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PRIME IDEALS IN GAMMA RINGS

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The notion of a Γ -ring was first introduced by Nobusawa. The class of Γ -rings contains not only all rings but also Hestenes ternary rings. Recently, the author proved the following two theorems: **THEOREM A.** Let M be a Γ -ring with right and left unities and R be the right operator ring. Then, the lattice of two-sided ideals of M is isomorphic to the lattice of two-sided ideals of R . **THEOREM B.** Let M be a Γ -ring such that $x \in M\Gamma x\Gamma M$ for every $x \in M$. If $\mathcal{S}(M)$ is the prime radical of the Γ -ring M , then $\mathcal{S}(M_{m,n}) = (\mathcal{S}(M))_{m,n}$. If a Γ -ring M has no unit elements, Theorem A is not, in general, the case. However, it is possible to establish for any Γ -ring M , with or without right and left unities, the result corresponding to Theorem A for a special type of ideals, namely, prime ideals. In this note, we prove Theorem 1. The set of all prime ideals of a Γ -ring M and the set of all prime ideals of the right (left) operator ring $R(L)$ of M are bijective. Applying this result to the matrix $\Gamma_{n,m}$ -ring $M_{m,n}$, we obtain Theorem 2. The prime ideals of the $\Gamma_{n,m}$ -ring $M_{m,n}$ are the sets $P_{m,n}$ corresponding to the prime ideals P of the Γ -ring M , and Corollary 2. If $\mathcal{S}(M)$ is the prime radical of the Γ -ring M , then $\mathcal{S}(M_{m,n}) = (\mathcal{S}(M))_{m,n}$. This corollary omits the assumption of Theorem B.

1. Preliminaries. Let M and Γ be additive abelian groups. If for $a, b, c \in M$ and $\gamma, \delta \in \Gamma$ the following conditions are satisfied,

$$(1) \quad a\gamma b \in M,$$

$$(2) \quad (a + b)\gamma c = a\gamma c + b\gamma c, \quad a(\gamma + \delta)b = a\gamma b + a\delta b, \quad a\gamma(b + c) = a\gamma b + a\gamma c,$$

$$(3) \quad (a\gamma b)\delta c = a\gamma(b\delta c),$$

then M is called a Γ -ring. If A and B are subsets of a Γ -ring M and $\theta \subseteq \Gamma$, we denote by $A\theta B$, the subset of M consisting of all finite sums of the form $\sum_i a_i \gamma_i b_i$, where $a_i \in A$, $b_i \in B$ and $\gamma_i \in \theta$. A right (left) ideal of a Γ -ring M is an additive subgroup I of M such that $I\Gamma M \subseteq I$ ($M\Gamma I \subseteq I$). If I is both a right and a left ideal, then we say that I is an ideal or a two-sided ideal of M . An ideal P of a Γ -ring M is prime if for any ideals $A, B \subseteq M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. The prime radical $\mathcal{S}(M)$ is defined to be the intersection of all prime ideals of M .

Let M be a Γ -ring and F be the free abelian group generated by $\Gamma \times M$. Then, $A = \{\sum_i n_i(\gamma_i, x_i) \in F \mid a \in M \Rightarrow \sum_i n_i a \gamma_i x_i = 0\}$ is a subgroup of F . Let $R = F/A$, the factor group, and denote the

coset $(\gamma, x) + A$ by $[\gamma, x]$. Clearly, every element of R can be expressed as a finite sum $\sum_i [\gamma_i, x_i]$. Also, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, $[x, \alpha] + [x, \beta] = [x, \alpha + \beta]$ and $[x, \alpha] + [y, \alpha] = [x + y, \alpha]$. We define a multiplication in R by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i \beta_j y_j].$$

Then, R forms a ring. If we define a composition on $M \times R$ into M by $a \sum_i [\alpha_i, x_i] = \sum_i \alpha \alpha_i x_i$ for $a \in M$, $\sum_i [\alpha_i, x_i] \in R$, then M is a right R -module, and we call R the right operator ring of the Γ -ring M .

