JOINS OF DOUBLE COSET SPACES

Hom Nath Bhattarai and James William Fernandez
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H. N. BHATTARAI AND J. W. FERNANDEZ

The double cosets of a group by a subgroup and the irreducible complex characters of a finite group have a structure which can be studied by means of hypergroups (alias convos, probability groups, and Pasch geometries). An external operation on these structures called the “join” is studied. Decomposition theorems are established in both the general hypergroup (Pasch geometry) case and the weighted hypergroup (probability group) case.

1. Geometries and Joins.

**Definition 1.1.** A Pasch Geometry (see (3)) is a triple \((A, \Delta, e)\) where \(A\) is a set, \(e \in A\) and \(\Delta \subseteq A \times A \times A\) satisfying:

(I) \(\forall a \in A, \exists \) unique \(b \in A\) with \((a, b, e) \in \Delta\). Denote \(b\) by \(a^e\).

(II) \(e^e = e\) and \((a^e)^e = a\forall a \in A\).

(III) \((a, b, c) \in \Delta \Rightarrow (b, c, a) \in \Delta\).

(IV) (Pasch’s axiom) \((a_1, a_2, a_3), (a_1, a_4, a_5) \in \Delta \Rightarrow \exists a_0 \in A\) with \((a_0, a_1, a_2), (a_0, a_3, a_5) \in \Delta\).

We may often write \(A\) for \((A, \Delta, e)\) and \(\Delta\) for \(\Delta^e\) if the context is clear. Also we use the word “geometry” for Pasch Geometry.

For \(B \subseteq A\), we denote \(B\backslash\{e\}\) by \(B^e\). If \(B \subseteq A\), \(B\) is called a sub-geometry of \(A\), and we write \(B < A\), iff (i) \(e \in B\) and (ii) \((b_1, b_2, a) \in \Delta\) with \(b_1, b_2 \in B\) implies that \(a \in B\). If \(A\) and \(C\) are geometries, a map \(f: A \rightarrow C\) is called a geometry morphism iff \(f(e) = e\) and \((x, y, z) \in \Delta\) implies \((f(x), f(y), f(z)) \in \Delta\).

1.2. Definition of \(A/B\). If \(A\) is a geometry and \(B < A\), it can be shown that the following relation \(~\) is an equivalence relation on \(A\):

\[x \sim y \text{ iff } \exists b_1, b_2 \in B, a \in A \text{ with } (x, b_1, a^e), (a, y^e, b_2) \in \Delta.\]

If the equivalence class of \(a \in A\) is denoted by \([a]\) and we let \(A/B\) denote \([a]: a \in A\), then in fact \((A/B, \Delta, [e])\) is a geometry where \(([x], [y], [z]) \in \Delta \iff \exists x' \in [x], y' \in [y], z' \in [z]\) with \((x', y', z') \in \Delta\).

If \(A\) is a group (with \(\Delta = \{x, y, z): xyz = 1\})\) and \(B\) is a subgroup of \(A\), then \(A/B\) defines a geometry structure on the set of \(B - B\) double cosets of \(A\).

The following result is routine from the definitions.

**Proposition 1.3.** Suppose that \(A\) and \(B\) are geometries and \(f: A \rightarrow B\) is a geometry morphism. Let \(K_f = \{a \in A: f(a) = e\}\) and
\[ \text{Im}_f = \bigcap_{(A) \subseteq C \subseteq B} C. \]

(1) \( K_f < A, \text{Im}_f < B. \)

(2) The inclusion map \( i: K_f \to A \) is a kernel of \( f \) (in the Geometry Category).

(3) The projection map \( p: B \to B/\text{Im}_f \) is a cokernel of \( f \).

1.4. Construction of \( A \vee B \). Suppose \( A \) and \( B \) are geometries. We construct a geometry \( A \vee B \) (see (4)) called the "join" of \( B \) by \( A \) as follows. Let \( A \vee B = A \cup B^* \). A function \( p: A \vee B \to B \) is defined via
\[
p(x) = \begin{cases} e & \text{if } x \in A, \\ x & \text{if } x \in B. \end{cases}
\]

Now let \( \Delta \) be defined by requiring \( (x, y, z) \in \Delta \) iff either \( \{x, y, z\} \subseteq A \) and \( (x, y, z) \in \Delta_A \), or \( \{x, y, z\} \nsubseteq A \) and \( (p(x), p(y), p(z)) \in \Delta_B \). It is a straightforward verification that \( A \vee B \) is in fact a geometry.

