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TOPOLOGICAL METHODS FOR C^*-ALGEBRAS II: GEOMETRIC RESOLUTIONS AND THE KÜNNETH FORMULA

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Let A and B be C^*-algebras with A in the smallest subcategory of the category of separable nuclear C^*-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by $\boldsymbol{Z}$ and by $\boldsymbol{R}$. Then there is a natural $\boldsymbol{Z}/2$-graded Künneth exact sequence

$$0 \longrightarrow K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) \longrightarrow \operatorname{Tor}(K_*(A), K_*(B)) \longrightarrow 0 .$$

Our proof uses the technique of geometric realization. The key fact is that given a unital C^*-algebra B, there is a commutative C^*-algebra F and an inclusion $F \to B \otimes \mathscr{K}$ such that the induced map $K_*(F) \to K_*(B)$ is surjective and $K_*(F)$ is free abelian.

1. Introduction. Let A and B be C^*-algebras. There is a $\boldsymbol{Z}/2$-graded pairing (defined in §2)

$$\alpha\colon K_p(A) \otimes K_q(B) \longrightarrow K_{p+q}(A \otimes B) \quad p, q \in \boldsymbol{Z}/2$$

where K_* denotes K-theory for Banach algebras [9, 17] and $\otimes = \otimes_{\min}$. Let $\mathfrak{N}$ be the smallest subcategory of the category of separable nuclear C^*-algebras which contains the separable Type I algebras and is closed under the operations of taking ideals, quotients, extensions, inductive limits, stable isomorphism, and crossed products by $\boldsymbol{Z}$ and by $\boldsymbol{R}$. We shall establish the following theorem.

KÜNNETH THEOREM. *Let A and B be C^*-algebras with $A \in \mathfrak{N}$. Then there is a natural short exact sequence*

$$0 \longrightarrow K_*(A) \otimes K_*(B) \xrightarrow{\alpha} K_*(A \otimes B) \xrightarrow{\beta} \operatorname{Tor}(K_*(A), K_*(B)) \longrightarrow 0 .$$

The sequence is $\boldsymbol{Z}/2$-graded with $\deg \alpha = 0$, $\deg \beta = 1$, *where $K_p \otimes K_q$ and* $\operatorname{Tor}(K_p, K_q)$ *are given degree $p + q$ $(p, q \in \boldsymbol{Z}/2)$.*

If $A = C(X)$ and $B = C(Y)$ with X and Y finite CW-complexes then the hypotheses are satisfied and we recover the classical Künneth Theorem for topological K-theory due to Atiyah [1]:

$$0 \longrightarrow K^*(X) \otimes K^*(Y) \xrightarrow{\alpha} K^*(X \times Y) \xrightarrow{\beta} \mathrm{Tor}(K^*(X), K^*(Y)) \longrightarrow 0 \tag{1.1}$$

Atiyah generalized (1.1) to the compact space setting [2] and further work was also done by him and by others (cf. [3]) regarding the splitting of (1.1).

An important step in Atiyah's proof of (1.1) was the technique of geometric realization of a projective resolution (i.e., a free presentation) of $K^*(Y)$. This idea has led to various developments within K-theory and bordism. Atiyah's theorem says that given a compact space Y, there is a compact space Y_1 and a continuous function $f_1: Y \to Y_1$ such that $K^*(Y_1)$ is free and $f_1^*: K^*(Y_1) \to K^*(Y)$ is surjective. A homotopy argument then yields a cofibration

$$Y \xrightarrow{f_1} Y_1 \xrightarrow{f_2} Y_2$$

such that the associated long exact sequence degenerates to a free presentation

$$0 \longrightarrow K^*(Y_2) \longrightarrow K^*(Y_1) \longrightarrow K^*(Y) \longrightarrow 0 \tag{1.2}$$

of the $Z/2$-graded group $K^*(Y)$. The space Y_1 used in Atiyah's proof is a product of Grassmann manifolds and their suspensions.

