RITT SCHEMES AND TORSION THEORY

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There is a natural way to associate a torsion theory to any differential ring. Using this tool, one may prove that there is a duality between the category of reduced affine Ritt schemes and a full subcategory of the category of Ritt algebras. As a consequence, a brief investigation is made concerning morphisms of differential finite type and a differential version of Chevalley's constructibility theorem is proved for such morphisms.

1. Introduction. The category $\text{Diff}$ has as its objects commutative rings $A$ with unit together with $m$ derivation operators $D_1, \ldots, D_m: A \to A$ which commute. A morphism $f: (A, D_1, \ldots, D_m) \to (\tilde{A}, \tilde{D}_1, \ldots, \tilde{D}_m)$ in $\text{Diff}$ is a ring homomorphism $f: A \to \tilde{A}$ with $\tilde{D}_i f = f D_i$ for every $i = 1, \ldots, m$. Recall from [5, p. 110] that an $\text{LDR}$-space is pair $(X, \mathcal{O}_X)$ where $X$ is a topological space and $\mathcal{O}_X$ is a sheaf in $\text{Diff}$ on $X$ such that for each $P \in X$ the ring $\mathcal{O}_{X,P}$ is local and its maximal ideal is differential. A morphism of $\text{LDR}$-spaces $f: (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair $(\psi, \theta)$ where $\psi: X \to Y$ is continuous and $\theta: \mathcal{O}_Y \to \psi_* \mathcal{O}_X$ is a morphism of sheaves in $\text{Diff}$ on $Y$ such that for each $P \in X$, the morphism $\mathcal{O}_{Y, \psi(P)} \to \mathcal{O}_{X,P}$ is local. The category of $\text{LDR}$-spaces is denoted by $\text{LDR}$.

There exist two fundamental functors $\text{Spec}_p: \text{Diff} \to \text{LDR}$ and $\Gamma_p: \text{LDR} \to \text{Diff}$ defined such as follows: for any differential ring $A$, the topological space $\text{Spec}_p A$ consists of the prime differential ideals of $A$, the topology being induced by the natural inclusion $j: \text{Spec}_p A \to \text{Spec} A$ and the defining sheaf $\mathcal{O}_{\text{Spec}_p A}$ being $\hat{A} = j^{-1}(\tilde{A})$, where $\hat{A}$ is the defining sheaf of the scheme $\text{Spec} A$, [3, p. 70]. Note that $\hat{A}$ (and consequently $\hat{A}$) has a natural structure of sheaf in $\text{Diff}$. Indeed the derivations $D_1, \ldots, D_m$ on $A$ canonically give derivations $D_{1,P}, \ldots, D_{m,P}$ on $A_P$ for any $P \in \text{Spec} A$; hence for any open set $U \subseteq \text{Spec} A$, the ring $\Gamma(U, \hat{A})$ becomes a differential ring with derivations $D_{1,U}, \ldots, D_{m,U}$ defined such as follows: for any $s \in \Gamma(U, A)$, $D_{i,U}(s)$ is the section defined by the family $\{D_i P(s_P))\}_{P \in U}$. On the other hand for any $\text{LDR}$-space $X$, $\Gamma_p(X)$ will denote the differential ring of global sections $\Gamma(X, \mathcal{O}_X)$.

It was proved in [5] that at least in the case of a single derivation, the functors $\text{Spec}_p$ and $\Gamma_p$ give an adjunction between $\text{Diff}$ and $\text{LDR}^{op}$. We shall prove in §2 that these functors give in fact an equivalence between sufficiently large subcategories.

For the remainder of this paper we shall suppose that all rings
contain the field of rationals $\mathbb{Q}$. A differential ring which contains
$\mathbb{Q}$ will be called a Ritt algebra. If $A$ is a Ritt algebra, an $LDR$-
space of the form $\text{Spec}_d A$ will be called an affine Ritt scheme. A
Ritt scheme will mean an $LDR$-space which may be covered by affine
Ritt schemes.

The equivalence proved in §2 makes possible an investigation of
morphisms of differential finite type between affine Ritt schemes
which will be made in §3. In §4 we make some remarks on the
differential affine space.

The necessary information on rings and modules of quotients
may be found in [11]. We will use this technique in the following
context: given a commutative ring $A$ and a subset $X$ of $\text{Spec} A$, one
may associate to $X$ an hereditary torsion class $\mathcal{T} = \{ M \in \text{Mod} A,$
$M_P = 0 \text{ for any } P \in X \}$, a Gabriel topology $F = \{ J \text{ ideal in } A,$
$A/J \in \mathcal{T} \} = \{ J \text{ ideal in } A, J \not\subseteq P \text{ for any } P \in X \}$
and a left exact radical $t: \text{Mod} A \rightarrow \mathcal{T}$, $t(M) = \{ x \in M, \text{ann}\,(x) \in F \}$. For any $M \in \text{Mod} A$ one
defines the module of quotients $M_F = \lim_{J \in F} \text{Hom}(J, M/t(M))$. Then
we have $\ker (\varphi_M: M \rightarrow M_F) = t(M)$, $\text{coker} (\varphi_M) \in \mathcal{T}$
and $M_F \rightarrow (M_F)_F$ is an isomorphism. For any ideal $I$ in $A$ one defines the ideal $I^e = \{ x \in A, I: x \in F \}$ and put $C_F(A) = \{ I \text{ ideal in } A, I^e = I \}$. Now the set
$F^e = \{ J \text{ ideal in } A_F, J \cap A \in F \}$ is a Gabriel topology on $A_F$ and there
is a one-to-one correspondence between $C_F(A)$ and $C_F(A_F)$ given by
$I \mapsto I_F$ and $J \mapsto J \cap A$. This correspondence induces a one-to-one cor-
respondence between $\text{Spec} A \cap C_F(A)$ and $\text{Spec} A_F \cap C_F(A_F)$. Note
that for any $P \in \text{Spec} A$, we have $P \in F$ or $P \in C_F(A)$.

