P_n(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{n} a_i e_i$, then $(\mathcal{T}_i P_n - P_{n+1} \mathcal{T}_i P_n)(x) \rightarrow 0$ for every $x \in X$, and as $\mathcal{T}_i$ is compact this convergence is uniform on the unit ball, hence $\|\mathcal{T}_i P_n - P_{n+1} \mathcal{T}_i P_n\| \rightarrow 0$ as $n \rightarrow \infty$]. As $\sum_{i=1}^{\infty} \sum_{j=1}^{n_i} t_{ij} e_j^* \otimes e_i$ is finite rank [hence compact] similar reasoning shows that there is an $\alpha_i > n_i$ so that $\|\sum_{i=1}^{\infty} \sum_{j=1}^{n_i} t_{ij} e_j^* \otimes e_i\| < \varepsilon_{i-1}$. Define $m_2 = \max\{\alpha_i, \beta_i\}$. Since $z_i \rightarrow 0$ weakly, we can use a standard gliding hump argument to find a $k_2 > 1$ such that $x_{2k} = z_{2k}$ has the property that if $x_2 = \sum_{k \geq 1} x_{2k} e_k$ then there exists a $p_2 \geq 1$ and $p_2' \geq 1$ such that $\|T(\sum_{k=m_2+1}^{m_2+p_2} x_{2k} e_k)\| > \delta$, and if $T(\sum_{k=m_2+p_2}^{m_2+p_2'} y_{2k} e_k) = \sum_{k \geq 1} y_{2k} e_k$, then also $\|\sum_{k=m_2+p_2}^{m_2+p_2'} y_{2k} e_k\| > \delta$. Let $n_2 = \max\{p_2, p_2'\}$ and let $\mathcal{T}_2 = \sum_{i=1}^{m_2+p_2} \sum_{j=1, j \neq m_2+p_2}^{n_i} t_{ij} e_j^* \otimes e_i$. Then $\delta < \|\mathcal{T}_2\| \leq 1$. Again find $\beta_2 > m_2 + n_2$ and $\alpha_2 > m_2 + n_2$ so that

$$\sum_{i=m_2+1}^{m_2+n_2} \sum_{j=m_2+1}^{m_2+n_2} t_{ij} e_j^* \otimes e_i < \varepsilon_3 \quad \text{and} \quad \sum_{i=m_2+1}^{m_2+n_2} \sum_{j=m_2+1}^{m_2+n_2} t_{ij} e_j^* \otimes e_i < \varepsilon_4.$$  

Let $m_3 = \max\{\alpha_2, \beta_2\}$ and repeat the process on $\sum_{i,j=m_3+1}^{m_3+n_3} t_{ij} e_j^* \otimes e_i$. Let $h = \sum_{i,j} t_{ij} e_j^* \otimes e_i$ be the hermitian element defined by

$$h_{ij} = \begin{cases} 1 & \text{if there is a } k \text{ so that } m_k + 1 \leq i = j \leq m_k + n_k \\ 0 & \text{otherwise} \end{cases}.$$
Then \( h'T \in M \). [Although the corollary to Lemma 1 need not hold here, what the proof of the corollary actually shows is that \( M \) is closed under multiplication by real diagonal matrices.] To see that 
\[
T' = \sum T_i \in M,
\]
choose \( \varepsilon > 0 \). There is an \( \varepsilon > 0 \) so that \( \sum \varepsilon_i < \varepsilon \).

Let \( K \) denote the compact operator represented by the first \( m_k + n_k \) rows and columns of \( h'T - K \). Then by the choice of \( \varepsilon \),
\[
\| h'T_1 - T' - K \| < \varepsilon
\]
and as \( M \) is closed we have that \( T' \in M \). If \( h' = \sum h'_i e_j \otimes e_i \) is defined by
\[
\begin{cases}
\frac{1}{\| T_k \|} & \text{if } m_k + 1 \leq i = j \leq m_k + n_k \\
0 & \text{otherwise},
\end{cases}
\]
then \( h' \) is \( B(\varepsilon) \), and \( h'T' \) is a direct sum of diagonal square blocks each having norm 1. Returning now to \( B(\varepsilon) \), we see that we may assume that if \( T \) is not compact and \( T \in M \), then \( w\log T = \sum T_i \) where each \( T_k = \sum T_i, j = m_k + 1 \to e_j \otimes e_i, \| T_k \| = 1 \), and \( m_k + n_k + 1 > m_k + 1 \).

Since \( \| T_k \| = 1 \), there exist \( x_l = (x_1^l, \ldots, x_n^l) \in \ell^\infty, y_i^l = (y_1^l, \ldots, y_n^l) \) and \( z_k = (z_1^k, \ldots, z_n^k) \in \ell^\infty \) all of norm 1 such that \( \langle T_k(x), y_i^l \rangle = 1 = \langle z_k, x^l \rangle \) for all \( k \).

