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# ON $g$-METRIZABILITY

LESLIE FOGED

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# ON $g$-METRIZABILITY

L. FOGED

**We show that a regular topological space is $g$-metrizable if and only if it is weakly first countable and admits a $\sigma$-locally finite $k$-network and that a $g$-metrizable space need not be $g$-developable.**

**0. Introduction.** $G$-metrizable spaces were defined in [8], where it was also shown that a space admits a countable weak base if and only if it is weakly first countable and has a countable $k$-network. In this paper we provide the corresponding result for $g$-metrizable spaces and give an example of a $g$-metrizable space which is not $g$-developable. The former result is in response to a question in [8], the latter answers a question in [6]. All spaces are at least regular.

## 1. Definition.

1.1. Let $X$ be a space. If $\Gamma$ is a family of subsets of $X$ and $\zeta: \Gamma \to \mathscr{P}(X)$ is a function, then the pair $\langle \Gamma, \zeta \rangle$ is a weak base for $X$ if, in addition, the following hold:

(a) For every member $G$ of $\Gamma$, $\zeta(G)$ is a subset of $G$.

(b) If $G_1$ and $G_2$ are members of $\Gamma$ and $x$ is an element of $\zeta(G_1) \cap \zeta(G_2)$, then there is a member $G_3$ of $\Gamma$ so that $x$ is in $\zeta(G_3)$ and $G_3$ is a subset of $G_1 \cap G_2$.

(c) A subset $U$ of $X$ is open if and only if for every element $x$ of $U$ there is a member $G$ of $\Gamma$ so that $x$ is in $\zeta(G)$ and $U$ contains. $G$.

This definition of weak base differs from that of [1], namely, a collection $\mathscr{B} = \cup\{T_x: x \in X\}$ is a weak base for $X$ if a set $U$ is open in $X$ precisely when for each point $x \in U$ there exists $B \in T_x$ such that $B \subset U$. It is easy to see that our definition is equivalent to this, for if $B$ is as above, we let $\Gamma = \mathscr{B}$ and for $G \in \Gamma$, let $\delta(G) = \{x: G \in T_x\}$ and if $\langle \Gamma, \delta \rangle$ is a weak base by 1.1, then we let $T_x = \{G: x \in \delta(G)\}$ and $\mathscr{B} = \cup\{T_x: x \in X\}$.

1.2. A space $X$ is $g$-metrizable if it has a weak base $\langle \Gamma, \zeta \rangle$ where $\Gamma$ is a $\sigma$-locally finite family. $X$ is weakly first countable if $X$ has a weak base $\langle \Gamma, \zeta \rangle$ so that the family $\{\zeta(G): G \in \Gamma\}$ is point countable or, equivalently, there is a function $B: \omega \times X \to \mathscr{P}(X)$ (called a wfc system for $X$) so that

(a) for all $n < \omega$ and $x \in X$, $B(n+1, x) \subset B(n, x)$;

(b) for all $x$ in $X$, $x \in \cap\{B(n, x): n < \omega\}$

(c) a subset $U$ of $X$ is open if and only if for every $x$ in $U$ there is an $n < \omega$ so that $U$ contains $B(n, x)$.

If $x$ is an element of a space $X$, then a subset $S$ of $X$ is said to be weak neighborhood of $x$ if every sequence converging to $x$ is eventually in $S$. One may show that if $X$ is weakly first countable with weak base $\langle \Gamma, \zeta \rangle$ so that $\{\zeta(G): G \in \Gamma\}$ is point countable, then $S$ is a weak neighborhood of $x$ if and only if $S$ contains a member $G$ of $\Gamma$ so that $x \in \zeta(G)$. Thus weakly first countable spaces are sequential [4].

1.3. If $X$ is a space, a collection $\Gamma$ of subsets of $X$ is said to be a $k$-network [7] for $X$ if for any compact subset $K$ of $X$ and any neighborhood $U$ of $K$, there is a finite subcollection $\Gamma'$ of $\Gamma$ so that $K \subset \cup \Gamma' \subset U$.