2. Duality given by $\text{Spec}_d$ and $\Gamma_d$. For any Ritt algebra $A$
let $\mathcal{T}_A$, $F_A$ and $t_A$ be the hereditary torsion class, the Gabriel
topology and the radical associated to the subset $X = \text{Spec}_d A$
of $\text{Spec} A$ as in §1. Put $A_d = \Gamma_d \text{Spec}_d A$.

**Proposition 2.1.** For any Ritt algebra $A$ we have $t_A(A) = \ker (A \rightarrow A_d)$ and the canonical morphism $A \rightarrow A/t_A(A)$ induces an
isomorphism of Ritt schemes $\text{Spec}_d A/t_A(A) \simeq \text{Spec}_d A$

**Proof.** Our first statement follows directly from definitions. To
prove the second statement, observe that for any $x \in t_A(A)$ and for
any $P \in \text{Spec}_d A$ we have $\text{ann}(x) \not\subseteq P$, hence $t_A(A)$ is contained in
every $P \in \text{Spec}_d A$. We get that $\text{Spec}_d A/t_A(A) \rightarrow \text{Spec}_d A$ is a homeo-
morphism. It is sufficient to prove that the morphisms induced on
the stalks are isomorphisms, i.e. that for any $P \in \text{Spec}_d A$ we have
$A_P \simeq (A/t_A(A))_P$. But this isomorphism holds since $(t_A(A))_P = 0$ for
any $P \in \text{Spec}_d A$. 
REMARK. If A is a Ritt domain with field of quotients K and U is an open subset of $X = \text{Spec}_A A$, then the ring $\Gamma(U, \mathcal{O}_X)$ is equal to the intersection (taken in $K$) of all the local rings $A_P$ as $P$ runs through $U$.

COROLLARY 2.2. Let $X$ be an affine Ritt scheme. The following statements are equivalent:

1. For any open subset $U \subseteq X$, the ring $\Gamma(U, \mathcal{O}_X)$ is reduced (or integral).
2. $X$ is isomorphic to $\text{spec}_D A$ where $A$ is a reduced (or integral) Ritt algebra.

Proof. (1) $\Rightarrow$ (2) If $X = \text{Spec}_D B$, it follows by Proposition 2.1 that $A = B/t_B(B)$ is a subring of $B_D = \Gamma(X, \mathcal{O}_X)$, hence $A$ is reduced (or integral). By Proposition 2.1 again, we get that $X$ is isomorphic to $\text{Spec}_D A$.

(2) $\Rightarrow$ (1) If $A$ is reduced then every local ring $A_P$ is reduced, hence every $\Gamma(U, \mathcal{O}_X)$ is reduced. If $A$ is integral then $\Gamma(U, \mathcal{O}_X)$ is integral by the remark above.

A Ritt scheme will be called reduced (or integral) if $\Gamma(U, \mathcal{O}_X)$ is reduced (or integral) for any open subset $U$ of $X$.

If $A$ is a Ritt algebra then $\text{Spec}_D A$ is quasi-compact by [6]. If $J$ is an ideal in $A$, then $r(J)$, $[J]$ and $\{J\}$ will denote the radical ideal, then differential ideal and the radical differential ideal respectively, which are generated by $J$. By [8, p. 13], $\{J\}$ is the intersection of all prime differential ideals which contain $J$ and by [4, Lemma 1.8] we have $\{J\} = r([J])$.

For any $A$-module $M$, put $\hat{M} = j^{-1}(\mathcal{M})$ where $j$ is the natural inclusion $\text{Spec}_D A \to \text{Spec} A$ and $\mathcal{M}$ is the sheaf on $\text{Spec} A$ defined by $M$ [3, p. 110]. The stalk of $\hat{M}$ at $P \in \text{Spec}_D A$ is $M_P = M \otimes_A A_P$. There exists a natural morphism of $A$-modules $\theta_M: M \to M_D = \Gamma(\text{Spec}_D A, \hat{M})$. It is apparent that $\ker(\varphi_M: M \to M_F) = \ker(\theta_M: M \to M_D) = t(M)$ where $F = F_A$ and $t = t_A$.

LEMMA 2.3. For every $A$-module $M$ there exists a natural injective morphism of $A$-modules $\varphi_M: M_F \to M_D$ such that $\varphi_M \varphi_M = \theta_M$.

Proof. Take $x \in M_F$. Since $\ker (\varphi_M) \in \mathcal{I}_A$ it follows that for every $P \in \text{Spec}_D A$ there exist elements $s_P \in A \setminus P$ and $x_P \in M$ such that $s_P x = \varphi_M(x_P)$. Since $\varphi_M(s_P x_P - s_Q x_P) = 0$ for every $P, Q \in \text{Spec}_D A$ it follows that $x_P / s_P = x_Q / s_Q$ in every $M_R$ with $R \in \text{Spec}_D A$, hence the elements $x_F / s_F \in M_{s_F}$ stick together and define a global section $s \in M_D$. It is apparent that $s$ depends only on $x$ and so we put $\varphi_M(x) = s$. 

$$\gamma_M$$ is injective because the condition $s = 0$ implies that for any $P \in \text{Spec}_A$ there exists $u_P \in A \setminus P$ such that $u_P x_P = 0$. It follows that $u_P s_P x = 0$ for any $P \in \text{Spec}_A$, hence $x \in t(M_P) = 0$.

