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The notion of a Γ-ring was first introduced by Nobusawa. The class of Γ-rings contains not only all rings but also Hestenes ternary rings. Recently, the author proved the following two theorems: THEOREM A. Let M be a Γ-ring with right and left unities and R be the right operator ring. Then, the lattice of two-sided ideals of M is isomorphic to the lattice of two-sided ideals of R. THEOREM B. Let M be a Γ-ring such that $x \in M\Gamma x\Gamma M$ for every $x \in M$. If $\mathscr{P}(M)$ is the prime radical of the Γ-ring M, then $\mathscr{P}(M_{m,n}) = (\mathscr{P}(M))_{m,n}$. If a Γ-ring M has no unit elements, Theorem A is not, in general, the case. However, it is possible to establish for any Γ-ring M, with or without right and left unities, the result corresponding to Theorem A for a special type of ideals, namely, prime ideals. In this note, we prove Theorem 1. The set of all prime ideals of a Γ-ring M and the set of all prime ideals of the right (left) operator ring $R(L)$ of M are bijective. Applying this result to the matrix $\Gamma_{n,m}$-ring $M_{m,n}$, we obtain Theorem 2. The prime ideals of the $\Gamma_{n,m}$-ring $M_{m,n}$ are the sets $P_{m,n}$ corresponding to the prime ideals P of the Γ-ring M, and Corollary 2. If $\mathscr{P}(M)$ is the prime radical of the Γ-ring M, then $\mathscr{P}(M_{m,n}) = (\mathscr{P}(M))_{m,n}$. This corollary omits the assumption of Theorem B.

1. Preliminaries. Let M and Γ be additive abelian groups. If for $a, b, c \in M$ and $\gamma, \delta \in \Gamma$ the following conditions are satisfied,

(1) $a\gamma b \in M$,

(2) $(a + b)\gamma c = a\gamma c + b\gamma c$, $a(\gamma + \delta)b = a\gamma b + a\delta b$, $a\gamma(b + c) = a\gamma b + a\gamma c$,

(3) $(a\gamma b)\delta c = a\gamma(b\delta c)$,

then M is called a Γ-ring. If A and B are subsets of a Γ-ring M and $\Theta \subseteq \Gamma$, we denote by $A\Theta B$, the subset of M consisting of all finite sums of the form $\sum_i a_i\gamma_i b_i$, where $a_i \in A$, $b_i \in B$ and $\gamma_i \in \Theta$. A right (left) ideal of a Γ-ring M is an additive subgroup I of M such that $I\Gamma M \subseteq I (M\Gamma I \subseteq I)$. If I is both a right and a left ideal, then we say that I is an ideal or a two-sided ideal of M. An ideal P of a Γ-ring M is prime if for any ideals $A, B \subseteq M$, $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. The prime radical $\mathscr{P}(M)$ is defined to be the intersection of all prime ideals of M.

Let M be a Γ-ring and F be the free abelian group generated by $\Gamma \times M$. Then, $A = \{\sum_i n_i(\gamma_i, x_i) \in F \mid a \in M \Rightarrow \sum_i n_i a\gamma_i x_i = 0\}$ is a subgroup of F. Let $R = F/A$, the factor group, and denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. Clearly, every element of R can be expressed as a finite sum $\sum_i [\gamma_i, x_i]$. Also, for all $x, y \in M$ and $\alpha, \beta \in \Gamma$, $[x, \alpha] + [x, \beta] = [x, \alpha + \beta]$ and $[x, \alpha] + [y, \alpha] = [x + y, \alpha]$. We define a multiplication in R by

$$\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i\beta_j y_j] .$$

Then, R forms a ring. If we define a composition on $M \times R$ into M by $a \sum_i [\alpha_i, x_i] = \sum_i a\alpha_i x_i$ for $a \in M$, $\sum_i [\alpha_i, x_i] \in R$, then M is a right R-module, and we call R the right operator ring of the Γ-ring M.