For the subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i [\gamma_i, x_i]$ in R , where $\gamma_i \in \Phi$, $x_i \in N$. Thus, in particular, $R = [\Gamma, M]$.

For a subset $Q \subseteq R$ we define $Q^* = \{a \in M \mid [\Gamma, a] = [\Gamma, \{a\}] \subseteq Q\}$. It follows that if Q is an ideal of R , then Q^* is an ideal of M . For a subset $P \subseteq M$, we define $P^{*'} = \{r \in R \mid Mr \subseteq P\}$. It follows that if P is an ideal of M , then $P^{*'}$ is an ideal of R , and $[\Gamma, P]$ is also an ideal of R .

Similarly, we can define the left operator ring L of M . For $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[N, \Phi]$, the set of all finite sums $\sum_i [x_i, \alpha_i]$ in L with $x_i \in N$ and $\alpha_i \in \Phi$. In particular, $L = [M, \Gamma]$.

For a subset $S \subseteq L$ we define $S^+ = \{a \in M \mid [a, \Gamma] = [\{a\}, \Gamma] \subseteq S\}$. If S is an ideal of L , then S^+ is an ideal of M . For $P \subseteq M$, we define $P^{+'} = \{l \in L \mid lM \subseteq P\}$. If P is an ideal of M , then $P^{+'}$ is an ideal of L , and $[P, \Gamma]$ is also an ideal of L .

Let a Γ -ring M be given. If $M_{m,n}$ (resp. $\Gamma_{n,m}$) is the additive abelian group of all m by n (resp. n by m) matrices over M (resp. Γ), $M_{m,n}$ forms a $\Gamma_{n,m}$ -ring. Denote the right operator ring of $M_{m,n}$ by $[\Gamma_{n,m}, M_{m,n}]$. Suppose R_n be the ring of all n by n matrices over the right operator ring R of M . Then, by the straightforward calculation on matrices one can verify that the right operator ring $[\Gamma_{n,m}, M_{m,n}]$ and the matrix ring R_n are isomorphic via the mapping

$$\phi: \sum_i [(\gamma_{jk}^{(i)}), (x_{uv}^{(i)})] \longmapsto \sum_i \left(\sum_{t=1}^m [\gamma_{jt}^{(i)}, x_{tv}^{(i)}] \right).$$

Similarly, the left operator ring $[M_{m,n}, \Gamma_{n,m}]$ of $M_{m,n}$ is isomorphic to the matrix ring L_m over the left operator ring L of M . Therefore, it may be considered that the right operator ring of the $\Gamma_{n,m}$ -ring $M_{m,n}$ is R_n and the left one L_m .

2. Prime ideals in gamma rings.

LEMMA 1. *Let P, Q and S be a prime ideal of a Γ -ring M , a*

prime ideal of the right operator ring R and a prime ideal of the left operator ring L respectively. Then, $P^{*'}$ is a prime ideal of R , $P^{+'}$ is a prime ideal of L , Q^* and S^+ are prime ideals of M .

Proof. Let U, V be ideals of R such that $UV \subseteq P^{*'}$, where $P^{*'} = \{r \in R \mid Mr \subseteq P\}$. Since $U(V)$ is an ideal, $U\Gamma MV = URV \subseteq UV$, and then $U\Gamma MV \subseteq P^{*'}$. Thus, $MU\Gamma MV \subseteq P$, but since P is prime, it follows that $MU \subseteq P$ or $MV \subseteq P$. Hence, $U \subseteq P^{*'}$ or $V \subseteq P^{*'}$, which proves $P^{*'}$ is prime.

Similarly, it can be verified that $P^{+'}$ is a prime ideal of L .