The join of geometries has an interesting categorial property. In any category \( \mathcal{C} \) with zero object, extensions are of interest where, if \( A \) and \( B \) are objects of \( \mathcal{C} \), an extension of \( B \) by \( A \) is a triple \( (f, M, g) \) such that \( M \) is an object of \( \mathcal{C} \), \( f \) and \( g \) are morphisms in \( \mathcal{C} \) satisfying:

\[
\begin{array}{ccc}
A & \xrightarrow{f} & M \\
\downarrow{f'} & & \downarrow{g'} \\
A \vee B & \xrightarrow{f \vee g} & B \\
\end{array}
\]

(1) \( f \) is a kernel of \( g \) and (2) \( g \) is a cokernel of \( f \). For \( A \) and \( B \) fixed objects of \( \mathcal{C} \), the class of exensions of \( B \) by \( A \) becomes a category \( \text{EXT}_{\mathcal{C}}(B, A) \) where a morphism from \( (f, M, g) \) to \( (f', N, g') \) is a \( \mathcal{C} \)-morphism \( \theta \) which makes the following diagram commute:

\[
\begin{array}{ccc}
M & \xrightarrow{f} & \phantom{M} \\
\downarrow{f} & & \downarrow{g} \\
A & \xrightarrow{\theta} & B \\
\phantom{M} & \downarrow{f'} & \phantom{M} \\
N & \xrightarrow{g'} & \phantom{N} \\
\end{array}
\]

A \( \mathcal{C} \)-join of \( B \) by \( A \) is defined as a terminal object in the category \( \text{EXT}_{\mathcal{C}}(B, A) \).

**Proposition 1.5.** Suppose \( A \) and \( B \) are geometries. With notation as above:

(1) \( (i, A \vee B, p) \) is an extension of \( B \) by \( A \) (in the geometry category).

(2) \( (i, A \vee B, p) \) is a \( \mathcal{C} \)-join of \( B \) by \( A \) (where \( \mathcal{C} \) denotes the geometry category).
Proof. Both assertions involve diagram chases using the definitions and are omitted.

The concept of 'join' can be extended. If $\Gamma$ is a linearly ordered set and $\{A_\lambda : \lambda \in \Gamma\}$ is a $\Gamma$-indexed family of geometries, a $\Gamma$-join of the geometries $\{A_\lambda\}$, denoted by $\bigvee_{\lambda \in \Gamma} A_\lambda$, can be defined in the following way: Let $\bigvee_{\lambda \in \Gamma} A_\lambda = \bigcup A_\lambda^* \cup \{e\}$. For $\sigma \in \Gamma$ a map $f_\sigma : \bigcup_{\tau \leq \sigma} A_\tau^* \cup \{e\} \to A_\sigma$ is defined via

$$f_\sigma(x) = \begin{cases} x & \text{if } x \in A_\sigma^* \\ e & \text{otherwise} \end{cases}$$

For $x, y, z \in \bigvee_{\lambda \in \Gamma} A_\lambda$, let $\Omega = \{\omega \in \Gamma : \{x, y, z\} \cap A_\omega^* \neq \emptyset\}$. Now we say $(x, y, z) \in A$ iff either $\Omega = \emptyset$ or $\Omega \neq \emptyset$ and $(f_\lambda(x), f_\lambda(y), f_\lambda(z)) \in A_\lambda(\lambda = \sup \Omega)$. A straightforward check verifies that $\bigvee_{\lambda \in \Gamma} A_\lambda$ is a geometry, and moreover if $\Gamma = \{1, 2\}$, then $\bigvee_{\lambda \in \Gamma} A_\lambda = A_1 \lor A_2$.

The operation $\lor$ is easily shown to be associative in the obvious sense.

