PRIME IDEALS IN GAMMA RINGS

Shoji Kyuno
The notion of a Γ-ring was first introduced by Nobusawa. The class of Γ-rings contains not only all rings but also Hestenes ternary rings. Recently, the author proved the following two theorems: Theorem A. Let $M$ be a Γ-ring with right and left unities and $R$ be the right operator ring. Then, the lattice of two-sided ideals of $M$ is isomorphic to the lattice of two-sided ideals of $R$. Theorem B. Let $M$ be a Γ-ring such that $x \in M \Gamma x \Gamma M$ for every $x \in M$. If $\mathcal{R}(M)$ is the prime radical of the Γ-ring $M$, then $\mathcal{R}(M_{m,n}) = (\mathcal{R}(M))_{m,n}$. If a Γ-ring $M$ has no unit elements, Theorem A is not, in general, the case. However, it is possible to establish for any Γ-ring $M$, with or without right and left unities, the result corresponding to Theorem A for a special type of ideals, namely, prime ideals. In this note, we prove Theorem 1. The set of all prime ideals of a Γ-ring $M$ and the set of all prime ideals of the right (left) operator ring $R(L)$ of $M$ are bijective. Applying this result to the matrix $\Gamma_{n,m}$-ring $M_{m,n}$, we obtain Theorem 2. The prime ideals of the $\Gamma_{n,m}$-ring $M_{m,n}$ are the sets $P_{m,n}$ corresponding to the prime ideals $P$ of the Γ-ring $M$, and Corollary 2. If $\mathcal{R}(M)$ is the prime radical of the Γ-ring $M$, then $\mathcal{R}(M_{m,n}) = (\mathcal{R}(M))_{m,n}$. This corollary omits the assumption of Theorem B.

1. Preliminaries. Let $M$ and $\Gamma$ be additive abelian groups. If for $a, b, c \in M$ and $\gamma, \delta \in \Gamma$ the following conditions are satisfied,

1. $a \gamma b \in M$,
2. $(a + b) \gamma c = a \gamma c + b \gamma c$, $a(\gamma + \delta)b = a \gamma b + a \delta b$, $a \gamma (b + c) = a \gamma b + a \gamma c$,
3. $(a \gamma b) \delta c = a \gamma (b \delta c)$,

then $M$ is called a Γ-ring. If $A$ and $B$ are subsets of a Γ-ring $M$ and $\theta \subseteq \Gamma$, we denote by $A \theta B$, the subset of $M$ consisting of all finite sums of the form $\sum a_i \gamma_i b_i$, where $a_i \in A$, $b_i \in B$ and $\gamma_i \in \theta$. A right (left) ideal of a Γ-ring $M$ is an additive subgroup $I$ of $M$ such that $I \Gamma M \subseteq I(M \Gamma I \subseteq I)$. If $I$ is both a right and a left ideal, then we say that $I$ is an ideal or a two-sided ideal of $M$. An ideal $P$ of a Γ-ring $M$ is prime if for any ideals $A, B \subseteq M$, $A \Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. The prime radical $\mathcal{P}(M)$ is defined to be the intersection of all prime ideals of $M$.

Let $M$ be a Γ-ring and $F$ be the free abelian group generated by $\Gamma \times M$. Then, $A = \{ \sum_i n_i(\gamma_i, x_i) \in F | a \in M \Rightarrow \sum_i n_i a \gamma_i x_i = 0 \}$ is a subgroup of $F$. Let $R = F/A$, the factor group, and denote the
coset \((\gamma, x) + A\) by \([\gamma, x]\). Clearly, every element of \(R\) can be expressed as a finite sum \(\sum_i [\gamma_i, x_i]\). Also, for all \(x, y \in M\) and \(\alpha, \beta \in \Gamma\), \([x, \alpha] + [x, \beta] = [x, \alpha + \beta]\) and \([x, \alpha] + [y, \alpha] = [x + y, \alpha]\). We define a multiplication in \(R\) by

\[
\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i \beta_j y_j].
\]

Then, \(R\) forms a ring. If we define a composition on \(M \times R\) into \(M\) by \(a \sum_i [\alpha_i, x_i] = \sum_i a \alpha_i x_i\) for \(a \in M\), \(\sum_i [\alpha_i, x_i] \in R\), then \(M\) is a right \(R\)-module, and we call \(R\) the right operator ring of the \(\Gamma\)-ring \(M\).