For the subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\sum_i [\gamma_i, x_i]$ in R, where $\gamma_i \in \Phi$, $x_i \in N$. Thus, in particular, $R = [\Gamma, M]$.

For a subset $Q \subseteq R$ we define $Q^* = \{a \in M | [\Gamma, a] = [\Gamma, \{a\}] \subseteq Q\}$. It follows that if Q is an ideal of R, then Q^* is an ideal of M. For a subset $P \subseteq M$, we define $P^{*\prime} = \{r \in R | Mr \subseteq P\}$. It follows that if P is an ideal of M, then $P^{*\prime}$ is an ideal of R, and $[\Gamma, P]$ is also an ideal of R.

Similarly, we can define the left operator ring L of M. For $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[N, \Phi]$, the set of all finite sums $\sum_i [x_i, \alpha_i]$ in L with $x_i \in N$ and $\alpha_i \in \Phi$. In particular, $L = [M, \Gamma]$.

For a subset $S \subseteq L$ we define $S^+ = \{a \in M | [a, \Gamma] = [\{a\}, \Gamma] \subseteq S\}$. If S is an ideal of L, then S^+ is an ideal of M. For $P \subseteq M$, we define $P^{+\prime} = \{l \in L | lM \subseteq P\}$. If P is an ideal of M, then $P^{+\prime}$ is an ideal of L, and $[P, \Gamma]$ is also an ideal of L.

Let a Γ-ring M be given. If $M_{m,n}$ (resp. $\Gamma_{n,m}$) is the additive abelian group of all m by n (resp. n by m) matrices over M (resp. Γ), $M_{m,n}$ forms a $\Gamma_{n,m}$-ring. Denote the right operator ring of $M_{m,n}$ by $[\Gamma_{n,m}, M_{m,n}]$. Suppose R_n be the ring of all n by n matrices over the right operator ring R of M. Then, by the straightforward calculation on matrices one can verify that the right operator ring $[\Gamma_{n,m}, M_{m,n}]$ and the matrix ring R_n are isomorphic via the mapping

$$\phi\colon \sum_i [(\gamma_{jk}^{(i)}),\ (x_{uv}^{(i)})] \longmapsto \sum_i \Big(\sum_{t=1}^{m} [\gamma_{jt}^{(i)}, x_{tv}^{(i)}] \Big) .$$

Similarly, the left operator ring $[M_{m,n}, \Gamma_{n,m}]$ of $M_{m,n}$ is isomorphic to the matrix ring L_m over the left operator ring L of M. Therefore, it may be considered that the right operator ring of the $\Gamma_{n,m}$-ring $M_{m,n}$ is R_n and the left one L_m.

2. Prime ideals in gamma rings.

LEMMA 1. *Let P, Q and S be a prime ideal of a Γ-ring M, a* *prime ideal of the right operator ring R and a prime ideal of the left operator ring L respectively. Then, $P^{*\prime}$ is a prime ideal of R, $P^{+\prime}$ is a prime ideal of L, Q^* and S^+ are prime ideals of M.*

Proof. Let U, V be ideals of R such that $UV \subseteqq P^{*\prime}$, where $P^{*\prime} = \{r \in R \mid Mr \subseteqq P\}$. Since $U(V)$ is an ideal, $U\Gamma MV = URV \subseteqq UV$, and then $U\Gamma MV \subseteqq P^{*\prime}$. Thus, $MU\Gamma MV \subseteqq P$, but since P is prime, it follows that $MU \subseteqq P$ or $MV \subseteqq P$. Hence, $U \subseteqq P^{*\prime}$ or $V \subseteqq P^{*\prime}$, which proves $P^{*\prime}$ is prime.

Similarly, it can be verified that $P^{+\prime}$ is a prime ideal of L.

Let A, B be ideals of M such that $A\Gamma B \subseteqq Q^*$, where $Q^* = \{x \in M \mid [\Gamma, x] \subseteqq Q\}$. Then, $[\Gamma, A][\Gamma, B] = [\Gamma, A\Gamma B] \subseteqq Q$, where $[\Gamma, A]$, $[\Gamma, B]$ are ideals of R. Since Q is prime, $[\Gamma, A] \subseteqq Q$ or $[\Gamma, B] \subseteqq Q$, which means $A \subseteqq Q^*$ or $B \subseteqq Q^*$. This proves Q^* is prime.