2. Decomposition of geometries. Throughout this section, $(A, \Delta, e)$ is a geometry.

**DEFINITION 2.1.** Let $B \subseteq A$. We call $B$ a weak subgeometry of $A$ if:

1. $e \in B$.
2. If $b \in B$, then $b^* \in B$.
3. If $b_1, b_2 \in B, b_1 \neq b_2$, and $x \in A$ with $(b_1, b_2, x) \in \Delta$, then $x \in B$.

The following result is straightforward.

**PROPOSITION 2.2.** If $B$ is a weak subgeometry of $A$, then $(B, \Delta_B, e)$ is a geometry, where $\Delta_B = (B \times B \times B) \cap \Delta_A$.

For $A$ a geometry, let $L_A = \{B : B$ is a subgeometry of $A$ and $A \setminus B^*$ is a weak subgeometry of $A\}$.

If $a \in A$, set $A : a = \{x : \text{whenever } (a, t, x) \in \Delta, t = a^t\}$.

**LEMMA 2.3.** With notation as above,

1. $L_A$ is linearly ordered by inclusion.
2. For $B \in L_A, B \subseteq A : a \forall a \in A \setminus B$.

**Proof.** (1) Let $B, C \in L_A$ with $B \nsubseteq C$. Then $\exists b \in B \setminus C$. Just suppose that $C \nsubseteq B$. Then $\exists c \in C \setminus B$. Let $t \in A$ with $(b, c, t) \in \Delta$ (such exists by an easy verification). It is easy to see that $(c, t, b) \in \Delta$. If $t \in C$, then $b \in C$ (since $B$ is a subgeometry), a contradiction. Hence $t \notin C$. Similarly if $t \in B$, then $c \in B$ since $(t, b, c) \in \Delta$, which is
again a contradiction. Thus $t \in B \cup C$. But then $t \neq c^t$ and hence $b \in A \setminus B^*$ (since $A \setminus B^*$ is a weak subgeometry of $A$), which is a contradiction.

(2) Let $a \in A \setminus B$, where $B \in L_A$. Let $b \in B$. Clearly $e \in A: a$, so we may assume that $b \neq e$. Suppose $t \in A$ with $(a, t, b) \in \Delta$. If $t \in B$, then $a \in B$ (since $B$ is a subgeometry and $(t, b, a) \in \Delta$), which is a contradiction. Hence $t \in A \setminus B$. If $t \neq a^t$, then $b \in A \setminus B$ (since $A \setminus B$ is a weak subgeometry of $a$), again a contradiction. Thus $t = a^t$.

For $B \in L_A$, let $\bar{B} = \{C: C \subseteq L_A$ and $C \subseteq B\}$. Let $L^*_A = \{B \in L_A: \bar{B} \neq B\}$.

**Lemma 2.4.** \(\forall x \in A^* \exists B \in L^*_A\) with $x \in B$ and if $C \in L_A$ such that $C \subseteq B$, then $x \in C$.

**Proof.** For $x \in A^*$, set $B = \bigcap_{D \in S(x)} D$, where $S(x) = \{D \in L_A: x \in D\}$. It is routine to show that $B \in L^*_A$ and has the desired property.

For $B \in L^*_A$, let $\bar{B} = B \setminus \bar{B}^*$.

**Theorem 2.5.** Let $A$ be a geometry.

1. $B^*$ is a weak subgeometry of $A \forall B \in L^*_A$.
2. $A = \bigcup_{B \in L^*_A} B^*$, and $B^* \cap C^* = \{e\}$ for $B, C \in L^*_A$ if $B \neq C$.
3. If $B, C \in L^*_A$ and $B \neq C$, then $B^* \subseteq A: c \forall C \in C^*$ iff $B \subseteq C$.
4. For the geometry $B^*$, where $B \in L^*_A$, $L_{B^*} = \{\{e\}, B\}$.

**Proof.** (1) is routine.

(2) $A = \bigcup B$ follows from 2.4. If $B, C \in L^*_A$ with $B \neq C$, we may assume $B \subseteq C$. Then $B \subseteq \bar{C}$, and hence $(B \setminus B^*) \cap (C \setminus C^*) = B^* \cap C^* = \{e\}$.