For any $A$-module $M$, $\text{Ass}(M)$ will denote the set of all primes $P$ in $A$ for which there exists $x \in M$ such that $P$ is minimal among the prime ideals containing $\text{ann}(x)$.

**Lemma 2.4.** Suppose that $\text{Ass}(M) \subseteq \text{Spec}_A$. Then $\varphi_M : M \to M_F$ is injective and $\psi_M : M_F \to M_D$ is bijective.

**Proof.** Since $\text{Ass}(M) \subseteq \text{Spec}_A$ it follows that for every $x \in M$ we have $\{\text{ann}(x)\} = r(\text{ann}(x))$. For the first assertion of the lemma it is sufficient to prove that $t(M) = 0$. If $x \in t(M)$, for every $P \in \text{Spec}_A$ there exists $s_P \in A \setminus P$ such that $s_P \in \text{ann}(x)$. Since the differential ideal generated by these $s_P$ as $P$ runs through $\text{Spec}_A$ is equal to $A$, it follows that $1 \in \{\text{ann}(x)\} = r(\text{ann}(x))$, hence $x = 0$.

To prove the second assertion it is sufficient to show that $\psi_M$ is surjective. Take $s \in M_D$. By quasi-compacity of $\text{Spec}_A$ there exist $f_1, \ldots, f_h \in A$ and $x_1, \ldots, x_k \in M$ such that $\{(f_1, \ldots, f_h)\} = A$ and the restriction of $s$ at $D(f_i) = \{P \in \text{Spec}_A, f_i \notin P\}$ is given by $x_i/f_i \in M_{f_i}$. Since $x_i/f_i = x_j/f_j$ in any $M_P$ with $P = D(f_i, f_j)$ it follows that for any such $P$ there exists $s_{ijP} \in A \setminus P$ with $s_{ijP} \in \text{ann}(f_i x_j - f_j x_i)$. Obviously, for a fixed pair $(i, j)$ the element $f_i f_j$ is contained in the radical differential ideal generated by all the $s_{ijP}$ as $P$ runs through $D(f_i, f_j)$, hence $f_i f_j \in \{\text{ann}(f_i x_j - f_j x_i)\} = r(\text{ann}(f_i x_j - f_j x_i))$. So there exists a common $N$ such that $(f_i f_j)^N (f_i x_j - f_j x_i) = 0$ for all $i$ and $j$. Replacing $x_i/f_i$ by $x_i f_i^{N+1}/f_i^{N+1}$ we may suppose $N = 0$. Consider the morphisms of $A$-modules $u : A^k \to J = f_1 A + \cdots + f_k A$ and $v : A^k \to M$ sending the elements of a basis of $A^k$ into $f_1, \ldots, f_k$ and $x_1, \ldots, x_k$ respectively. Notice that $v(\ker(u)) \subseteq t(M)$. Indeed if we have $a_1, \ldots, a_k \in A$ such that $\sum_{i=1}^k a_i f_i = 0$ we get that

$$f_j \left( \sum_{i=1}^k a_i x_i \right) = \sum_{i=1}^k a_i f_i x_j = 0$$

for all $j$, hence $\sum_{i=1}^k a_i x_i \in t(M)$. So $v$ induces a morphism of $A$-modules $\overline{v} \in \text{Hom}(J, M/t(M))$. Since $J \in F$ we may consider the image of $\overline{v}$ in $M_F = \lim I \in F \text{Hom}(I, M/t(M))$ and denote it by $x$. It is apparent that $\overline{\varphi_M}(x) = s$.

**Lemma 2.5.** Suppose $A$ satisfies one of the conditions:

(1) $A$ is reduced

(2) $A$ is Noetherian

Then $\text{Ass}(A) \subseteq \text{Spec}_A$. 

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**ALEXANDRU BUIUM**
Proof. Suppose first that \( A \) is reduced and take \( P \in \text{Ass}(A) \). Since \( PA_P \in \text{Ass}(A_P) \) we may suppose that \( A \) is \( P \)-local. Since \( P \) is minimal among the primes which contain \( \text{ann}(x) \) for a certain \( x \in A \), it follows that \( P = r(\text{ann}(x)) \). On the other hand since the ideal \((0)\) is radical and differential, so is \((0):(x)\) [4, Lemma 1.4] hence \( P \) is differential.

If \( A \) is supposed to be Noetherian, our statement follows for instance from [10].

We say that a Ritt algebra \( A \) is closed if the morphism \( \Theta_A : A \rightarrow A_D \) is an isomorphism. Every local Ritt algebra whose maximal ideal is differential is closed by [7, Proposition 3.3]. The following result shows that any ring of global sections of a reduced affine Ritt scheme is closed. In §4 we will also show that an algebra of differential polynomials over a closed Ritt domain is closed.

**Theorem 2.6.** Let \( A \) be a Ritt algebra such that \( \text{Ass}(A) \subseteq \text{Spec}_D A \). Then the morphism \( \Theta_A : A \rightarrow A_D \) is injective and induces an isomorphism of Ritt schemes \( \text{Spec}_D A_D \rightarrow \text{Spec}_D A \). Consequently, \( A_D \) is closed.