Our proof of the Künneth Theorem parallels Atiyah's argument. The key step, and a result of independent interest (as we shall explain) is the following geometric realization theorem. Let $\mathscr{K}$ denote the C^*-algebra of compact operators (on a possibly inseparable Hilbert space) and let M_n denote the complex n by n matrix ring. Note that $K_*(A) \cong K_*(A \otimes M_n) \cong K_*(A \otimes \mathscr{K})$.

THEOREM (*Geometric Realization*). *Let B be a unital C^*-algebra. Then there exists a commutative C^*-algebra $F = C_0(Y)$ with Y a disjoint union of points and lines and an inclusion of C^*-algebras*

$$\mu: F \longrightarrow B \otimes \mathscr{K}$$

such that $K_*(F)$ is free abelian and

$$\mu_*: K_*(F) \longrightarrow (B \otimes \mathscr{K}) \cong K_*(B)$$

is surjective. If $K_*(B)$ is free abelian then μ_* is an isomorphism.

The theorem and a homotopy argument imply that there is a short exact sequence

$$0 \longrightarrow B \otimes \mathscr{K} \otimes C_0(\boldsymbol{R}) \longrightarrow C \xrightarrow{\nu} F \longrightarrow 0 \tag{1.3}$$

whose associated long exact sequence in K-theory becomes a free resolution

$$0 \longrightarrow K_*(C) \xrightarrow{\nu_*} K_*(F) \xrightarrow{\mu_*} K_*(B) \longrightarrow 0$$

of $K_*(B)$.

Our geometric realization theorem does not quite imply Atiyah's geometric realization theorem. The distinction lies in the fact that not every map $f: C(X) \to C(X') \otimes M_n$ or $f: C(X) \to C(X') \otimes \mathscr{K}$ arises via a map of spaces $X' \to X$. The spaces which do arise from our results are much simpler than those required by Atiyah's result. A moment's reflection should convince the reader that there is no hope in trying to prove Atiyah's theorem when the spaces Y_i are restricted to subsets of the plane.

In geometric realization applications there seems to be no situation where having a map only at the level $F \to B \otimes \mathscr{K}$ is a handicap. The Künneth Theorem follows from the geometric realization theorem in just the same way as Atiyah's Künneth theorem (1.1) follows from his geometric realization theorem. Note, however, that if $Y^+ \subset \boldsymbol{C}$ then $K^*(Y)$ has a trivial ring structure and so $\mu_*: K^*(Y) \to K_*(B)$ has no chance of being a ring map when B is commutative. Maps $X \to X'$ induce ring maps $K^*(X') \to K^*(X)$; maps $C(X') \to C(X) \otimes \mathscr{K}$ do not in general do so.

Despite the title, this paper is mathematically independent of [16]. Philosophically they are linked. Künneth theorems are a necessary prelude to product structures and to the introduction of coefficients into homology and cohomology theories on C^*-algebras. As all theories studied to date are closely tied to (and frequently determined by) K-theory, it seems reasonable to study K-theory first. Moreover, the geometric realization technique has already been useful elsewhere (see Remark 3.6).

The remainder of the paper is organized as follows. In §2 we show that $\alpha: K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$ is an isomorphism when $A \in \mathfrak{N}$ and $K_*(B)$ is torsionfree. Section 3 is devoted to the geometric realization theorem. In §4 the results of the two previous sections are combined to prove the Künneth Theorem.

It is a pleasure to acknowledge valuable conversations and correspondence with L. G. Brown, J. Cuntz, and J. Rosenberg and to thank the UCLA mathematical community for its warm hospitality during my visit there.

2. The Künneth formula: special cases. In this section the map α is defined and is shown to be an isomorphism in certain special cases.