Define norm-1 matrices \( A, X, Y, \) and \( Z \) in \( B(\varepsilon) \) by
\[
A = \sum_{k \geq 1} e_{m_k+1} \otimes e_{m_k+1}, \quad X = \sum_{k \geq 1} X_k, \quad Y = \sum_{k \geq 1} Y_k, \quad \text{and}
\]
\[
Z = \sum_{k \geq 1} Z_k,
\]
where
\[
X_k = \sum_{j \leq n_k} x_j^k e_{m_k+1} \otimes e_{m_k+1}, \quad Y_k = \sum_{j \leq n_k} y_j^k e_{m_k+1} \otimes e_{m_k+1}, \quad \text{and}
\]
\[
Z_k = \sum_{j \leq n_k} z_j^k e_{m_k+1} \otimes e_{m_k+1}.
\]

Then \( ZX = YTX = A \). Claim: If \( X \in M \), then \( M = B(\varepsilon) \). For if not, choose \( \varphi \in \ell^\infty \) so that \( \| \varphi \| = 1 \). Define \( \gamma \in B(\ell^\infty) \) by \( \gamma(N) = \varphi((n_{m_k+n_{k+1}+m_k+1})_{k \geq 1}) \) where \( N = \sum n_i e_j \otimes e_i \). We may assume that \( \gamma \in M \), or else \( M \) contains an element with a sequence of entries in \( \ell^\infty \), hence \( M = B(\varepsilon) \)

If \( X \in M \), then the functional \( \gamma \) is defined by \( \gamma_1(N) = \varphi((ZN)_{m_k+n_{k+1}+m_k+1})_{k \geq 1} \) is in \( \tilde{M} \), as \( \gamma_1(X) = 1 \) and as has been noted before, any functional attaining its norm at a norm-1 element of \( M \) is in \( \tilde{M} \). Therefore \( 2 = \| \gamma + \gamma_1 \| \). However for any \( N \in B(\varepsilon) \) of norm-1, we have that
\[
\| \gamma(N) + \gamma_1(N) \| = \| \varphi((n_{m_k+n_{k+1}+m_k+1} + \sum z_j^k n_{m_k+n_{k+1}+m_k+1})_{k \geq 1}) \|
\]
\[
\leq \| (z_1^k, z_2^k, \ldots, z_n^k, 1) \|_{\ell^2} = 2^{1/2},
\]
a contradiction implying that \( M = B(\varepsilon) \). What this argument in fact shows is that if \( M \) contains any element with the same form as \( X \) then \( M = B(\varepsilon) \). In particular the functional \( \varphi_2 \) defined by
\[ \varphi_2(N) = \varphi\left( (YN)_{m_k+1, m_k+n_k+1} \right) \] is in \( M_1 \). [For if there is an \( m = \sum m_i \epsilon_j \otimes e_i \in M \) such that \( \varphi_2(m) \neq 0 \), then there exists \( \epsilon > 0 \) such that \( \| m_k \| > \epsilon \) for infinitely many \( k \) where \( m_k = \sum_{j \leq n_k} m_{m_k+j, m_k+n_k+1} \otimes e_{m_k+j} \). Reasoning as in Lemma 2 we may pass to a subsequence if necessary to get \( \sum_{k \geq 1} m_k \in M \), which up to normalization of the blocks \( \bar{m}_k \) has the same form as \( X \).] Finally define \( \varphi_1 \in B(\ell_p) \) by
\[ \varphi_1 = \varphi\left( (YNX)_{m_k+1, m_k+n_k+1} \right) \]. As \( \varphi_1(T) = 1 \), \( \varphi_1 \in \hat{M} \), and so \( 2 = \| \varphi_1 + \varphi_2 \| \). But for any norm-1 \( N \in B(\ell_p) \), we have that
\[
\| \varphi_1(N) + \varphi_2(N) \| \leq \sup_k \left\{ \sum_{j \leq n_k} (YN)_{m_k+j, m_k+n_k+1} \right\}
\leq \sup_k \| (x_{i_1}^{m_k}, \ldots, x_{i_n}^{m_k}, 1) \|_p = 2^{1/p}\]
a contradiction showing that if \( T \in M \) then \( M = B(\ell_p) \). \( \square \)

The properties of \( \ell_p \) used to prove this theorem are the existence of a symmetric basis and of certain convexity conditions in the space and its dual.

J. Hennefeld recently announced the following result [AMS Notices Volume 25, Number 6, 760-B8].

**Theorem.** The only 1-symmetric spaces \( X \) for which \( K(X) \) is an \( M \)-ideal in \( B(X) \) are \( c_0 \) and \( \ell_p \), \( 1 < p < \infty \).

Hence combining these theorems we have that if \( X \) is not \( c_0 \) or \( \ell_p \), \( 1 < p < \infty \), has a symmetric basis in \( X \) and \( X^* \) and satisfies the required convexity conditions, then there are no nontrivial \( M \)-ideals in \( B(X) \).

**References**


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