Proof. By Lemma 2.3, \( \varphi_A : A_F \rightarrow A_D \) is an isomorphism, hence it is sufficient to prove that \( \text{Spec}_D A \cong \text{Spec}_D A \) is an isomorphism of Ritt schemes. Let us prove first that \( \text{Spec}_D A \) is a homeomorphism between the underlying topological spaces. Since \( \text{Spec}_D A \cap F = \emptyset \) we get that \( \text{Spec}_D A \subseteq C_F(A) \). Applying this remark to the ring \( B = A_F \) we get that \( \text{Spec}_D B \subseteq C_F(B) \). On the other hand we see that \( F^e \subseteq F^e \). Indeed if \( J \in F^e \) then \( J \) cannot be contained in any \( P \in \text{Spec}_D B \) because if we had \( J \subseteq P \) for such a \( P \), we would get \( J \cap A \subseteq P \cap A \in \text{Spec}_D A \) which is a contradiction. Hence \( C_F(B) \subseteq C_F(B) \) and finally we deduce that \( \text{Spec}_D B \subseteq C_F(B) \). Let us also observe that for any \( P \in \text{Spec}_D A \) we have \( P_F \in \text{Spec}_D B \). Indeed there exists a canonical morphism of \( A \)-algebras \( A_F \rightarrow A_P \) and since \( PA_P \in \text{Spec}_D A_P \) it follow that \( \bar{P} = PA_P \cap A_F \in \text{Spec}_D A_F \subseteq C_F(A_F) \). Since \( \bar{P} \cap A = P_F \cap A = P \), by the one-to-one correspondence between \( C_F(A) \) and \( C_F(A_F) \) we get \( P_F = \bar{P} \), hence \( P_F \in \text{Spec}_D A_F \). We may conclude that the one-to-one correspondence between \( \text{Spec} A \cap C_F(A) \) and \( \text{Spec} A_F \cap C_F(A_F) \) induces one-to-one correspondence between \( \text{Spec}_D A \) and \( \text{Spec}_D A_F \). One knows that \( O \mapsto O \cap A \) is continuous. To prove that \( P \mapsto P_F \) is continuous, take \( y \in A_F \) and calculate \( h^{-1}(D(y)) \) where \( D(y) \) is the principal open subset of \( \text{Spec}_D A_F \) defined by \( y \). There exist \( f_1, \ldots, f_k \in A \) and \( y_1, \ldots, y_k \in A \) such that \( \{ (f_i, \ldots, f_k) \} = A \) and \( \varphi_A(f_i) = \varphi_A(y_i) \) for every \( i \). Then we have
\[ h^{-1}(D(y)) = \bigcup_i h^{-1}(D(\varphi_A(f_i)y)) = \bigcup_i D(y_i) \]

which is an open set in \( \text{Spec}_D A \).

Now we only have to prove that \( \text{Spec}_D \varphi_A \) gives an isomorphism on each stalk, i.e. that for every \( P \in \text{Spec}_D A \) we have \( A_P \cong (A_F)_{P_F} \).

We have the commutative diagram:

\[
\begin{array}{ccc}
A & \xrightarrow{\alpha} & A_F \\
\delta & & \downarrow{\gamma} \\
A_P & \xrightarrow{\alpha} & (A_F)_{P_F} \\
& & \downarrow{\beta} \\
& & (A_F)_{P_{AP}}
\end{array}
\]

where \( \alpha = \varphi_A \) and \( \beta \alpha \) is the identity of \( A_P \). It is sufficient to prove that \( \beta \) is injective. Suppose we have \( y \in (A_F)_{P_F} \) with \( \beta(y) = 0 \). After multiplying \( y \) with a unit in \( (A_F)_{P_F} \) we may suppose \( y = \gamma(x) \) with \( x \in A_F \). Since \( \text{coker}(\alpha) \in \mathcal{O}_A \) it follows that there exist \( s \in A \setminus P \) and \( a \in A \) such that \( \alpha(s)x = \alpha(a) \). Applying \( \beta \gamma \) to this equality we get \( \delta(a) = 0 \), hence there exists \( u \in A \setminus P \) such that \( ua = 0 \). We get that \( \alpha(us)x = 0 \) and since \( \alpha(us) \not\in P \) we get \( y = 0 \).

**Corollary 2.7.** The functors \( \text{Spec}_D \) and \( \Gamma_D \) give a duality between the category \( \mathcal{R}_A \) of reduced affine Ritt schemes and the category \( \mathcal{R}_C \) of reduced closed Ritt algebras. Consequently, the category \( \mathcal{R}_C \) is a coreflective subcategory of the category \( \mathcal{R} \) of reduced Ritt algebras.

**Proof.** The Corollary is a consequence of Lemma 2.5 and Theorem 2.6.

**Remark.** Our Theorem 2.6 may be stated in more general terms. If in the definition of the category \( \mathcal{LDR} \) we forget the condition “maximal ideals in the stalks of \( \mathcal{O} \) are differential”, we get, by [5], the category of differential local ringed spaces which is denoted by \( \text{Diff} \text{loc} \). There are two standard functors \( \text{Spec}: \text{Diff} \to \text{Diff} \text{loc} \) and \( \Gamma: \text{Diff} \text{loc} \to \text{Diff} \), where \( \text{Spec} A \), as topological space, is the set of all prime ideals of \( A \) with the Zariski topology, \( A \) being any differential ring. Suppose we have another functor \( S: \text{Diff} \to \text{Diff} \text{loc} \) which satisfies the follows:

1) There exists a functorial morphism \( j: S \to \text{Spec} \) such that for any differential ring \( A \), the morphism \( j_A: S(A) \to \text{Spec} A \) is an inclusion of sets, the topology and the defining sheaf on \( S(A) \) being obtained by inverse image from the structures of \( \text{Spec} A \).

2) For any differential ring \( A \) and for any \( P \in S(A) \) we have \( PA_F \in S(A_F) \).
Notice that the functor Spec\(_d\) in arbitrary characteristic satisfies these axioms. The functor Spec\(_q\) considered in [7] satisfies them too. It is easy to see that our method leads in fact to the following result:

**Theorem 2.8.** If \(A\) is a differential ring such that \(S(A)\) is quasi-compact and \(\text{Ass}(A) \subseteq S(A)\), then the morphism \(A \to \Gamma S(A)\) is injective and induces an isomorphism \(\Gamma S(A) \to S(A)\).