Let A and B be C^*-algebras. The map α is a $\mathbf{Z}/2$-graded pairing

$$\alpha: K_p(A) \otimes K_q(B) \longrightarrow K_{p+q}(A \otimes B) \quad p, q \in \mathbf{Z}/2 \tag{2.1}$$

defined as follows. Suppose that A and B are unital. The natural mappings

$$M_r(A) \otimes M_s(B) \longrightarrow M_{r+s}(A \otimes B)$$

induce a pairing

$$K_0(A) \otimes K_0(B) \longrightarrow K_0(A \otimes B) \tag{2.2}$$

which is natural with respect to pairs of maps $A \to A'$, $B \to B'$. Karoubi [10, Theorem 5.1] shows that if one specifies a sign convention, then (2.2) extends uniquely to a pairing α of the form (2.1), retaining naturality and respecting suspensions and boundary maps.

Next extend to nonunital C^*-algebras. If A is nonunital but B is unital then the diagram with exact rows

$$\begin{array}{ccccccc} K_p(A) \otimes K_q(B) & \longrightarrow & K_p(A^+) \otimes K_q(B) & \longrightarrow & K_p(\boldsymbol{C}) \otimes K_q(B) & \longrightarrow & 0 \\ \vdots & & \downarrow \alpha & & \downarrow \alpha & & \\ K_{p+q}(A \otimes B) & \longrightarrow & K_{p+q}(A^+ \otimes B) & \longrightarrow & K_{p+q}(\boldsymbol{C} \otimes B) & & \end{array}$$

extends α. A similar argument when both A and B are nonunital yields the pairing (2.1) in the general case.

DEFINITION 2.3. If $\alpha: K_*(A) \otimes K_*(B) \to K_*(A \otimes B)$ is a $\mathbf{Z}/2$-graded isomorphism then α is said to be an isomorphism for the pair (A, B); this we abbreviate to "$\alpha(A, B)$ is an isomorphism".

To be completely transparent regarding the gradings, we insist that both of the maps

$$(K_0(A) \otimes K_0(B)) \oplus (K_1(A) \otimes K_1(B)) \xrightarrow{\alpha} K_0(A \otimes B)$$

and

$$(K_0(A) \otimes K_1(B)) \oplus (K_1(A) \otimes K_0(B)) \xrightarrow{\alpha} K_1(A \otimes B)$$

be isomorphisms.

The class of pairs (A, B) for which $\alpha(A, B)$ is an isomorphism is closed under inductive limits (over directed sets).

PROPOSITION 2.4. *Suppose that $\{A_\gamma\}$ is a directed system of nuclear C^*-algebras and suppose that $\alpha(A_\gamma, B)$ is an isomorphism for all γ. Then $\alpha(\varinjlim A_\gamma, B)$ is an isomorphism.*

Proof. The nuclearity assumption implies that

$$(\varinjlim A_\gamma) \otimes B \cong \varinjlim (A_\gamma \otimes B)$$

and hence α for the pair $(\varinjlim A_\gamma, B)$ factors as the composite of isomorphisms

$$\begin{array}{ccc}
K_*\varinjlim (A_\gamma) \otimes K_*(B) & \dashrightarrow & K_*((\varinjlim A_\gamma) \otimes B) \\
\uparrow \cong & & \uparrow \cong \\
(\varinjlim K_*A_\gamma) \otimes K_*(B) & & K_*(\varinjlim (A_\gamma \otimes B)) \\
\uparrow \cong & & \uparrow \cong \\
\varinjlim ((K_*(A_\gamma) \otimes K_*(B)) & \longrightarrow & \varinjlim K_*(A_\gamma \otimes B)
\end{array}$$

which completes the argument. □

Similarly, the class of pairs (A, B) for which α is an isomorphism is closed under crossed products by $\boldsymbol{R}$:

PROPOSITION 2.5. *Suppose that $\alpha(A, B)$ is an isomorphism and suppose that $\boldsymbol{R}$ acts on A as a group of automorphisms. Then $\alpha(A \times_\varphi \boldsymbol{R}, B)$ is an isomorphism.*

Proof. This is an immediate consequence of the Thom isomorphism theorem of A. Connes [4]. He proves that there is a natural isomorphism $K_*(A \times_\varphi \boldsymbol{R}) \cong K_*(A)$ (with a degree shift) from which the proposition follows. □

In the following few propositions we assume that $K_*(B)$ is torsionfree. This implies that $\mathrm{Tor}(K_*(A), K_*(B)) = 0$ for any A, so that the Künneth formula is just the statement that α is an isomorphism.