Similarly, it can be verified that S^+ is prime.

We now prove the analogous result to Theorem 2 in [2].

THEOREM 1. *The sets of all prime ideals of a Γ-ring M and its right (left) operator ring $R(L)$ are bijective via the mapping $P \mapsto P^{*\prime}(P \mapsto P^{+\prime})$, where P denotes a prime ideal of M.*

Proof. Let P be a prime ideal of M. By the definitions of ${}^{*\prime}$ and * we have

$$(P^{*\prime})^* = \{x \in M \mid [\Gamma, x] \subseteqq P^{*\prime}\} = \{x \in M \mid M\Gamma x \subseteqq P\} .$$

Since P is an ideal of M $M\Gamma P \subseteqq P$, and then $P \subseteqq (P^{*\prime})^*$. On the other hand, since $M\Gamma(P^{*\prime})^* \subseteqq P$ and P is prime, $M \subseteqq P$ or $(P^{*\prime})^* \subseteqq P$. Then, in either case, $(P^{*\prime})^* \subseteqq P$. Therefore, $(P^{*\prime})^* = P$.

Let Q be a prime ideal of R. Then we have

$$(Q^*)^{*\prime} = \{r \in R \mid Mr \subseteqq Q^*\} = \{r \in R \mid [\Gamma, Mr] \subseteqq Q\} .$$

Since Q is an ideal of R $[\Gamma, M]Q \subseteqq Q$, and then $Q \subseteqq (Q^*)^{*\prime}$. But, since $[\Gamma, M](Q^*)^{*\prime} = R(Q^*)^{*\prime} \subseteqq Q$ and Q is prime, $(Q^*)^{*\prime} \subseteqq Q$. Hence, $(Q^*)^{*\prime} = Q$. This proves that the sets of all prime ideals of M and R are bijective.

Similarly, it can be verified that $(P^{+\prime})^+ = P$ and $(S^+)^{+\prime} = S$, where S is a prime ideal of L. Thus, the sets of all prime ideals of M and L are bijective.

COROLLARY 1. *Let R and L be the right operator ring and the left one of a Γ-ring M respectively. Then, the sets of all prime ideals of R and L are bijective via the mapping $Q \mapsto (Q^*)^{+\prime}$, where Q is a prime ideal of R.*

Proof. Let Q and S be prime ideals of R and L respectively. Then, $(Q^*)^{+\prime}$ is a prime ideal of L. Set $(Q^*)^{+\prime} = T$. By Theorem 1, we have $(T^+)^{*\prime} = Q$, that is, $(((Q^*)^{+\prime})^+)^{*\prime} = Q$. Similarly, we have $(((S^+)^{*\prime})^*)^{+\prime} = S$.

3. Prime ideals in matrix gamma rings. We note that Lemma 1 and Theorem 1 hold also for the matrix $\Gamma_{n,m}$-ring $M_{m,n}$.

For any ring R with or without an unit element, Sand proved the following fact.

LEMMA 2 ([4] *Theorem* 1). *The prime ideals of R_n are the sets A_n corresponding to prime ideals A of R.*

We prepare the following lemma.

LEMMA 3. *Let Q be a subset of the right operator ring R of a Γ-ring M. Then, $(Q_n)^* = (Q^*)_{m,n}$.*

Proof. Recall $(Q_n)^* = \{(x_{ij}) \in M_{m,n} \mid [\Gamma_{n,m}, (x_{ij})] \subseteqq Q_n\}$ and $Q^* = \{x \in M \mid [\Gamma, x] \subseteqq Q\}$.