(3) This assertion is shown using 2.3(2).

(4) Suppose $B \in L^*_A$ and $C \in L^*_A$ with $C \neq B$ and $C \neq \{e\}$. Let $B_1 = C \cup \bar{B}$. Then $\bar{B} \subseteq B_1 \subseteq B$. A contradiction to the definition of $\bar{B}$ is obtained after it is easily established that $B_1 \in L_A$.

A non-trivial geometry $A$ is called join-indecomposable iff whenever $A \cong B \vee C$ for some geometries $B$ and $C$, then either $B = \{e\}$ or $C = \{e\}$.

**Proposition 2.6.** A non-trivial geometry $A$ is join-indecomposable iff $L_A = \{\{e\}, A\}$.

**Proof.** ($\Leftarrow$). If $f: B \vee C \rightarrow A$ is a geometry isometry isomorphism, it is obvious from the definition of the join that $f(B) \in L_A$.

($\Rightarrow$) If $B \in L_A$ with $B \neq \{e\}$ and $B \neq A$, then it is routine to verify that $A = B \vee (A \setminus B^*)$.

If $\Gamma$ is a partially ordered set with zero element $\lambda_0 \in \Gamma$, let $\Gamma^*$
Theorem 2.7. Let $A$ be a nontrivial geometry. There exists a linearly ordered set $\Gamma$ and a $\Gamma^*$-family of weak subgeometries of $A$, $\{A_\lambda\}_{\lambda \in \Gamma^*}$, for which

1. $A_\lambda \neq \emptyset$ for all $\lambda \in \Gamma^*$.
2. $A_1$ is a join-indecomposable geometry $\forall \lambda \in \Gamma^*$.
3. $A = \bigvee_{\lambda \in \Gamma^*} A_\lambda$.

Moreover, if $\Omega$ is a linearly ordered set and $\{B_\omega\}$ is a $\Omega$-family of nontrivial join-indecomposable geometries such that $A = \bigvee B_\omega$, then there is an isomorphism of partially ordered sets $f: \Gamma^* \rightarrow \Omega$ and a $\Gamma^*$-family of geometry isomorphisms $\{f_\lambda\}$ such that $f_\lambda: A_\lambda \rightarrow B_{f(\lambda)}$.

3. Probability groups. Many examples of geometries, such as the double cosets of finite groups, the set of irreducible complex characters of a finite group, and finite projective geometries, can be given additional structure, which is reflected in the following concept (see (3) and (6)):

Definition 3.1. A discrete probability group is a pair $(A, p_A)$ where $A$ is a set and $p_A: A \times A \times A \rightarrow [0, 1]$ is a map which we will denote by $(a, b, c) \rightarrow p^A(a, b)$, and which satisfies the following axioms:

0. For any $a, b, c \in A$, $p^A(a, b, c) = 0$ for all but finitely many $c \in A$ and $\sum_{c \in A} p^A(a, b) = 1$.

1. (Associativity) For any $a, b, c, d \in A$,

$$\sum_{x \in A} p^A(a, b)p^A(x, c) = \sum_{y \in A} p^A(a, y)p^A(b, c).$$

2. (Identity) $\exists e \in A$ such that $p^A_i(a, e) = \delta(a, e) \forall a, c \in A$, where $\delta$ is the Kronecker delta.

3. (Inverse) For each $a \in A$, there exists a unique $b \in A$ such that $p^A_i(b, a) \neq 0$. We denote $b$ by $a^i$.

4. $(a^i)^i = a$ for all $a \in A$.

5. $p^A_i(a, b) = p^A_i(b^i, a^i) \forall a, b, c \in A$.

Note that the $e \in A$ in (2) above is unique. The term "discrete probability group" will be abbreviated to simply "probability group" or even "prob. group". Also $p$ will be used to denote $p_A$ if the context is clear. For $B \subseteq A$, we use $B^*$ to denote the set $B \setminus \{e\}$.

Useful properties of probability groups which follow easily from the axioms appear in the following statement.

Proposition 3.2. Let $(A, p)$ be a prob. group. Then

1. $p^A_i(c^i, a) = p^A_i(c, a) \forall a, c \in A$. 