PROPOSITION 2.6. *Suppose that $K_*(B)$ is torsionfree, J is an ideal of A with J, A, and A/J nuclear and that α is an isomorphism for two of the pairs (J, B), (A, B), $(A/J, B)$. Then α is an isomorphism for the third pair as well.*

Proof. Tensor the long exact sequence

$$\longrightarrow K_q(J) \longrightarrow K_q(A) \longrightarrow K_q(A/J) \xrightarrow{\partial} K_{q-1}(J) \longrightarrow$$

with $K_*(B)$. The resulting sequence remains exact (since torsionfree groups are flat). An easy argument involving the Five Lemma and

the naturality of α completes the proof. □

PROPOSITION 2.7. *Suppose that $K_*(B)$ is torsionfree, A is nuclear, and that $\alpha(A, B)$ is an isomorphism. Let $\boldsymbol{Z}$ act on A as a group of automorphisms. Then $\alpha(A \times_\varphi \boldsymbol{Z}, B)$ is an isomorphism.*

Proof. Pimsner and Voiculescu [12] show that there is an exact sequence of the form

$$\begin{array}{ccc} K_*(A) & \xrightarrow{D_A} & K_*(A) \\ & {\scriptstyle\partial}\nwarrow \quad \swarrow{\scriptstyle i_*} & \\ & K_*(A \times_\varphi \boldsymbol{Z}) & \end{array} \tag{2.8}$$

which is natural in the appropriate senses, where $D_A = id_A - (\varphi(-1))_*$. Since $K_*(B)$ is torsionfree, there is an exact triangle

$$\begin{array}{ccc} K_*(A) \otimes K_*(B) & \xrightarrow{D_A \otimes 1} & K_*(A) \otimes K_*(B) \\ & {\scriptstyle\partial\otimes 1}\nwarrow \quad \swarrow{\scriptstyle i_*\otimes 1} & \\ & K_*(A \times_\varphi \boldsymbol{Z}) \otimes K_*(B) & \end{array} \tag{2.9}$$

The C^*-algebra $(A \otimes B) \times_{(\varphi\times 1)} \boldsymbol{Z} \cong (A \times_\varphi \boldsymbol{Z}) \otimes B$ when plugged into (2.8) yields an exact triangle

$$\begin{array}{ccc} K_*(A \otimes B) & \xrightarrow{D_{A\otimes B}} & K_*(A \otimes B) \\ & {\scriptstyle\partial}\nwarrow \quad \swarrow{\scriptstyle (i\otimes 1)_*} & \\ & K_*((A \times_\varphi \boldsymbol{Z}) \otimes B) & \end{array} \tag{2.10}$$

and α induces a morphism of exact triangles (from 2.9 to 2.10). The hypotheses imply that α is an isomorphism on two of the three vertices; the Five Lemma implies that α is an isomorphism on the third as well. Thus

$$\alpha\colon K_*(A \times_\varphi \boldsymbol{Z}) \otimes K_*(B) \longrightarrow K_*((A \times_\varphi \boldsymbol{Z}) \otimes B)$$

is an isomorphism. □

To this point we have no examples of C^*-algebras (A, B) for which α is an isomorphism. The following proposition remedies that deficiency.