3. Morphisms of differential finite type. We say that a morphism \(f: X \to Y\) of affine Ritt schemes is of differential finite type if there exists a morphism of Ritt algebras \(u: A \to B\) with \(B\) finitely generated over \(A\) as a differential algebra [8, p. 59] and there exist isomorphisms \(X \cong \text{Spec}_d B\), \(Y \cong \text{Spec}_d A\) such that \(\text{Spec}_d u\) induces \(f\) via these isomorphisms. If \(b_1, \ldots, b_s\) generate \(B\) as an \(A\)-differential algebra we use the notation \(B = A[b_1, \ldots, b_s]\). A morphism of Ritt schemes \(f: X \to Y\) is called dominant if \(f(X)\) is a dense subset in \(Y\). One may easily check that given a morphism of Ritt algebras \(u: A \to B\), then \(\text{Spec}_d u: \text{Spec}_d B \to \text{Spec}_d A\) is dominant if and only if \(\ker(u) \subseteq \text{nil}(A)\).

**Lemma 3.1.** If \(u: A \to B\) is an injective morphism of reduced Ritt algebras, then the morphism \(u_d: A_d \to B_d\) is also injective.

**Proof.** Since \(u\) is injective, it follows that the morphism \(\text{Spec}_d B \to \text{Spec}_d A\) is dominant. By Theorem 2.6 we get that the morphism \(\text{Spec}_d B_d \to \text{Spec}_d A_d\) is dominant, hence \(\ker(u_d) \subseteq \text{nil}(A_d) = 0\).

**Lemma 3.2.** Let \(f: X \to Y\) be a morphism of reduced affine Ritt schemes. Put \(A = \Gamma(Y, \mathcal{O}_Y)\) and \(B = \Gamma(X, \mathcal{O}_X)\). Then \(f\) is of differential finite type if and only if there exists a differential sub-\(A\)-Algebra \(C\) of \(B\) such that \(C\) is finitely generated over \(A\) as a differential algebra and \(C_d = B\).

**Proof.** By Corollary 2.2 we have \(Y \cong \text{Spec}_d A_1\) where \(A_1\) is reduced. By Theorem 2.6 we get that \(\text{Spec}_d A_1 \cong \text{Spec}_d (A_1)_d = \text{Spec}_d A\), hence \(Y \cong \text{Spec}_d A\). In the same way we get \(X \cong \text{Spec}_d B\). If we suppose there exists an algebra \(C\) as in the statement of our lemma, then by Theorem 2.6 we get that \(\text{Spec}_d C \cong \text{Spec}_d B\) and so \(f\) is of differential finite type. Conversely, suppose that \(f\) is of differential finite type. Let \(u: \bar{A} \to \bar{B}\) be a morphism of Ritt algebras such that \(\bar{B}\) is finitely generated over \(\bar{A}\) as a differential algebra and suppose there exist isomorphisms \(Y \cong \text{Spec}_d \bar{A}\), \(X \cong \text{Spec}_d \bar{B}\).
such that $f$ is given by $\text{Spec}_D u$ via these isomorphisms. Since $t_A(\overline{A}) \subseteq t_B(\overline{B})$, by Proposition 2.1 we may replace $\overline{A}$ and $\overline{B}$ by $\overline{A}/t_A(\overline{A})$ and $\overline{B}/t_B(\overline{B})$ which are reduced being subrings in $\overline{A}_D = A$ and $\overline{B}_D = B$ respectively. Hence we may suppose that $\overline{A}$ and $\overline{B}$ are reduced. If $u_D$ denotes the morphism $A \to B$, put $C = u_D(A)[\overline{B}] \subseteq B$. Applying Lemma 3.1 to $\overline{B} \subseteq C \subseteq B$ we get $B = \overline{B}_D \subseteq C_D \subseteq B$, hence $C_D = B$. Since $C$ is finitely generated over $A$ as a differential algebra, the lemma is proved.

**Theorem 3.3.** If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are morphisms of differential finite type between reduced affine Ritt schemes, then $gf$ is also of differential finite type.

**Proof.** Put $A = \Gamma(Z, \mathcal{O}_Z)$, $B = \Gamma(Y, \mathcal{O}_Y)$, $C = \Gamma(X, \mathcal{O}_X)$. By Lemma 3.2 there exist morphisms $A \xrightarrow{a} E \subseteq B = E_D$ and $B \xrightarrow{b} G \subseteq C = G_D$ where $E$ and $G$ are finitely generated over $A$ and $B$ respectively, as differential algebras. Put $T = b(E)[x_i, \ldots, x_k] \subseteq C$, where $x_i, \ldots, x_k$ generate $G$ over $B$ as a differential algebra. Applying the functor $\Gamma_D \text{Spec}_D$ to $E \to T \subseteq C$ we get by Lemma 3.1 $B = E_D \to T_D \subseteq C_D = C$, hence $T_D$ contains the ring $b(B)$ and the elements $x_i, \ldots, x_k$. Consequently $T_D$ contains $G$ and so applying again Lemma 3.1 to $G \subseteq T_D \subseteq C$ we get $C = G_D \subseteq T_D \subseteq C$ hence $T_D = C$. Since $T$ is finitely generated over $A$ as a differential algebra, we may apply Lemma 3.2 and we get that $gf$ is of differential finite type.

Now we prove the following differential version of Chevalley’s constructibility theorem:

**Theorem 3.4** Let $X \xrightarrow{f} Y$ be a morphism of differential finite type between ordinary affine Ritt schemes (“ordinary” means there is a single derivation). Suppose that $Y$ has a Noetherian underlying topological space. Then $f$ is constructible.