(2) \( p_\epsilon(a, a) \neq 1 \ \forall a \in A^* \).

(3) \( p_\epsilon(a, b)p_\epsilon(c, b) = p_\epsilon(a, c)p_\epsilon(b, c) \). In particular, if \( b = c^t \),
\( p_\epsilon(a, b)p_\epsilon(c, b^t) = p_\epsilon(a, c)p_\epsilon(b, b^t) \), which implies that whenever \( p_\epsilon(a, b) = 1 \), \( p_\epsilon(b, b^t) = p_\epsilon(a, a^t)/p_\epsilon(a, a)^t \).

(4) \( p_\epsilon(a, b) \neq 0 \) iff \( p_\epsilon(b, c^t) \neq 0 \).

If \( A \) is a finite prob. group, let \( n_\Delta = \sum_{x \in A} 1/p_\epsilon(x, x^t) \). \( n_\Delta \) is well defined since \( p_\epsilon(x, x^t) \neq 0 \) and \( A \) is finite. Two probability groups \( (A, p^A) \) and \( (B, p^B) \) are called isomorphic iff there exists a bijective map \( f: A \rightarrow B \) such that \( p^B(x, y) = p^B(f(x), f(y)) \ \forall x, y, z \in A \).

The notions of geometries and probability groups are related by the following result which can be obtained from the definitions (see (2)).

**Proposition 3.3.** If \( (A, p) \) is a prob. group, and \( \Delta = \{(a, b, c): p_\epsilon(a, b) \neq 0\} \), then \( (A, A, e) \) is a geometry, called the geometry induced by \( (A, p) \).

If \( A \) is a prob. group, a subset \( B \) of \( A \) is called a sub-prob. group of \( A \) iff \( e \in B \) and \( (B, p_B^B) \) is a prob. group, where \( p_B^B \) is the restriction of \( p \) to triples from \( B \). It is easily seen that \( B \) is a sub-prob. group of \( A \) iff \( B \) is a subgeometry of the geometry \( A \). In particular, \( B \) is a sub-prob. group of \( A \) iff \( \forall b, c \in B \), whenever \( p_\epsilon(b, c) \neq 0 \), then \( a \in B \) (assuming \( B \) non-empty).

4. Joins of probability groups. Suppose that \( (A, p^A) \) and \( (B, p^B) \) are probability groups, and also suppose that the set \( A \) is finite. \( A \) and \( B \) acquire geometry structures according to 3.3. Consider the following map (see (6)). \( p: (A \cup B^*) \times (A \cup B^*) \times (A \cup B^*) \rightarrow [0, 1] \) is defined via

\[
p_\epsilon(x, y, z, z^*) = \begin{cases} 
p^A_\epsilon(x, y) & \text{if } x, y, z \in A . \\
0 & \text{if } x, y \in A \text{ and } z \in B^* . \\
\delta_{x}(z) & \text{if } x \in B^* \text{ and } y \in A . \\
\delta_{y}(z) & \text{if } x \in A \text{ and } y \in B^* . \\
p^B_\epsilon(x, y) & \text{if } x, y, z \in B^* . \\
0 & \text{if } x, y \in B^* , z \in A \text{ and } y \neq x^* . \\
\frac{p^B_\epsilon(x, x^*)}{n_A p^A_\epsilon(z, z^*)} & \text{if } x \in B^* , y = x^* , z \in A . 
\end{cases}
\]

A long but straightforward check shows the following:

**Proposition 4.1.** If \( A \) and \( B \) are probability groups with \( A \)
finite, then \((A \cup B^*, p)\) is a prob. group whose induced geometry is \((A \lor B, \Delta, e)\).

The probability group \((A \cup B^*, p)\) is called the join of \((A, p^A)\) and \((B, p^B)\), and is denoted by \((A \lor B, p)\) or simply by \(A \lor B\). It will now be shown that there is uniqueness contained in the definition of the join.