Let A, B be ideals of M such that $A\Gamma B \subseteq Q^*$, where $Q^* = \{x \in M \mid [\Gamma, x] \subseteq Q\}$. Then, $[\Gamma, A][\Gamma, B] = [\Gamma, A\Gamma B] \subseteq Q$, where $[\Gamma, A], [\Gamma, B]$ are ideals of R . Since Q is prime, $[\Gamma, A] \subseteq Q$ or $[\Gamma, B] \subseteq Q$, which means $A \subseteq Q^*$ or $B \subseteq Q^*$. This proves Q^* is prime.

Similarly, it can be verified that S^+ is prime.

We now prove the analogous result to Theorem 2 in [2].

THEOREM 1. *The sets of all prime ideals of a Γ -ring M and its right (left) operator ring $R(L)$ are bijective via the mapping $P \mapsto P^{*'}(P \mapsto P^{+'})$, where P denotes a prime ideal of M .*

Proof. Let P be a prime ideal of M . By the definitions of $^{*'}$ and $^{+'}$ we have

$$(P^{*'})^* = \{x \in M \mid [\Gamma, x] \subseteq P^{*'}\} = \{x \in M \mid M\Gamma x \subseteq P\}.$$

Since P is an ideal of M $M\Gamma P \subseteq P$, and then $P \subseteq (P^{*'})^*$. On the other hand, since $M\Gamma(P^{*'})^* \subseteq P$ and P is prime, $M \subseteq P$ or $(P^{*'})^* \subseteq P$. Then, in either case, $(P^{*'})^* \subseteq P$. Therefore, $(P^{*'})^* = P$.

Let Q be a prime ideal of R . Then we have

$$(Q^*)^{*'} = \{r \in R \mid Mr \subseteq Q^*\} = \{r \in R \mid [\Gamma, Mr] \subseteq Q\}.$$

Since Q is an ideal of R $[\Gamma, M]Q \subseteq Q$, and then $Q \subseteq (Q^*)^{*'}$. But, since $[\Gamma, M](Q^*)^{*' } = R(Q^*)^{*' } \subseteq Q$ and Q is prime, $(Q^*)^{*' } \subseteq Q$. Hence, $(Q^*)^{*' } = Q$. This proves that the sets of all prime ideals of M and R are bijective.

Similarly, it can be verified that $(P^{+'})^+ = P$ and $(S^+)^{+' } = S$, where S is a prime ideal of L . Thus, the sets of all prime ideals of M and L are bijective.

COROLLARY 1. *Let R and L be the right operator ring and the left one of a Γ -ring M respectively. Then, the sets of all prime ideals of R and L are bijective via the mapping $Q \mapsto (Q^*)^{+'}$, where Q is a prime ideal of R .*

Proof. Let Q and S be prime ideals of R and L respectively. Then, $(Q^*)^{+'}$ is a prime ideal of L . Set $(Q^*)^{+'} = T$. By Theorem 1, we have $(T^+)^{*'} = Q$, that is, $((Q^*)^{+'})^{+'*'} = Q$. Similarly, we have $((S^+)^{*'}*)^{+'} = S$.

3. Prime ideals in matrix gamma rings. We note that Lemma 1 and Theorem 1 hold also for the matrix $\Gamma_{n,m}$ -ring $M_{m,n}$.

For any ring R with or without an unit element, Sand proved the following fact.

LEMMA 2 ([4] Theorem 1). *The prime ideals of R_n are the sets A_n corresponding to prime ideals A of R .*

We prepare the following lemma.

LEMMA 3. *Let Q be a subset of the right operator ring R of a Γ -ring M . Then, $(Q_n)^* = (Q^*)_{m,n}$.*

Proof. Recall $(Q_n)^* = \{(x_{ij}) \in M_{m,n} \mid [I_{n,m}, (x_{ij})] \subseteq Q_n\}$ and $Q^* = \{x \in M \mid [I, x] \subseteq Q\}$.

