

Pacific Journal of Mathematics

**EVENLY DISTRIBUTED SUBSETS OF S^n AND A
COMBINATORIAL APPLICATION**

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A family \mathcal{F} of nonempty subsets of the n -sphere S^n is said to be evenly distributed if every open hemisphere contains at least one set of \mathcal{F} . This paper first proves an antipodal theorem for evenly distributed families of nonempty closed subsets of S^n , and then applies it to improve a recent combinatorial result of Kneser-Lovász-Bárány.

For a positive integer n , let S^n denote the n -sphere $\{x \in \mathbf{R}^{n+1} : \|x\| = 1\}$ in the Euclidean $(n+1)$ -space \mathbf{R}^{n+1} . For a subset A of S^n , $-A$ denotes the antipodal set of A : $-A = \{-x : x \in A\}$. For each $x \in S^n$, let $H(x)$ be the open hemisphere $H(x) = \{y \in S^n : (x, y) > 0\}$, where (x, y) is the inner product of x and y . Following Gale [5], we say that a family \mathcal{F} of nonempty subsets of S^n is *evenly distributed*, if for every $x \in S^n$, the open hemisphere $H(x)$ contains at least one set of \mathcal{F} .

THEOREM 1. *Let n, m be two positive integers. Let \mathcal{F} be an evenly distributed family of nonempty closed subsets of S^n . Let \mathcal{F} be partitioned into m subfamilies $\mathcal{F} = \bigcup_{i=1}^m \mathcal{F}_i$ such that for each i and for any two subsets A', A'' in the same subfamily \mathcal{F}_i , $A' \cup (-A'')$ is not contained in any open hemisphere. Then m is necessarily $\geq n + 2$. Furthermore, there exist $n + 2$ indices $1 \leq \nu_1 < \nu_2 < \dots < \nu_{n+2} \leq m$ and $n + 2$ sets $A_j \in \mathcal{F}_{\nu_j}$ ($1 \leq j \leq n + 2$) such that the union $\bigcup_{j=1}^{n+2} (-1)^j A_j$ is contained in an open hemisphere.*

Proof. For each $i = 1, 2, \dots, m$, let G_i be the set of those points $x \in S^n$ for which the open hemisphere $H(x)$ contains at least one set of \mathcal{F}_i . Clearly G_i is open in S^n . As $\mathcal{F} = \bigcup_{i=1}^m \mathcal{F}_i$ is evenly distributed, we have $S^n = \bigcup_{i=1}^m G_i$. Furthermore, G_i contains no pair of antipodal points. In fact, $x \in G_i$ and $-x \in G_i$ would mean the existence of $A' \in \mathcal{F}_i$ and $A'' \in \mathcal{F}_i$ such that $A' \subset H(x)$ and $A'' \subset H(-x)$. Then we would have $A' \cup (-A'') \subset H(x)$, against our hypothesis.

The open covering $S^n = \bigcup_{i=1}^m G_i$ can be shrunken to a closed covering, i.e., we can find closed sets $F_i \subset G_i$ ($1 \leq i \leq m$) such that $S^n = \bigcup_{i=1}^m F_i$. Then none of the F_i 's contains a pair of antipodal points. By the classical antipodal theorem of Lusternik-Schnirelmann-Borsuk [2], [3], [8], m is necessarily $\geq n + 2$. Moreover, by a result in our paper [4], which asserts slightly more than the

Lusternik-Schnirelmann-Borsuk theorem, there exist $n + 2$ indices $1 \leq \nu_1 < \nu_2 < \cdots < \nu_{n+2} \leq m$ such that $\bigcap_{j=1}^{n+2} (-1)^j F_{\nu_j} \neq \emptyset$. Then for any point z in this intersection, we have $-z \in \bigcap_{j \text{ odd}} F_{\nu_j} \subset \bigcap_{j \text{ odd}} G_{\nu_j}$ and $z \in \bigcap_{j \text{ even}} F_{\nu_j} \subset \bigcap_{j \text{ even}} G_{\nu_j}$. Hence there exist $n + 2$ sets $A_j \in \mathcal{F}_{\nu_j}$ ($1 \leq j \leq n + 2$) such that $A_j \subset H(-z)$ for odd j , and $A_j \subset H(z)$ for even j . In other words, the union $\bigcup_{j=1}^{n+2} (-1)^j A_j$ is contained in the open hemisphere $H(z)$. This completes the proof.