## 2. $g$-metrizability and $k$-networks.

LEMMA 2.1. *If $X$ is a space in which points are $G_\delta$ and if $\langle \Gamma, \zeta \rangle$ is a weak base for $X$, then $\Gamma$ is a $k$-network for $X$.*

*Proof.* Let $K$ be a compact subset of $X$ and $U$ an open neighborhood of $K$. As $K$ is closed, $\langle \Gamma', \zeta' \rangle$ given by $\Gamma' = \{G \cap K: G \in \Gamma\}$ and $\zeta'(G \cap K) = \zeta(G) \cap K$ for all $G$ in $\Gamma$, is a weak base for $K$. Thus since $K$ is Fréchet, for every $G$ in $\Gamma$ $\zeta'(G \cap K) \subset \mathrm{int}_K (G \cap K)$. Consequently if $\Gamma^*$ is a subcollection of $\Gamma$ so that $K \subset \cup \{\zeta(G): G \in \Gamma^*\}$ and $\cup \Gamma^* \subset U$, then a finite subfamily of $\Gamma^*$ convers $K$.

THEOREM 2.2 [3]. *A regular space with a $\sigma$-locally finite $k$-network has a $\sigma$-dicrete $k$-network.*

LEMMA 2.3. *Suppose $X$ hase $\langle \Gamma, \zeta \rangle$ so that $\Gamma = \cup \{\Gamma_n: n < \omega\}$ where every $\Gamma_n$ ts a closure-preserving family of closed sets. If $\{F_\alpha: \alpha \in I\}$ is a discrete collection of subsets of $X$, then there is a pairwise disjoint collection $\{N_\alpha: \alpha \in I\}$ so that for every $\alpha \in I$ and $x \in F_\alpha$, there is a $G$ in $\Gamma$ so that $x \in \zeta(G)$ and $G \subset N_\alpha$.*

*Proof.* For each $n < \omega$ and each $\alpha \in I$, let

$$G(n, \alpha) = \cup \{G \in \Gamma_n: G \cap (\cup \{F_\beta: \beta \neq \alpha\}) = \emptyset\}$$

For each $\alpha \in I$, let

$$N_\alpha = \bigcup_{n<\omega} [G(n, \alpha) \setminus \cup \{G(m, \beta): m \leqq n, \beta \neq \alpha\}] .$$

Of course $\{N_\alpha: \alpha \in I\}$ is pairwise disjoint; we now verify that $\{N_\alpha: \alpha \in I\}$ is the desired collection. Let $\alpha \in I$ and let $x \in F_\alpha$. Find an

$n<\omega$ and a $G_1$ in $\Gamma_n$ so that $x \in \zeta(G_1)$ and so that $G_1$ misses the closed set $\cup\{F_\beta: \beta \neq \alpha\}$. Pick $G_2 \in \Gamma$ so that $x \in \zeta(G_2)$ and so that $G_2$ misses the closed set $\cup\{G(m, \beta): m \leqq n, \beta \neq \alpha\}$. Now there is a $G_3 \in \Gamma$ with $x \in \zeta(G_3)$ so that $G_3$ is a subset of $G_1 \cap G_2$, hence $G_3 \subset N_\alpha$, as desired.

We are now in a position to prove the main result of this section.

THEOREM 2.4. *A regular space is g-metrizable if and only if it is weakly first countable and admits a σ-locally finite k-network.*

*Proof.* The necessity follows from Lemma 2.1. For the sufficiency: by Theorem 2.2, for each $n<\omega$ let $\Lambda_n$ be a discrete collection of closed subsets of $X$ so that $\Lambda = \cup\{\Lambda_n: n<\omega\}$ is closed under finite intersections and is a $k$-network for $X$. Let

$$\Gamma = \{\cup \Lambda^*: \Lambda^* \text{ is a finite subset of } \Lambda \text{ so that } \cap \Lambda^* \neq \emptyset\} .$$

For $\Lambda^*$ a finite subset of $\Lambda$ with $\cap \Lambda^* \neq \emptyset$, let

$$\zeta(\cup \Lambda^*) \simeq \{x \in \cap \Lambda^*: \cup \Lambda^* \text{ is a weak neighborhood of } x\} .$$