PROPOSITION 2.11. *Suppose that $K_*(B)$ is torsionfree. Then for every locally compact space Y, $\alpha(C_0(Y), B)$ is an isomorphism.*

Proof. If $Y = \boldsymbol{R}^n$ then α is an isomorphism by Bott periodicity. Similarly the proposition holds for Y a sphere (by 2.6) or, by additivity, a finite wedge of spheres. Induction on the number of cells of Y shows that α is an isomorphism whenever Y is a finite complex. If Y is compact (Hausdorff) then it may be written as an inverse limit of finite complexes Y_γ (cf. Eilenberg-Steenrod [8, Theorem X. 10.1]). Then $C(Y) \cong \lim C(Y_\gamma)$, and Proposition 2.4 implies that α is an isomorphism. In the general case, application of Proposition 2.6 to the sequence

$$0 \longrightarrow C_0(Y) \longrightarrow C(Y^+) \longrightarrow \boldsymbol{C} \longrightarrow 0$$

completes the argument, where Y^+ is the one-point compactification of Y. □

Recall that a C^*-algebra A is *solvable* if $A = \overline{\bigcup_n A_n}$ for some ascending sequence of closed ideals A_n with

$$A_p/A_{p-1} \cong C_0(Y_p) \otimes \mathscr{K}(\mathscr{H}_p)$$

for some locally compact spaces Y_p and some (finite or infinite-dimensional) Hilbert spaces $\mathscr{H}_p$.

PROPOSITION 2.12. *If $K_*(B)$ is torsionfree and A is solvable then $\alpha(A, B)$ is an isomorphism.*

Proof. This follows from (2.4) and (2.6), but we prefer to give a spectral sequence proof. Define functors L_q and M_q by

$$L_q(-) = (K_*(-) \otimes K_*(B))_q$$
$$M_q(-) = K_q((-) \otimes B) .$$

Each of these functors satisfy the exactness axiom and α induces a natural transformation $\alpha: L_* \to M_*$ which is an isomorphism when restricted to C^*-algebras of the form (commutative) $\otimes \mathscr{K}$, by Proposition 2.11. A spectral sequence comparison theorem [16, Theorem 4.2] implies that $\alpha: L^*(A) \to M_*(A)$ is an isomorphism for any solvable C^*-algebra A. □

THEOREM 2.13. *If $K_*(B)$ is torsionfree and A is separable Type I then $\alpha(A, B)$ is an isomorphism.*

Proof. Since A is a separable Type I C^*-algebra, A has a countable composition series (J_ρ) with each $J_{\rho+1}/J_\rho$ of continuous trace [13, § 2]. By repeated use of (2.4) and (2.6) we may assume that A is of the form $A =$ (continuous trace) $\otimes \mathscr{K}$. Such a con-

tinuous trace C^*-algebra is homogeneous of degree $\aleph_0$ and hence the associated continuous family of elementary C^*-algebras over $\hat{A}$ is locally trivial [13, §4]. Let $\{U_i\}$ be a countable open cover for $\hat{A}$ such that $A|_{U_i}$ is trivial for each i. The sequence of closed ideals $\{L_i\}$ of A corresponding to the sequence of open sets

$$U_1 \subseteq U_1 \cup U_2 \subseteq U_1 \cup U_2 \cup U_3 \subseteq \cdots$$

is an increasing sequence with $L_{i+1}/L_i \cong C_0(Y_i) \otimes \mathscr{K}$. Hence A is solvable. □

The following theorem summarizes the principal results of this section. It has been established (independently) by J. Cuntz using similar methods.

THEOREM 2.14. *Let A and B be C^*-algebras with $K_*(B)$ torsion-free and $A \in \mathfrak{N}$. Then there is a natural isomorphism*

$$\alpha: K_*(A) \otimes K_*(B) \longrightarrow K_*(A \otimes B) . \qquad \square$$

REMARK 2.15. It is plausible that this theorem (and hence the full Künneth Theorem) holds for A separable nuclear. We know of no case where the Künneth Theorem fails for D^*-algebras. There is a slight chance that $\mathfrak{N}$ is itself the category of separable nuclear C^*-algebras or that one might get up to that category by allowing crossed products by a few more groups.