As an application of Theorem 1, we have the following combinatorial result.

THEOREM 2. *Let k, n, m be three positive integers. Let E be a finite set with at least $2k + n$ elements, and let \mathcal{F} denote the family of those subsets of E which have exactly k elements. If \mathcal{F} is partitioned into m subfamilies $\mathcal{F} = \bigcup_{i=1}^m \mathcal{F}_i$ such that for each i , no two subsets in the same subfamily \mathcal{F}_i are disjoint, then $m \geq n + 2$. Furthermore, there exist $n + 2$ indices $1 \leq \nu_1 < \nu_2 < \cdots < \nu_{n+2} \leq m$ and $n + 2$ sets $A_j \in \mathcal{F}_{\nu_j}$ ($1 \leq j \leq n + 2$) such that the union $\bigcup_{j \text{ odd}} A_j$ is disjoint from the union $\bigcup_{j \text{ even}} A_j$.*

Proof. According to a theorem of Gale [5], there exist $2k + n$ points on S^n such that every open hemisphere contains at least k of these points. As E has at least $2k + n$ elements, E can be regarded as a subset of S^n such that the family \mathcal{F} (of all subsets of E with k elements) is evenly distributed. For each i and for any two subsets A', A'' in the same subfamily \mathcal{F}_i , we have $A' \cap A'' \neq \emptyset$ and therefore $A' \cup (-A'')$ is not contained in any open hemisphere. By Theorem 1, m is necessarily $\geq n + 2$. Furthermore, there exist $n + 2$ indices $1 \leq \nu_1 < \nu_2 < \cdots < \nu_{n+2} \leq m$ and $n + 2$ sets $A_j \in \mathcal{F}_{\nu_j}$ ($1 \leq j \leq n + 2$) such that $\bigcup_{j=1}^{n+2} (-1)^j A_j$ is contained in an open hemisphere $H(z)$. Then $\bigcup_{j \text{ odd}} A_j$ and $\bigcup_{j \text{ even}} A_j$ are contained in $H(-z)$ and $H(z)$ respectively, and therefore are disjoint.

Obviously Theorem 2 can be interpreted as a result on coloring (with m colors) of the $(k - 1)$ -dimensional faces of a simplex of dimension $\geq 2k + n - 1$ such that no two $(k - 1)$ -dimensional faces of the same color are disjoint.

The partial conclusion $m \geq n + 2$ in Theorem 2 was conjectured by Kneser [6] in 1955, and proved recently by Lovász [7] and Bárány [1]. In proving $m \geq n + 2$, both these authors use the Lusternik-Schnirelmann-Borsuk theorem. Bárány's proof depends also on Gale's theorem.

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Received April 15, 1981. Work supported in part by the National Science Foundation.

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The *Pacific Journal of Mathematics* is issued monthly as of January 1966. Regular subscription rate: \$102.00 a year (6 Vols., 12 issues). Special rate: \$51.00 a year to individual members of supporting institutions.

Subscriptions, orders for numbers issued in the last three calendar years, and changes of address should be sent to Pacific Journal of Mathematics, P.O. Box 969, Carmel Valley, CA 93924, U.S.A. Old back numbers obtainable from Kraus Periodicals Co., Route 100, Millwood, NY 10546.

PUBLISHED BY PACIFIC JOURNAL OF MATHEMATICS, A NON-PROFIT CORPORATION

Printed at Kokusai Bunken Insatsusha (International Academic Printing Co., Ltd.).

8-8, 3-chome, Takadanobaba, Shinjuku-ku, Tokyo 160, Japan.

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