Note that $\{(G): G \in \Gamma\}$ is point-countable. We now show that $\langle \Gamma, \zeta \rangle$ is a weak base for $X$. One easily verifies that (a) and (b) of 1.1 are satisfied. For (c), observe that if $U$ is a subset of $X$ so that for every $x \in U$ there is a $G \in \Gamma$ so that $x \in \zeta(G)$ and $U$ contains $G$, then $U$ is sequentially open, hence open. Conversely, suppose $U$ is open and there is an element $x$ of $U$ so that $U$ contains no member $G$ of $\Gamma$ such that $x \in \zeta(G)$, i.e. the union of no finite subset of $\{L_j: j<\omega\} = \{L \in \Lambda: x \in L, L \subset U\}$ is a weak neighborhood of $x$. Let $B$ a wfc system for $X$ so that $B(1, x) \subset U$. Inductively pick a sequence $\{x_n: n<\omega\}$ so that $x_n \in B(n, x) \setminus \cup\{L_j: j \leqq n\}$. The sequence $\{x_n: n<\omega\}$ converges to $x$, hence $\{x\} \cup \{x_n: n<\omega\}$ is compact. Let $\Lambda'$ be a finite subset of $\Lambda$ so that $\{x\} \cup \{x_n: n<\omega\} \subset \cup \Lambda' \subset U$ and let $\Lambda^* = \{L \in \Lambda': x \in L\}$. The closed set $\cup(\Lambda' \setminus \Lambda^*)$ omits $x$, so there is an $m<\omega$ so that $\{x\} \cup \{x_n: n \geqq m\} \subset \cup \Lambda^*$. Also $\Lambda^* \subset \{L \subset \Lambda: x \in L, L \subset U\}$, so there is an $r \geqq m$ so that $\Lambda^* \subset \{L_j: j \leqq r\}$, which implies that $x_r \in \cup \Lambda^* \subset \cup\{L_j: j \leqq r\}$. This contradicts the fact that $x_r$ was picked in the complement of $\cup\{L_j: j \leqq r\}$. Thus if $U$ is open, then for all $x \in U$, $U$ contains a $G \in \Gamma$ so that $x \in \zeta(G)$; so $\langle \Gamma, \zeta \rangle$ is a weak base for $X$.

Note that if $n<\omega$,

$$\Gamma_n = \{\cup \Lambda^*: \Lambda^* \text{ is a finite subset of } \cup\{\Lambda_j: j \leqq n\} \text{ so that } \cap \Lambda^* \neq \emptyset\}$$

is a closure-preserving collection, hence $\Gamma = \cup\{\Gamma_n: n<\omega\}$ is $\sigma$-conservative.

For every finite subset $S$ of $\omega$, let

$$\Lambda_S = \{\Lambda^*: \text{ for } n < \omega \ \Lambda^* \cap \Lambda_n \neq \emptyset \text{ iff } n \in S; \ \cap \Lambda^* \neq \emptyset\}$$

and write $\Lambda_S = \{\Lambda_\alpha^*: \alpha \in I(S)\}$. Further, as $\{\cap \Lambda_\alpha^*: \alpha \in I(S)\}$ is a discrete collection, use Lemma 2.3 to find a pairwise disjoint collection $\{N_\alpha: \alpha \in I(S)\}$ so that for every $\alpha$ in $I(S)$ $N_\alpha$ is a weak neighborhood of $\cap \Lambda_\alpha^*$.

Now if $n < \omega$, $S$ is a finite subset of $\omega$, and if $\alpha \in I(S)$, let

$$G(n, \alpha) = \cup\{G \in \Gamma_n: G \subset (\cup \Lambda_\alpha^*) \cap N_\alpha\} .$$

and let

$$\zeta'(G(n, \alpha)) = \cup\{\zeta(G) \cap \zeta(\cup \Lambda_\alpha^*): G \in \Gamma_n, G \subset (\cup \Lambda_\alpha^*) \cap N_\alpha\} .$$

If $n < \omega$ and if $S$ is a finite subset of $\omega$, let

$$\Gamma(n, S) = \{G(n, \alpha): \alpha \in I(S)\} .$$

The collections $\Gamma(n, S)$ are conservative and, since $G(n, \alpha) \subset N_\alpha$ for every $\alpha \in I(S)$, pairwise disjoint, hence discrete. Let $\Gamma'$ be the family of all intersections of finite subcollections of $\cup\{\Gamma(n, S): n < \omega$, $S$ a finite subset of $\omega\}$ and extend $\zeta'$ to $\Gamma'$ by $\zeta'(\bigcap_{i=1}^k G(n_i, \alpha_i)) = \bigcap_{i=1}^k \zeta'(G(n_i, \alpha_i))$. Observe that $\Gamma'$ is $\sigma$-discrete; we will show that $\langle \Gamma', \zeta' \rangle$ is a weak base for $X$, completing the proof.