3. Geometric realization. Let B be a unital C^*-algebra. Then for each q it is easy to find free abelian groups G'_q, G''_q such that

$$0 \longrightarrow G'_q \longrightarrow G''_q \longrightarrow K_q(B) \longrightarrow 0$$

is a free resolution of $K_q(B)$. With slightly more effort we can find C^*-algebras B', B'' with

$$K_q(B') = G'_q$$
$$K_q(B'') = G''_q$$

and hence exact sequences

$$0 \longrightarrow K_q(B') \xrightarrow{\nu} K_q(B'') \xrightarrow{\eta} K_q(B) \longrightarrow 0 , \quad q \in \mathbf{Z}/2 ,$$

This is essentially useless for the study of $K_*(B)$, however, since the maps ν and η constructed above do not arise from maps at the level of C^*-algebras; such resolutions are not "geometric".

The principal result of this section is the construction of geometric resolutions of the type

$$0 \longrightarrow K_q(C) \longrightarrow K_q(F) \longrightarrow K_q(B) \longrightarrow 0$$

with various useful properties. The existence of such resolutions will enable us (in §4) to prove the full Künneth theorem. (Those with less than complete faith in the necessity of §3 may prefer to read §4 first to see how the proof is completed.)

The construction is done in two steps. The first step is Lemma 3.1 which shows that there are enough projectives of the required sort. The second step is a homotopy argument which uses a mapping cone construction.

LEMMA 3.1. *Let B be a unital C^*-algebra. Then there is a commutative C^*-algebra $F = C_0(Y)$ and an inclusion $\mu: F \to B \otimes \mathscr{K}(\bar{\mathscr{H}})$ such that the induced map*

$$\mu_*: K_*(F) \longrightarrow K_*(B)$$

is surjective. The space Y is a disjoint union of points and copies of $\boldsymbol{R}$. If $K_*(B)$ is free abelian then μ_* is an isomorphism. If $K_*(B)$ is countably generated (e.g., *if B is separable*) *then $\bar{\mathscr{H}}$ is separable and Y has countably many path components, so Y embeds in the plane.*

Proof. Select a minimal family of generators for $K_*(B)$. Each generator of $K_0(B)$ is of the form

$$[p_s] - r(s)[1] \qquad r(s) \in \boldsymbol{Z}$$

where $[1]$ represents the class of the identity of B and $p_s \in B \otimes \mathscr{L}(\mathscr{H}_s)$ with $\mathscr{H}_s$ a finite-dimensional Hilbert space and p_s a self-adjoint projection. Each generator of $K_1(B)$ may be represented by some unitary $u_t \in B \otimes \mathscr{L}(\mathscr{H}_t)$ with $\mathscr{H}_t$ also of finite dimension. Let $\bar{\mathscr{H}}$ be the Hilbert space direct sum of the (possibly uncountably many) Hilbert spaces $\{\mathscr{H}_s\}$ and $\{\mathscr{H}_t\}$. Define elements in $B \otimes \mathscr{K}(\bar{\mathscr{H}})$ by $\bar{p}_s = p_s \oplus 0$ with respect to $B \otimes \mathscr{L}(\bar{\mathscr{H}}) \supseteq [B \otimes \mathscr{L}(\mathscr{H}_s)] \oplus [B \otimes \mathscr{L}(\mathscr{H}_s^{\perp})]$ and $\bar{w}_t = (u_t - 1) \oplus 0$ with respect to $B \otimes \mathscr{L}(\bar{\mathscr{H}}) \supseteq [B \otimes \mathscr{L}(\mathscr{H}_t)] \oplus [B \otimes \mathscr{L}(\mathscr{H}_t^{\perp})]$.

Then $\bar{p}_s \bar{p}_{s'} = \bar{p}_s \bar{w}_t = \bar{w}_t \bar{w}_{t'} = 0$ for $s \neq s'$, $t \neq t'$. Further,

$$[\bar{p}_s] = [p_s] \quad \text{in} \quad K_0(B \otimes \mathscr{K}) \cong K_0(B)$$

and

$$[\bar{w}_t + 1] = [u_t] \quad \text{in} \quad K_1((B \otimes \mathscr{K})^+) \cong K_1(B) .$$

Let F be the C^*-subalgebra of $B \otimes \mathscr{K}(\bar{\mathscr{H}})$ generated by $\{\bar{p}_s\} \cup \{\bar{w}_t\}$. Then F is commutative. Its maximal space is easily determined.