**Proof.** Suppose that $f = \text{Spec}_D u$ where $u$ is a morphism of Ritt algebras $A \to B$ such that $B$ is finitely generated over $A$ as a differential algebra.

Suppose for the beginning that $B$ is finitely generated over $A$ as an algebra (in the nondifferential sense!). It is sufficient to prove that $f(\text{Spec}_D B)$ is constructible. Since $\text{Spec}_D A$ is a Noetherian topological space, it is sufficient, applying classical criterion [9, 6. C], to prove that whenever the morphism $\text{Spec}_D (B/\overline{PB}) \xrightarrow{g} \text{Spec}_D (A/P)$ is dominant for a certain $P \in \text{Spec}_D A$, it follows that the image of $g$ contains a nonempty open subset in $\text{Spec}_D (A/P)$. But if $g$ is dominant, we get that $A/P \to B/\overline{PB}$ is injective. So we may suppose
that $A$ is a domain and $A \subseteq B$, all we have to show being that $f(\text{Spec}_D B)$ contains a nonempty open subset in $\text{Spec}_D A$. But $f(\text{Spec} B)$ contains a nonempty open subset in $\text{Spec} A$ by [9, proof of 6. E] and our statement follows from the general formula

$$f(\text{Spec}_D B) = f(\text{Spec} B) \cap \text{Spec}_D A$$

and from the fact that $\text{Spec}_D A$ is dense in $\text{Spec} A$.

Now come back to the general case and suppose that $B$ is finitely generated over $A$ as a differential algebra. It is sufficient to prove that $f(\text{Spec}_D B)$ is constructible. Applying [9, 6. C] again, we reduce ourselves to the case $A$ domain, $A \subseteq B$ and we have to prove that the image of $f$ contains a nonempty open set. Suppose $B = A[y_1, \cdots, y_n]$. Let $y_1, \cdots y_N$ be a maximal family of differentially algebraically independent elements over $A[8, \text{p. 69}]$ and put $C = A[y_1, \cdots, y_N]$. Since $\text{Spec}_D A$ is a Noetherian topological space and $B$ is finitely generated over $A$ as a differential algebra, it follows from [8, Theorem 1, p. 126] that $\text{Spec}_D B$ is also Noetherian and hence by [8, Theorem 1, p. 14] every radical differential ideal in $B$ is a finite intersection of prime differential ideals. Consequently, $\text{nil}(B) = P_1 \cap \cdots \cap P_r$, $P_i \in \text{Spec}_D B$ for all $i$ and so we get $(0) = \text{nil}(B) \cap C = (P_1 \cap C) \cap \cdots (P_r \cap C)$. Hence there exists an index $i$ such that the morphism $C \to B/P_i$ is injective. Put $z_j = y_j \mod P_i$ for all $j \geq N + 1$. For any such $j$ take a differential polynomial $[8, \text{p. 70}] F_j \in C(Y)$, $F_j \neq 0$, $F_j(z_j) = 0$. Suppose that we have chosen each $F_j$ of minimum order $n_j$ and of minimum degree in $Y^{(n_j)}$. Consider $S_j = \partial F_j / \partial Y^{(n_j)}$ the separant of $F_j$ [8, p. 75]. We have $S_j \neq 0$ (because of the characteristic) and $S_j(z_j) \neq 0$ by the minimality of $F_j$. Put $S = \prod S_j$ which is a nonzero element in $E = B/P_i$. We claim that $E[1/S]$ is finitely generated as a $C$-algebra (in the nondifferential sense). Indeed, if

$$F_j = \sum_k G_{kj}(Y^{(n_j)})^k, \quad G_{kj} \in C[Y, Y', \cdots, Y^{(n_j-1)}]$$

we get

$$0 = (F_j(z_j))' = \sum_k (G_{kj}(z_j))'(z_j^{(n_j)})^k + S_j(z_j)z_j^{(n_j+1)}$$

We get then by induction that for any $q \geq 0$

$$z_j^{(q)} \in C[z_{N+1}, \cdots, z_{N+1}^{(q)}, \cdots, z_n, \cdots, z_n^{(M)}, 1/S], \quad M = \max_j n_j$$

By the first part of our proof, the morphism

$$\text{Spec}_D (E[1/S]) \longrightarrow \text{Spec}_D C$$

is constructible and since it is dominant we get that its image contains a principal open set $D(H) \subseteq \text{Spec}_D C$ with $H \neq 0$. Now if
4. Remarks on the differential affine space. For any Ritt algebra $A$, let $B = A(Y_1, \ldots, Y_n)$ be the ring of differential polynomials over $A$. The Ritt scheme $A^*_n = \text{Spec}_B B$ will be called the differential $n$-affine space over $A$. For all $a = (a_1, \ldots, a_m) \in \mathbb{N}^m$ and for all $F \in B$ we shall write $F^{(a)}$ instead of $D_1^{a_1} \cdots D_n^{a_m} F$. Order the set of all indeterminates $Y_i^{(a)}$ lexicographically [8, p. 75]. Then for any $F \in B$, the leader $u_F$ of $F$ denotes the highest derivative $Y_i^{(a)}$ present in $F$ [8, p. 75].

**Lemma 4.1.** Let $A$ be a Ritt domain, $B = A(Y_1, \ldots, Y_n)$ and $0 \neq F \in B$. Then any element $x \in \Gamma(D(F), \hat{B})$ may be written as $x = H/G$, $H, G \in B$ and $u_G \leq u_F$.