Conditions (a) and (b) of 1.1 are easily verified. Recalling that $\{\zeta(G): G \in \Gamma\}$ is point countable, the remarks in 1.2 give that for all $G \in \Gamma$ $G$ is a weak neighborhood of $\zeta(G)$ so that if $n < \omega$, $S$ is a finite subset of $\omega$ and if $\alpha \in I(S)$, then $G(n, \alpha)$ is a weak neighborhood of $\zeta'(G(n, \alpha))$. Consequently if $G' \in \Gamma'$, then $G'$ is a weak neighborhood of $\zeta(G')$. Hence if $U$ is a subset of $X$ such that for every member $x$ of $U$ there is a member $G'$ of $\Gamma'$ with $x \in \zeta(G')$ and $G' \subset U$, then $U$ is a weak neighborhood of each of its elements, thus sequentially open, and so $U$ is open. To complete the proof of (c), let $U$ be an open subset of $X$, and let $x \in U$. Since $\langle \Gamma, \zeta \rangle$ is a weak base for $X$, there is a finite subset $\Lambda^*$ of $\Lambda$ so that $x \in \zeta(\cup \Lambda^*) \subset \cap \Lambda^* \subset \cup \Lambda^* \subset U$. Find a finite subset $S$ of $\omega$ and an $\alpha \in I(S)$ so that $\Lambda^* = \Lambda_\alpha^*$. Since $\cup \Lambda_\alpha^*$ is a member of $\Gamma$, $\cup \Lambda_\alpha^*$ is a weak neighborhood of $\zeta(\cup \Lambda_\alpha^*)$, hence of $x$; $N_\alpha$ is a weak neighborhood of $\cap \Lambda_\alpha^*$, hence of $x$; thus $(\cup \Lambda_\alpha^*) \cap N_\alpha$ is a weak neighborhood of $x$. Again since $\{\zeta(G): G \in \Gamma\}$ is point-countable, we have that there is an $n < \omega$ and a $G \in \Gamma_n$ so that $x \in \zeta(G)$ and $G \subset (\cup \Lambda_\alpha^*) \cap N_\alpha$. Thus $x \in \zeta'(G(n, \alpha))$ and $G(n, \alpha) \subset \cup \Lambda_\alpha^* \subset U$. Thus (c) is established.

**3. $g$-developable spaces.** Generalizing a characterization of developability given in [5], Lee [6] defined $g$-developable spaces to be those weakly first countable spaces $X$ which have a wfc system satisfying the following: if $x \in X$ and if $\{x_n: n < \omega\}$ and $\{y_n: n < \omega\}$ are sequences in $X$ so that for every $n < \omega$ $x$ and $x_n$ are elements of $B(n, y_n)$, then the sequence $\{x_n: n < \omega\}$ converges to $x$.

PROPOSITION 3.1. *A $\sigma$-discrete weakly first countable space $X$ is $g$-developable.*

*Proof.* Write $X = \dot{\bigcup}\{D_n: n < \omega\}$, where $D_n$ is a closed discrete set for every $n < \omega$. $X$ is symmetrizable [1], so let $d$ be a compatible symmetric function. We define $B: w \times X \to \mathscr{P}(X)$ as follows: if $m$ and $n$ are finite ordinals and if $x \in D_m$, let

$$B(n, x) = \{y \in X: d(x, y) < 1/n\} \backslash \bigcup \{D_k: k < m\} .$$

One easily checks that $B$ is a wfc system for the topology of $X$. To see that $B$ satisfies the defining condition for $g$-developability let $x \in X$ and let $\{x_n: n < \omega\}$ and $\{y_n: n < \omega\}$ be sequences in $X$ so that for every $n < \omega$ $x$ and $x_n$ are in $B(n, y_n)$. If $m < \omega$ so that $x \in D_m$, then there is a $j < \omega$ so that $\{y \in X: d(x, y) < 1/j\} \cap (\bigcup \{D_k: k \leqq m\}) = \{x\}$. The fact that $x \notin \bigcup \{B(j, y): y \neq x\}$ implies that if $n \geqq j$, then $y_n = x$. Thus for all $n \geqq j$ we have $x_n \in B(n, x)$, hence $\{x_n: n < \omega\}$ converges to $x$, as desired.