Each projection $\bar{p}_s$ contributes a discrete point. Each element $\bar{w}_t$ contributes a copy of $\boldsymbol{R}$. This is true since $\bar{w}_t + 1$ is unitary (forcing the maximal ideal space of $C^*\{\bar{w}_t\}^+$ to be a closed subset of the unit circle) and $u_t \neq 0$ in $K_1(B)$ (so that $C^*\{\bar{w}_t\}^+ \cong C(S^1)$).

Let $\mu: F \to B \otimes \mathscr{K}(\bar{\mathscr{H}})$ be the inclusion. Then $\mu^+: F^+ \to (B \otimes \mathscr{K}(\bar{\mathscr{H}}))^+$ is unital, and so

$$\mu_*([\bar{p}_s] - r(s)[1]) = [p_s] - r(s)[1]$$

and

$$\mu_*([\bar{w}_t + 1]) = [u_t] .$$

Thus μ_* is surjective. If $K_*(B)$ is free abelian then it is clear that μ_* is an isomorphism. Finally, if $K_*(B)$ is countably generated then $\bar{\mathscr{H}}$ is a separable Hilbert space and Y has countably many path components. □

REMARK 3.2. An obvious modification of the above argument allows the image of μ_* to be *any* subgroup of $K_*(B)$.

REMARK 3.3. Let B be unital. Cuntz has shown that each element of $K_0(B \otimes \mathscr{O}_\infty)$ is represented by a projection. As $K_*(B) \cong K_*(B \otimes \mathscr{O}_\infty)$ in a natural manner, it would be possible to recast the above argument to avoid differences of projections. If B is not unital then the situation is unclear.

We proceed to the construction of the geometric resolution. Suppose given a unital C^*-algebra B. Lemma 3.1 implies that there is a commutative C^*-algebra $F = C_0(Y)$ and a C^*-inclusion $\mu: F \to B \otimes \mathscr{K}$ such that the induced map

$$\mu_*: K_*(F) \longrightarrow K_*(B \otimes \mathscr{K}) \cong K_*(B)$$

is surjective. Let C be the mapping cone of μ; that is,

$$C = \{(g, x) \mid g: [0, 1] \longrightarrow B \otimes \mathscr{K},\ x \in F,\ g(0) = 0,\ g(1) = \mu(x)\} .$$

This is a C^*-algebra in the evident manner. Let $\upsilon: C \to F$ be defined by $\upsilon(g, x) = x$. Then υ is surjective and

$$\begin{aligned} \operatorname{Ker}(\upsilon) &= \{(g, x) \in C \mid x = 0\} \\ &= \{g: [0, 1] \longrightarrow B \otimes \mathscr{K} \mid g(0) = g(1) = 0\} \\ &= B \otimes \mathscr{K} \otimes C_0(\boldsymbol{R}) . \end{aligned}$$

So there is a short exact sequence

$$0 \longrightarrow B \otimes \mathscr{K} \otimes C_0(\boldsymbol{R}) \longrightarrow C \xrightarrow{\upsilon} F \longrightarrow 0 . \tag{3.4}$$

In the associated K-theory long exact sequence the boundary homomorphism ∂ corresponds to μ_* in the sense that the diagrams

$$\begin{array}{ccc} K_q(F) & \xrightarrow{\partial} & K_{q-1}(B \otimes \mathscr{K} \otimes C_0(\boldsymbol{R})) \\ & {\scriptstyle \mu_*}\searrow \quad \nearrow{\scriptstyle \cong \text{(suspension)}} & \\ & K_q(B) & \end{array} \tag{3.5}$$

commute for all q. Thus ∂ is surjective and the long exact sequence associated to (3.4) degenerates to two short exact sequences of the form

$$0 \longrightarrow K_q(C) \xrightarrow{\upsilon_*} K_q(F_1) \xrightarrow{\partial} K_{q-1}(B \otimes \mathscr{K} \otimes C_0(\boldsymbol{R})) \longrightarrow 0\,. \tag{3.6}$$

This implies in particular that $K_q(C)$ is a free abelian group for each $q \in \boldsymbol{Z}/2$ and thus (3.4) does yield a geometric resolution of $K_*(B)$.