For any $\sum_{k=1}^{q}[(\gamma_{ij}^{(k)}), (x_{uv}^{(k)})] \in [\Gamma_{n,m}, (Q^*)_{m,n}]$, where $(\gamma_{ij}^{(k)}) \in \Gamma_{n,m}$ and $(x_{uv}^{(k)}) \in (Q^*)_{m,n}$, $1 \leqq k \leqq q$, we have

$$\sum_{k=1}^{q} [(\gamma_{ij}^{(k)}),\ (x_{uv}^{(k)})] = \sum_{k=1}^{q} \left(\sum_{t=1}^{m} [\gamma_{it}^{(k)},\ x_{tv}^{(k)}] \right) \in ([\Gamma,\ Q^*])_n \subseteqq Q_n\ .$$

This means that $[\Gamma_{n,m}, (Q^*)_{m,n}] \subseteqq Q_n$, which proves $(Q^*)_{m,n} \subseteqq (Q_n)^*$. Conversely, for any $(x_{uv}) \in (Q_n)^*$, we have $[\Gamma_{n,m}, (x_{uv})] \subseteqq Q_n$. For any $\gamma \in \Gamma$, $[(\gamma)^{1,u}, (x_{uv})]$ is a matrix of $[\Gamma_{n,m}, (x_{uv})]$ which has the element $[\gamma, x_{uv}]$ as its $(1, v)$th component, where $(\gamma)^{1,u}$ denotes the matrix which has γ in the first row and uth column and zero elsewhere. Hence, $[\gamma, x_{uv}] \in Q$. This is true for each element $\gamma \in \Gamma$; hence $[\Gamma, x_{uv}] \subseteqq Q$, and then $x_{uv} \in Q^*$. Hence, $(x_{uv}) \in (Q^*)_{m,n}$, which proves $(Q_n)^* \subseteqq (Q^*)_{m,n}$. Therefore, $(Q_n)^* = (Q^*)_{m,n}$.

THEOREM 2. *The prime ideals of the $\Gamma_{n,m}$-ring $M_{m,n}$ are the sets $P_{m,n}$ corresponding to the prime ideals P of the Γ-ring M.*

Proof. Let A be a prime ideal of $M_{m,n}$. Apply Theorem 1 to the $\Gamma_{n,m}$-ring $M_{m,n}$. Then,

$$\begin{aligned} A &= (A^{*\prime})^* && (A^{*\prime} \text{ is a prime ideal of } R_n) \\ &= (Q_n)^* && (\text{by Lemma 2, } A^{*\prime} = Q_n, \text{ where } Q \text{ is a prime ideal of } R) \end{aligned}$$

$$\begin{aligned} &= (Q^*)_{m,n} && \text{(by Lemma 3)} \\ &= P_{m,n} && (Q^* = P, \text{ and by Lemma 1 } P \text{ is a prime ideal of } M). \end{aligned}$$

Conversely, let P be a prime ideal of M. By Theorem 1, $P=(P^{*\prime})^*$, where $P^{*\prime}$ is a prime ideal of R. Set $P^{*\prime} = Q$. Lemma 2 implies Q_n is a prime ideal of R_n. Then Lemma 1 yields $(Q_n)^*$ is a prime ideal of $M_{m,n}$. By Lemma 3, $(Q_n)^* = (Q^*)_{m,n} = ((P^{*\prime})^*)_{m,n} = P_{m,n}$. Hence, $P_{m,n}$ is a prime ideal of $M_{m,n}$. This proves the theorem.

COROLLARY 2. *If $\mathscr{P}(M)$ is the prime radical of the Γ-ring M, then $\mathscr{P}(M_{m,n}) = (\mathscr{P}(M))_{m,n}$.*

Proof. If $\{P_i \mid i \in \mathfrak{A}\}$ is the set of all prime ideals in M, Theorem 2 implies $\mathscr{P}(M_{m,n}) = \bigcap_{i \in \mathfrak{A}} (P_i)_{m,n} = (\bigcap_{i \in \mathfrak{A}} P_i)_{m,n} = (\mathscr{P}(M))_{m,n}$.

Corollary 2 omits the assumption of Theorem 8 in [1].

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