The definition of $g$-developable inspires the question to which the following is a negative answer.

THEOREM 3.2. *There is a $g$-metrizable space which is not $g$-developable.*

*Proof.* Let $\boldsymbol{R}$ denote the set of real numbers $\boldsymbol{Q}$ the set of rationals. Choose a countable quasibase $\Lambda$ for the Euclidean topology of $\boldsymbol{R}$ consisting of closed sets. Let $X = \{\langle x, y\rangle \in R^2$: either $y = 0$, or $x \in \boldsymbol{Q}$ & $1/y \in \omega\}$, and view $\boldsymbol{R}$ as $\{\langle x, y\rangle \in X: y = 0\}$. For every $q \in \boldsymbol{Q}$ and $m < \omega$, define $A(m, q) = \{r \in \boldsymbol{R}: |r - q| \leqq 1/m\} \cup \{\langle q, 1/n\rangle: n > m\}$. Let

$$\Gamma = \{A(m, q): m < \omega, q \in \boldsymbol{Q}\} \cup \Lambda \cup \{\{\langle q, 1/n\rangle\}: q \in \boldsymbol{Q}, n < \omega\} ,$$

and define

$$\zeta(A(m, q)) = \{q\} , \quad \text{if } m < \omega \text{ and } q \in \boldsymbol{Q} ;$$
$$\zeta(L) = \{r \in \boldsymbol{R}\backslash\boldsymbol{Q}: r \text{ is in the Euclidean interior of } L\} , \quad \text{if } L \in \Lambda ;$$
$$\zeta(\{\langle q, 1/n\rangle\}) = \{\langle q, 1/n\rangle\} , \quad \text{if } q \in \boldsymbol{Q} \text{ and } n < \omega .$$

Give $X$ the topology for which $\langle \Gamma, \zeta\rangle$ is a weak base. Certainly $\Gamma$ is countable, so, as $X$ is easily seen to be regular, $X$ is $g$-metrizable. To show that $X$ is not $g$-developable, assume that $B$ is a wfc-system for $X$ satisfying the defining condition for $g$-developability.

Define a function $\phi: \boldsymbol{R}\backslash\boldsymbol{Q} \rightarrow \omega$ so that if $r \in \boldsymbol{R}\backslash\boldsymbol{Q}$, then $r \notin \bigcup\{B(\phi(r), q): q \in \boldsymbol{Q}\}$. This is possible, for if there is an $r \in \boldsymbol{R}\backslash\boldsymbol{Q}$ so that for every $n < \omega$ there is a $q_n \in \boldsymbol{Q}$ so that $r \in B(n, q_n)$, then find, for each $n < \omega$, an $x_n \in X\backslash\boldsymbol{R} \cap B(n, q_n)$. This would imply that for every $n < \omega$, $r$ and $x_n$ are in $B(n, q_n)$, but $\{x_n: n < \omega\}$ does not converge to $r$, a contradiction.

Since $\boldsymbol{R}\backslash\boldsymbol{Q} = \bigcup\{\{r \in \boldsymbol{R}\backslash\boldsymbol{Q}: \phi(r) \leqq n\}: n < \omega\}$, there is an $m < \omega$ so that the Euclidean closure $\mathrm{cl}_{\boldsymbol{R}}\, \{r \in \boldsymbol{R}\backslash\boldsymbol{Q}: \phi(r) \leqq m\}$ contains a Euclidean open set $U$. Choose a $p \in \boldsymbol{Q} \cap U$. As $B(m, p) \cap \boldsymbol{R}$ is a Euclidean neighborhood of $p$ in $R$, there is an $r \in \boldsymbol{R}\backslash\boldsymbol{Q} \cap B(m, p)$ so that $\phi(r) \leqq m$, that is $r \notin \bigcup\{B(m, q): q \in \boldsymbol{Q}\}$; this contradiction completes the proof.

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UNIVERSITY OF TEXAS
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