For the subsets \(N \subseteq M\), \(\Phi \subseteq \Gamma\), we denote by \([\Phi, N]\) the set of all finite sums \(\sum_i [\gamma_i, x_i]\) in \(R\), where \(\gamma_i \in \Phi\), \(x_i \in N\). Thus, in particular, \(R = [\Gamma, M]\).

For a subset \(Q \subseteq R\) we define \(Q^* = \{a \in M | [\gamma, a] = [\gamma, \{a\}] \subseteq Q\}\). It follows that if \(Q\) is an ideal of \(R\), then \(Q^*\) is an ideal of \(M\). For a subset \(P \subseteq M\), we define \(P^{*\prime} = \{r \in R | Mr \subseteq P\}\). It follows that if \(P\) is an ideal of \(M\), then \(P^{*\prime}\) is an ideal of \(R\), and \([\Gamma, P]\) is also an ideal of \(R\).

Similarly, we can define the left operator ring \(L\) of \(M\). For \(N \subseteq M\), \(\Phi \subseteq \Gamma\), we denote by \([N, \Phi]\), the set of all finite sums \(\sum_i [x_i, \alpha_i]\) in \(L\) with \(x_i \in N\) and \(\alpha_i \in \Phi\). In particular, \(L = [M, \Gamma]\).

For a subset \(S \subseteq L\) we define \(S^+ = \{a \in M | [a, \Gamma] = [(a), \Gamma] \subseteq S\}\). If \(S\) is an ideal of \(L\), then \(S^+\) is an ideal of \(M\). For \(P \subseteq M\), we define \(P^{+\prime} = \{l \in L | LM \subseteq P\}\). If \(P\) is an ideal of \(M\), then \(P^{+\prime}\) is an ideal of \(L\), and \([P, \Gamma]\) is also an ideal of \(L\).

Let a \(\Gamma\)-ring \(M\) be given. If \(M_{m,n}\) (resp. \(\Gamma_{n,m}\)) is the additive abelian group of all \(m\) by \(n\) (resp. \(n\) by \(m\)) matrices over \(M\) (resp. \(\Gamma\)), \(M_{m,n}\) forms a \(\Gamma_{n,m}\)-ring. Denote the right operator ring of \(M_{m,n}\) by \([\Gamma_{n,m}, M_{m,n}]\). Suppose \(R_n\) be the ring of all \(n\) by \(n\) matrices over the right operator ring \(R\) of \(M\). Then, by the straightforward calculation on matrices one can verify that the right operator ring \([\Gamma_{n,m}, M_{m,n}]\) and the matrix ring \(R_n\) are isomorphic via the mapping

\[
\phi: \sum_i [(\gamma_i)_{jk}, \ (x_i)^{(j)}_{uv}] \longmapsto \sum_i \left( \sum_{i=1}^m [\gamma_i^{(j)}, x_i^{(j)}_{uv}] \right).
\]

Similarly, the left operator ring \([M_{m,n}, \Gamma_{n,m}]\) of \(M_{m,n}\) is isomorphic to the matrix ring \(L_n\) over the left operator ring \(L\) of \(M\). Therefore, it may be considered that the right operator ring of the \(\Gamma_{n,m}\)-ring \(M_{m,n}\) is \(R_n\) and the left one \(L_n\).

2. Prime ideals in gamma rings.

**Lemma 1.** Let \(P, Q\) and \(S\) be a prime ideal of a \(\Gamma\)-ring \(M\), a
prime ideal of the right operator ring \( R \) and a prime ideal of the left operator ring \( L \) respectively. Then, \( P^{**} \) is a prime ideal of \( R \), \( P^{*'} \) is a prime ideal of \( L \), \( Q^* \) and \( S^+ \) are prime ideals of \( M \).

**Proof.** Let \( U, V \) be ideals of \( R \) such that \( UV \subseteq P^{*'} \), where \( P^{*'} = \{ r \in R | Mr \subseteq P \} \). Since \( U(V) \) is an ideal, \( U \Gamma MV = URV \subseteq UV \), and then \( U \Gamma MV \subseteq P^{*'} \). Thus, \( M U \Gamma MV \subseteq P \), but since \( P \) is prime, it follows that \( M U \subseteq P \) or \( M V \subseteq P \). Hence, \( U \subseteq P^{**} \) or \( V \subseteq P^{*'} \), which proves \( P^{*'} \) is prime.