For any $\sum_{k=1}^q [(\gamma_{ij}^{(k)}), (x_{uv}^{(k)})] \in [I_{n,m}, (Q^*)_{m,n}]$, where $(\gamma_{ij}^{(k)}) \in \Gamma_{n,m}$ and $(x_{uv}^{(k)}) \in (Q^*)_{m,n}$, $1 \leq k \leq q$, we have

$$\sum_{k=1}^q [(\gamma_{ij}^{(k)}), (x_{uv}^{(k)})] = \sum_{k=1}^q \left(\sum_{t=1}^m [(\gamma_{it}^{(k)}), (x_{tv}^{(k)})] \right) \in ([I, Q^*])_n \subseteq Q_n.$$

This means that $[I_{n,m}, (Q^*)_{m,n}] \subseteq Q_n$, which proves $(Q^*)_{m,n} \subseteq (Q_n)^*$. Conversely, for any $(x_{uv}) \in (Q_n)^*$, we have $[I_{n,m}, (x_{uv})] \subseteq Q_n$. For any $\gamma \in \Gamma$, $[(\gamma)^{1,u}, (x_{uv})]$ is a matrix of $[I_{n,m}, (x_{uv})]$ which has the element $[\gamma, x_{uv}]$ as its $(1, v)$ th component, where $(\gamma)^{1,u}$ denotes the matrix which has γ in the first row and u th column and zero elsewhere. Hence, $[\gamma, x_{uv}] \in Q$. This is true for each element $\gamma \in \Gamma$; hence $[I, x_{uv}] \subseteq Q$, and then $x_{uv} \in Q^*$. Hence, $(x_{uv}) \in (Q^*)_{m,n}$, which proves $(Q_n)^* \subseteq (Q^*)_{m,n}$. Therefore, $(Q_n)^* = (Q^*)_{m,n}$.

THEOREM 2. *The prime ideals of the $\Gamma_{n,m}$ -ring $M_{m,n}$ are the sets $P_{m,n}$ corresponding to the prime ideals P of the Γ -ring M .*

Proof. Let A be a prime ideal of $M_{m,n}$. Apply Theorem 1 to the $\Gamma_{n,m}$ -ring $M_{m,n}$. Then,

$$\begin{aligned} A &= (A^{*'})^* && (A^{*'} \text{ is a prime ideal of } R_n) \\ &= (Q_n)^* && (\text{by Lemma 2, } A^{*'} = Q_n, \text{ where } Q \text{ is a} \\ &&& \text{prime ideal of } R) \end{aligned}$$

$$\begin{aligned}
&= (Q^*)_{m,n} \quad (\text{by Lemma 3}) \\
&= P_{m,n} \quad (Q^* = P, \text{ and by Lemma 1 } P \text{ is a prime} \\
&\quad \text{ideal of } M).
\end{aligned}$$

Conversely, let P be a prime ideal of M . By Theorem 1, $P = (P^{*'})^*$, where $P^{*'}$ is a prime ideal of R . Set $P^{*' } = Q$. Lemma 2 implies Q_n is a prime ideal of R_n . Then Lemma 1 yields $(Q_n)^*$ is a prime ideal of $M_{m,n}$. By Lemma 3, $(Q_n)^* = (Q^*)_{m,n} = ((P^{*' })^*)_{m,n} = P_{m,n}$. Hence, $P_{m,n}$ is a prime ideal of $M_{m,n}$. This proves the theorem.

COROLLARY 2. *If $\mathcal{P}(M)$ is the prime radical of the Γ -ring M , then $\mathcal{P}(M_{m,n}) = (\mathcal{P}(M))_{m,n}$.*

Proof. If $\{P_i \mid i \in \mathfrak{A}\}$ is the set of all prime ideals in M , Theorem 2 implies $\mathcal{P}(M_{m,n}) = \bigcap_{i \in \mathfrak{A}} (P_i)_{m,n} = (\bigcap_{i \in \mathfrak{A}} P_i)_{m,n} = (\mathcal{P}(M))_{m,n}$.

Corollary 2 omits the assumption of Theorem 8 in [1].

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