**Theorem 4.2.** Suppose that \((A, p^A)\) and \((B, p^B)\) are probability groups with \(A\) finite. Suppose further that \((A \cup B^*, p')\) is a probability group such that

1. The geometry induced from \((A \cup B^*, p')\) is \(A \lor B\),
2. \(p'_a(x, y) = p^B_b(x, y) \quad \forall b, x, y \in B^*\),
3. \(p'_a(x, y) = p^A_e(x, y) \quad \forall a, x, y \in A\).

Then \(p' = p\), where \(p\) is the map defined above.

**Proof.** Clearly \(p' = p\) except possibly in the last case of the definition of \(p\). Hence we need only check that \(p'_a(b, b^*) = p^A_e(b, b^*)\) for \(a \in A, b \in B^*\). But we have

\[
\sum_{a \in A} p'_a(b, b^*) = 1 - \sum_{y \in B^*} p'_y(b, b^*) = 1 - \sum_{y \in B^*} p^B_b(b, b^*) = p^B_b(b, b^*) .
\]

Now by 3.2(3),

\[
p'_a(b, b^*) = \frac{p'_a(b, b^*)}{p'_a(a, a^*)} \cdot p'_a(a, a^*) = \frac{p'_a(b, b^*)}{p'_a(a, a^*)} (1).
\]

Hence

\[
\sum_{a \in A} p'_a(b, b^*) = \sum_{a \in A} \frac{p'_a(b, b^*)}{p'_a(a, a^*)} = p'_a(b, b^*) \cdot n_A .
\]

Combining the two expressions for the same sum, we obtain that

\[
p'_a(b, b^*) = \frac{p^A_e(b, b^*)}{n_A} .
\]

Thus

\[
p'_a(b, b^*) = \frac{p'(b, b^*)}{n_A} = \frac{p^A_e(b, b^*)}{n_A p^A_e(a, a^*)} = p^A_e(b, b^*) .
\]

**Corollary 4.3.** Suppose \((A_1, p^1), (A_2, p^2), \ldots, (A_n, p^n)\) are probability groups with \(A_1, A_2, \ldots, A_{n-1}\) finite. For \(i = 1, \ldots, n\) \(A_i\) has a geometry structure induced by the map \(p^i\), and hence we may form the join of the geometries \(\bigvee_{i=1}^n A_i\). There exists a unique probability
map $p$ on the set $\bigcup_{i=1}^{\infty} A_i^* \cup \{e\}$ which satisfies (i) $p$ induces the join geometry $V_{i=1}^{\infty} A_i$ and (ii) $p_x(y, z) = p_x^i(y, z)$ whenever $x, y, z \in A_i^*$.

Proof. A routine induction argument on $n$ using the theorem establishes the corollary.

**Corollary 4.4.** Suppose that $\{(A_i, p^i)\}_{i=1}^{\infty}$ is a family of finite probability groups. There exists a unique probability map $p$ defined on $\bigcup_{i=1}^{\infty} A_i^* \cup \{e\}$ which satisfies (1) $p$ induces the join geometry $V_{i=1}^{\infty} A_i$, and (2) for $x, y, z \in A_i^*$, $p_x(y, z) = p_x^i(y, z)$.

Proof. This corollary follows immediately from 4.3, since $V_{i=1}^{\infty} A_i$ is a limit of $V_{i=1}^{\infty} A_i$.

Two notes on the above construction are important.

1) The condition that all the probability groups be finite (except possibly the last one) is necessary because of the definition of a probability group, which stated that $p_c(a, b)$ is nonzero for only finitely many $c$. However, if $b = a^* \in A_i^*$ and $c \in A_\ell$ where $\ell < i$, then the construction has $p_c(a, a^*) \neq 0$. (2) The map $p$ described in 4.3 and 4.4 (which is a generalization of the original construction) can be explicitly defined as follows, where $n_k$ denotes $\sum_{x \in A_k} 1/p^k(x, x^t)$:

\[
p_c(a, b) = \begin{cases} 
p^i(a, b) & \text{if } a, b, c \in A_i \text{ and either } c \neq e \text{ or } b \neq a^* \\
\left(\prod_{k < i} \frac{1}{n_k}\right) \cdot p^i(a, a^*) & \text{if } a \in A_i^*, b = a^*, c = e . \\
\delta_c(c) & \text{if } a \in A_i, b \in A_j^*, \text{ and } i < j . \\
\delta_c^a(a) & \text{if } a \in A_i^*, b \in A_j, \text{ and } j < i . \\
0 & \text{if } a, b \in A_i^*, c \in A_j^*, b \neq a^*, \text{ and } i \neq j . \\
\left(\prod_{i \geq k < i} \frac{1}{n_k}\right) \cdot \frac{p^i(a, a^*)}{p^i(c, c^*)} & \text{if } a^* = b \in A_i^*, c \in A_j^* \text{ and } j < i . \\
\delta_c(e) & \text{if } a = b = e .
\end{cases}
\]

Combining the above results with the decomposition of geometries established in §2, a decomposition theorem is easily obtained.

**Theorem 4.5.** Let $A$ be a probability group which is nontrivial. There exists a linearly ordered set $\Gamma$ and a $\Gamma$-indexed family of nontrivial probability groups $\{(A_i, p^i)\}$ such that

1) $A_i$ is a join-indecomposable geometry $\forall i \in \Gamma$.
2) $\Gamma$ is either finite or order isomorphic to $Z^+$.
3) The geometry $A$ is the join of the geometries $\{A_i\}_{i \in \Gamma}$.
Also, if $\Gamma$ is finite then $A_i$ is finite except possibly $A_n$, where $\Gamma$ is identified with $\{1, 2, \ldots, n\}$, while if $\Gamma$ is infinite, $A_i$ is finite $\forall i \in \Gamma$. Finally, the original probability map $p$ on $A$ is given by the formula $(\dagger)$.

**Proof.** The main statements are merely a recapitulation of Theorem 2.7 and the calculations which appear above. The only statement which needs to be proved is that $A_i$ has a probability structure which is "compatible" with the original map $p$. If we define the map $p^i$ via:

$$p^i(x, y) = \begin{cases} 
p_i(x, y) & \text{if } x, y, z \in A_i^* \\
\delta_z(x) & \text{if } y = e \\
\delta_y(z) & \text{if } x = e \\
0 & \text{if } z = e \text{ and } y \neq x^i \\
1 - \sum_{a \in A_i} p_a(x, y) & \text{if } z = e \text{ and } y = x^i,
\end{cases}$$

it is an easy exercise that $(A_i^*, p^i)$ is the probability group that is needed.

Several examples of joins (of probability groups and geometries) can be given.

(1) The complex irreducible characters of a finite group $G$ form a probability group if one defines the map $p$ via the following procedure. If $G = \{\chi_1, \ldots, \chi_n\}$ is the set of complex irreducible characters, one has

$$\chi_i \cdot \chi_j = \sum_{k=1}^{n} n_k^{ij} \chi_k \text{ for } i, j = 1, \ldots, n.$$ 

Let $p_{ij}(\chi_i, \chi_j) = n_k^{ij}\chi_k(1)/\chi_i(1)\chi_j(1)$. Then $(\hat{G}, p)$ is a probability group (See (6) and (3)). If $G$ is a nonabelian group of order $p^i$ for some prime $p$, it can be shown that $G$ is the join of $Z_p \times Z_p$ and $Z_p$. Several other groups can be decomposed into joins, but it is not known to the authors which finite groups (non-simple, since a join does give a normal subgroup) decompose into joins.

(2) Projective geometries have a structure which may be studied as a hypergroup (see (5)) or as a Pasch geometry or probability group (3). A necessary and sufficient condition that a geometry decompose into a join of projective geometries is given in (1).

(3) Totally ordered groups (from which valuations are defined) can be given a geometry structure (although not a probability group structure except in the trivial case) which breaks up into the join of geometries consisting of two elements. If one considers more
general geometries in the same setting, a "generalized valuation" concept can be studied in the study of Rings and Fields (See (3)).

REFERENCES


Received March 15, 1977 and in revised form February 27, 1981.

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The Pacific Journal of Mathematics is issued monthly as of January 1966. Regular subscription rate: $102.00 a year (6 Vols., 12 issues). Special rate: $51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION
Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.). 8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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