**Proof.** Let $K$ be the field of quotients of $A$. There exist $F_1, \ldots, F_r, G_1, \ldots, G_r \in B$ such that $F = \{ (G_1, \ldots, G_r) \}$ and $x = F_i/G_i$ for all $i = 1, \ldots, r$. Let $H, G \in B$ be such that $x = H/G$ and $H, G$ have no common prime divisor in the ring $K\{Y_1, \ldots, Y_n\}$ which is factorial. Since $HG_i = F_iG$ for all $i$, it follows that $G$ divides $G_i$ in $K\{Y_1, \ldots, Y_n\}$ and so there exist $g_1, \ldots, g_r \in A$ such that $g_i G_i \in (G)B$ for all $i$. If $g = g_1g_2 \cdots g_r$, we have $\{ (G) \} \cong \{ (gG_1, \ldots, gG_r) \} = (g) \cap \{ (G_1, \ldots, G_r) \} \ni gF$. Now take a decomposition of $G$ into prime factors $G = E_1^{i_1} \cdots E_k^{i_k}$ in the ring $K\{Y_1, \ldots, Y_n\}$. Since $gF \in \{ E_i \}$, we cannot have $u_{gF} < u_{E_i}$ because if we had such an inequality, $gF$ would be reduced with respect to $E_i$ [8, p. 77] and hence $gF$ would be divisible by $E_i$ [8, p. 155] which is a contradiction. Hence $u_F = u_{gF} \geq u_{E_i}$, for all $i = 1, \ldots, r$ and since $u_G = \max \{ u_{E_i}, 1 \leq i \leq r \}$ we get $u_F \geq u_G$.

**Theorem 4.2.** Let $A$ be a Ritt domain. There is a natural isomorphism

$$(A\{Y_1, \ldots, Y_n\})_D \cong A_D\{Y_1, \ldots, Y_n\}$$

Consequently, if $A$ is closed, so is $A\{Y_1, \ldots, Y_n\}$.

**Proof.** Both rings are subrings in field of quotients of $B = A\{Y_1, \ldots, Y_n\}$. Obviously, $A_D\{Y_1, \ldots, Y_n\} \subseteq B_D$. Conversely, if $x \in B_D = \Gamma(\text{Spec}_B B, \hat{B})$, Lemma 4.1 $x$ may be written as $F/g$ with $F \in B$ and $g \in A$. Let us prove that $F/g \in A_D\{Y_1, \ldots, Y_n\}$ by induction on the number of monomials in $F$. Take $p \in \text{Spec}_D A$ and put $P = pB + [Y_1, \ldots, Y_n] \in \text{Spec}_B B$. Since $F/g \in B_P$ we get that there exist
Let \( W, H \in B \) with \( W(0, \ldots, 0) = w \in p \) and \( FW = gH \). Let \( fM = f \prod_{i,a}(Y_i^{(a)})^{k_{ia}} \) be a monomial of minimum degree in \( F \) where \( f \in A \). Identifying the coefficients of \( M \) we get \( fw = gh \) with \( h \in A \), hence \( f/g \in A_p \). Since \( p \) runs through \( \text{Spec}_D A \) it follows that \( f/g \in A_D \). Hence \( fM/g \in A_D \{ Y_1, \ldots, Y_n \} \) and so \( F/g \in A_D \{ Y_1, \ldots, Y_n \} \).

Let \( K \) be a field of characteristic zero. A consequence of Theorem 4.2 is the fact that the algebra \( K \{ Y_1, \ldots, Y_n \} \) is closed. It is natural to look for its factorizations which remain closed. Suppose the derivation operators are independent on \( K[\delta, \eta] \) and that \( K \) has an uncountable field of constants.

**Proposition 4.3.** Let \( P \) be a differential prime ideal in \( K \{ Y_1, \ldots, Y_n \} \) such that the algebra \( K \{ Y_1, \ldots, Y_n \}/P \) is closed. If the field of quotients \( Q(A) \) is differentially algebraic over \( K \), then \( A \) is a field, algebraic extension of \( K \).

**Proof.** Suppose there exists \( y \in A \) which is transcendental over \( K \). Since \( y \) is differentially algebraic over \( K \), there exists a differential polynomial \( 0 \neq F \in K \{ Z \} \) such that \( F(y) = 0 \). Since the derivation operators are independent on \( K \), there exists by [8, p. 99] \( v \in K \) such that \( F(v) \neq 0 \). By Taylor's formula,

\[
F(v + T) = F(v) + \sum_a (\partial F/\partial Z^{(a)})(v)T^{(a)} + \cdots
\]

Put

\[
G(T) = F(v) + (\partial F/\partial Z)(v)T + \cdots
\]

Since \( G \) is a nonzero polynomial in \( K[T] \), it has only a finite set of roots (perhaps non) in the field of constants \( K_0 \) of \( K \). Consequently, there is an uncountable set \( \Omega \subseteq K_0 \) such that for every \( c \in \Omega \) we have \( F(v + c) = G(c) \neq 0 \). Put \( z = y - v \) which is also transcendental over \( K \). Since \( F((z - c) + (v + c)) = 0 \) we get that for every \( c \in \Omega \) the differential ideal generated by \( z - c \) in \( A \) is the whole ring \( A \). Consequently, for every \( c \in \Omega \) we have \( 1/(z - c) \in A_D = A \). But the family \( \{ 1/(z - c), c \in \Omega \} \) is uncountable and linearly independent over \( K \), which contradicts the obvious fact that \( A \) is generated as a vector space by a countable set. Consequently \( A \) is algebraic over \( K \), hence \( A \) is a field.