Similarly, it can be verified that \( P^{*'} \) is a prime ideal of \( L \).

Let \( A, B \) be ideals of \( M \) such that \( A \Gamma B \subseteq Q^* \), where \( Q^* = \{ x \in M | [\Gamma, x] \subseteq Q \} \). Then, \( [\Gamma, A][\Gamma, B] = [\Gamma, A \Gamma B] \subseteq Q \), where \( [\Gamma, A], [\Gamma, B] \) are ideals of \( R \). Since \( Q \) is prime, \( [\Gamma, A] \subseteq Q \) or \( [\Gamma, B] \subseteq Q \), which means \( A \subseteq Q^* \) or \( B \subseteq Q^* \). This proves \( Q^* \) is prime.

Similarly, it can be verified that \( S^+ \) is prime.

We now prove the analogous result to Theorem 2 in \([2]\).

**THEOREM 1.** The sets of all prime ideals of a \( \Gamma \)-ring \( M \) and its right (left) operator ring \( R (L) \) are bijective via the mapping \( P \mapsto P^{**}(P \mapsto P^{*'}) \), where \( P \) denotes a prime ideal of \( M \).

**Proof.** Let \( P \) be a prime ideal of \( M \). By the definitions of \( ^* \) and \( ^* \) we have

\[
(P^{**})^* = \{ x \in M | [\Gamma, x] \subseteq P^{*'} \} = \{ x \in M | M \Gamma x \subseteq P \}.
\]

Since \( P \) is an ideal of \( M \), \( M \Gamma P \subseteq P \), and then \( P \subseteq (P^{**})^* \). On the other hand, since \( M \Gamma (P^{**})^* \subseteq P \) and \( P \) is prime, \( M \subseteq P \) or \( (P^{**})^* \subseteq P \). Then, in either case, \( (P^{**})^* \subseteq P \). Therefore, \( (P^{**})^* = P \).

Let \( Q \) be a prime ideal of \( R \). Then we have

\[
(Q^*)^{*'} = \{ r \in R | Mr \subseteq Q^* \} = \{ r \in R | [\Gamma, M r] \subseteq Q \}.
\]

Since \( Q \) is an ideal of \( R \), \( [\Gamma, M]Q \subseteq Q \), and then \( Q \subseteq (Q^*)^{*'} \). But, since \( [\Gamma, M](Q^*)^{*'} = R(Q^*)^{*'} \subseteq Q \) and \( Q \) is prime, \( (Q^*)^{*'} \subseteq Q \). Hence, \( (Q^*)^{*'} = Q \). This proves that the sets of all prime ideals of \( M \) and \( R \) are bijective.

Similarly, it can be verified that \( (P^{*'})^+ = P \) and \( (S^+)^{*'} = S \), where \( S \) is a prime ideal of \( L \). Thus, the sets of all prime ideals of \( M \) and \( L \) are bijective.

**COROLLARY 1.** Let \( R \) and \( L \) be the right operator ring and the left one of a \( \Gamma \)-ring \( M \) respectively. Then, the sets of all prime ideals of \( R \) and \( L \) are bijective via the mapping \( Q \mapsto (Q^*)^{*'} \), where \( Q \) is a prime ideal of \( R \).
Proof. Let \( Q \) and \( S \) be prime ideals of \( R \) and \( L \) respectively. Then, \((Q^*)^{+}r\) is a prime ideal of \( L \). Set \((Q^*)^{+}T = T\). By Theorem 1, we have \((T^r)^{S'} = Q\), that is, \(((Q^*)^{+}r)^{+}s = Q\). Similarly, we have \(((S^*)^{+}r)^{+}s = S\).

3. Prime ideals in matrix gamma rings. We note that Lemma 1 and Theorem 1 hold also for the matrix \( \Gamma_{n,m} \)-ring \( M_{m,n} \).

For any ring \( R \) with or without a unit element, Sand proved the following fact.

**Lemma 2 ([4] Theorem 1).** The prime ideals of \( R_n \) are the sets \( A_m \) corresponding to prime ideals \( A \) of \( R \).

We prepare the following lemma.

**Lemma 3.** Let \( Q \) be a subset of the right operator ring \( R \) of a \( \Gamma \)-ring \( M \). Then, \((Q_n)^* = (Q^*)_{m,n} \).