**Corollary 4.4.** If \( P \) is a nonzero differential prime ideal prime ideal in \( K \{ Y \} \), then \( K \{ Y \}/P \) cannot be closed unless it is a field.
For any topological space $X$, $H^i(X, \mathcal{O})$ will denote the derived functors of $\Gamma(X, \mathcal{O}) : \text{Ab}(X) \to \text{Ab}$, [3, p. 207]. The following result shows that there is a great difference from the cohomological point of view between schemes and Ritt schemes.

**Theorem 4.5.** Let $A$ be a Ritt domain, $n \geq 1$ and $A^n$ the differential $n$-affine space over $A$, $\mathcal{O}$ being its defining sheaf. Then we have:

$$H^i(U, \mathcal{O}) \neq 0$$

for any nonempty open subset $U$ of $A^n$.

**Proof.** Suppose $H^i(U, \mathcal{O}) = 0$ where $U = D(I)$, $I$ being a nonzero ideal in $B = A\{Y_1, \cdots, Y_n\}$. Replacing $A\{Y_1, \cdots, Y_{n-1}\}$ by $A$ we may suppose that $n = 1$ and put $Y = Y_1$. Choose $F \in I$, $F' \in A$, $u_F = Y^{(b)}$. Let us also take $a \in N^n$, $a > b$ in the lexicographic order and take $c \in N^n$, $c \neq (0, \cdots, 0)$. Put $y = Y^{(a)} - Y^{(a+c)}$ and consider the exact sequence of $B$-modules

$$0 \longrightarrow B \longrightarrow B/yB = M \longrightarrow 0$$

where $w$ is the multiplication by $y$. This sequence induces an exact sequence of $\hat{B} = \mathcal{O}$-modules

$$0 \longrightarrow \mathcal{O} \longrightarrow \mathcal{O} \longrightarrow \tilde{M} \longrightarrow 0$$

We get an exact sequence

$$\Gamma(U, \mathcal{O}) \longrightarrow \Gamma(U, \tilde{M}) \longrightarrow H^1(U, \mathcal{O}) = 0$$

Put $F_1 = Y^{(a)} - 1$ and $F_2 = Y^{(a+c)} - 1$. Since $F_1^{(c)} - F_2 = 1$ we get that $D(F_1) \cup D(F_2) = \text{Spec}_B B$. Put

$$s_1 = \bar{x}/F_1 \in M_{F_1} \quad s_2 = \bar{x}/F_2 \in M_{F_2}$$

where for all $x \in B$, $\bar{x}$ is the image of $x$ in $M$. Since $s_1$ and $s_2$ stick together, we get a section $s \in \Gamma(U, \tilde{M})$. Since $p$ is surjective there exists $t \in \Gamma(U, \mathcal{O})$ such that $p(t) = s$. By Lemma 4.1, we may write $t = W/H$ with $W, H \in B$ and $u_W \leq u_H$. Since $W/H$ and $1/F_1$ have the same image in any $M_P$ with $P \in D(F_1) \cap U = D(F_1I)$, we get that for any such $P$, there exist $T_P \in B\setminus P$ and $G_P \in B$ such that

$$T_P(Y^{(a)} - 1)W - H = G_P(Y^{(a)} - Y^{(a+c)})$$

But $Y^{(a)} - Y^{(a+c)}$ cannot divide the polynomial $E = (Y^{(a)} - 1)W - H$ because if it did, making in $E$ the substitution $Y^{(a)} = Y^{(a+c)} = 1$ we would get $H = 0$ (since $H$ does not change under this substitution).
Consequently, $Y^{(a)} - Y^{(a+e)}$ divides $T_P$ and so $T_P \in [Y^{(a)}]$ for any $P \in D(F'I)$. Let $J$ be the radical differential ideal generated by all $T_P$ as $P$ runs through $D(F'I)$. Obviously we have $F'I \subseteq J$ and so

$$F'_1F \in [Y^{(a)}]$$

On the other hand $F'_1 = Y^{(a)} - 1 \in [Y^{(a)}]$ and $F \in [Y^{(a)}]$ because $u_F = Y^{(b)}$ and $b < a$. We have obtained a contradiction, because the ideal $[Y^{(a)}]$ is prime.

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Thomas E. Armstrong, Barycentric simplicial subdivision of infinite-dimensional simplexes and octahedra ........................................... 251
Hom Nath Bhattarai and James William Fernandez, Joins of double coset spaces ................................................................. 271
Alexandru Buium, Ritt schemes and torsion theory .......................... 281
Jacob Burbea, Operator-valued Pick’s conditions and holomorphicity ...... 295
Su-Shing Chen, Duality condition and property (S) .......................... 313
Ky Fan, Evenly distributed subsets of $S^n$ and a combinatorial application ... 323
Leslie Foged, On $g$-metrizability .................................................. 327
John Groves Heywood, An error estimate uniform in time for spectral Galerkin approximations of the Navier-Stokes problem ................. 333
Aggie Ho, The Kreĭ n-Milman property and complemented bushes in Banach spaces ................................................................. 347
David R. Jackett, Rings on certain mixed abelian groups ..................... 365
Shoji Kyuno, Prime ideals in gamma rings ........................................ 375
George Lucius O’Brien, Zero-inducing functions on finite abelian groups ...................................................................................... 381
P. Robba, Sur les équations différentielles linéaires $p$-adiques. II .................. 393
Wolfgang Ruess, [Weakly] compact operators and DF spaces .................. 419
Claude Schochet, Topological methods for $C^*$-algebras. II. Geometry resolutions and the Künneth formula .................................... 443
Harry F. Smith, Jr., Equivalent nilpotencies in certain generalized right alternative rings ................................................................. 459
Max Warshauer, Diagonalization up to Witt ....................................... 469
Hugh C. Williams, A class of primality tests for trinomials which includes the Lucas-Lehmer test .................................................... 477