**Proof.** Recall \((Q_n)^* = \{(x_{ij}) \in M_{m,n} | \{I_{n,m}, (x_{ij}) \} \subseteq Q_n \} \) and \( Q^* = \{x \in M | \{I, x \} \subseteq Q \} \).

For any \( \sum_{k=1}^{q} (\gamma_{ij}^{(k)}), (x_{uv}^{(k)}) \in I_{n,m}, (Q^*)_{m,n} \), where \( (\gamma_{ij}^{(k)}) \in I_{n,m} \) and \( (x_{uv}^{(k)}) \in (Q^*)_{m,n} \), \( 1 \leq k \leq q \), we have

\[ \sum_{k=1}^{q} [(\gamma_{ij}^{(k)}), (x_{uv}^{(k)})] = \sum_{k=1}^{q} \left( \sum_{l=1}^{m} [\gamma_{il}^{(k)}, x_{lv}^{(k)}] \right) \in \{I, Q^*\}_n \subseteq Q_n . \]

This means that \( I_{n,m}, (Q^*)_{m,n} \subseteq Q_n \), which proves \((Q^*)_{m,n} \subseteq (Q_n)^* \). Conversely, for any \( (x_{uv}) \in (Q_n)^* \), we have \([I_{n,m}, (x_{uv})] \subseteq Q_n \). For any \( \gamma \in I \), \([\gamma_{ij}^{(k)}], (x_{uv})\) is a matrix of \([I_{n,m}, (x_{uv})]\) which has the element \( \gamma_{ij} \) as its \( (1, v) \)th component, \( \gamma_{ij}^{(k)} \) denotes the matrix which has \( \gamma \) in the first row and \( u \)th column and zero elsewhere. Hence, \( [\gamma, x_{uv}] \in Q \). This is true for each element \( \gamma \in I \); hence \([I, x_{uv}] \subseteq Q \), and then \( x_{uv} \in Q^* \). Hence, \( (x_{uv}) \in (Q^*)_{m,n} \), which proves \((Q_n)^* \subseteq (Q^*)_{m,n} \). Therefore, \((Q_n)^* = (Q^*)_{m,n} \).

**Theorem 2.** The prime ideals of the \( \Gamma_{n,m} \)-ring \( M_{m,n} \) are the sets \( P_{m,n} \) corresponding to the prime ideals \( P \) of the \( \Gamma \)-ring \( M \).

**Proof.** Let \( A \) be a prime ideal of \( M_{m,n} \). Apply Theorem 1 to the \( \Gamma_{n,m} \)-ring \( M_{m,n} \). Then,

\[ A = (A^*)^{+} \quad (A^* \text{ is a prime ideal of } R_n) \]

\[ = (Q_n)^* \quad (\text{by Lemma 2, } A^{+r} = Q_n, \text{ where } Q \text{ is a prime ideal of } R) \]
\[= (Q^*)_{m,n} \quad \text{(by Lemma 3)}
\]
\[= P_{m,n} \quad (Q^* = P, \text{ and by Lemma 1 } P \text{ is a prime ideal of } M).\]

Conversely, let \(P\) be a prime ideal of \(M\). By Theorem 1, \(P = (P^{**})^*\), where \(P^{**}\) is a prime ideal of \(R\). Set \(P^{**} = Q\). Lemma 2 implies \(Q_n\) is a prime ideal of \(R_n\). Then Lemma 1 yields \((Q_n)^*\) is a prime ideal of \(M_{m,n}\). By Lemma 3, \((Q_n)^* = (Q^*)_{m,n} = ((P^{**})^*)_m = P_{m,n}\). Hence, \(P_{m,n}\) is a prime ideal of \(M_{m,n}\). This proves the theorem.

**Corollary 2.** If \(\mathcal{P}(M)\) is the prime radical of the \(\Gamma\)-ring \(M\), then \(\mathcal{P}(M_{m,n}) = (\mathcal{P}(M))_{m,n}\).

**Proof.** If \(\{P_i | i \in \mathfrak{A}\}\) is the set of all prime ideals in \(M\), Theorem 2 implies \(\mathcal{P}(M_{m,n}) = \bigcap_{i \in \mathfrak{A}} (P_i)_{m,n} = (\bigcap_{i \in \mathfrak{A}} P_i)_{m,n} = (\mathcal{P}(M))_{m,n}\).

Corollary 2 omits the assumption of Theorem 8 in [1].

**References**


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