REMARK 3.7. J. Rosenberg has shown that our results plus an additional argument imply the existence of geometric *injective* resolutions

$$0 \longrightarrow K_*(B) \longrightarrow K_*(I_1) \longrightarrow K_*(I_2) \longrightarrow 0$$

with $K_*(I_j)$ injective (i.e., divisible) abelian groups. We have established Künneth formulas and Universal Coefficient Theorems for the Kasparov groups [11] $\mathrm{Ext}(A, B)$ which classify extension of the form

$$0 \longrightarrow B \otimes \mathscr{K} \longrightarrow (\ \ ) \longrightarrow A \longrightarrow 0$$

up to stable equivalence using geometric realization techniques provided that $A \in \mathfrak{N}$ [14], [15].

4. The Künneth formula: the general case.

THEOREM 4.1 (KÜNNETH THEOREM). *Let A and B be C^*-algebras with $A \in \mathfrak{N}$. Then there is a short exact sequence*

$$0 \longrightarrow K_*(A) \otimes K_*(B) \xrightarrow{\alpha} K_*(A \otimes B) \xrightarrow{\beta} \mathrm{Tor}(K_*(A), K_*(B)) \longrightarrow 0$$

with α of degree 0 and β of degree 1. The sequence is natural for maps of pairs $(A, B) \to (A', B')$.

The following proof does not use the fact that $A \in \mathfrak{N}$, but only that $\alpha(A, B)$ is an isomorphism whenever $K_*(B)$ is torsionfree. Thus

any generalization of Theorem 2.4 to a class of C^*-algebras which is larger than $\mathfrak{N}$ yields the full Künneth Theorem on that larger class of C^*-algebras.

Proof. Suppose initially that B is unital. Form a geometric resolution of the form

$$0 \longrightarrow B \otimes \mathscr{K} \otimes C_0(\boldsymbol{R}) \longrightarrow C \xrightarrow{\nu} F \longrightarrow 0$$

and tensor it with the (nuclear) C^*-algebra A. The resulting short exact sequence

$$(4.3) \qquad 0 \longrightarrow A \otimes B \otimes \mathscr{K} \otimes C_0(\boldsymbol{R}) \longrightarrow A \otimes C \xrightarrow{1\otimes\nu} A \otimes F \longrightarrow 0$$

has associated K-theory long exact sequence

$$\to K_q(A \otimes C) \xrightarrow{(1\otimes\nu)_*} K_q(A \otimes F) \to K_q(A \otimes B) \to K_{q-1}(A \otimes C) \xrightarrow{(1\otimes\nu)_*} K_{q-1}(A \otimes F)$$

Unsplice this sequence to obtain the short exact sequence

$$(4.4) \qquad 0 \to \mathrm{Coker}((1 \otimes \nu)_*)_q \xrightarrow{\hat{\alpha}} K_q(A \otimes B) \to \mathrm{Ker}((1 \otimes \nu)_*)_{q-1} \to 0 \ .$$

The sequence 4.4 is the Künneth formula; it remains to demonstrate that fact.

Consider the commutative diagram

$$(4.5) \qquad \begin{array}{ccccccccc} 0 \to \mathrm{Tor}(K_*(A), K_*(B)) & \to & K_*(A) \otimes K_*(C) & \xrightarrow{1\otimes\nu_*} & K_*(A) \otimes K_*(F) & \to & K_*(A) \otimes K_*(B) & \to & 0 \\ & & \downarrow{\scriptstyle \alpha(A,C)} & & \downarrow{\scriptstyle \alpha(A,F)} & & & & \\ & & K_*(A \otimes C) & \xrightarrow{(1\otimes\nu)_*} & K_*(A \otimes F) & & & & \end{array}$$

The row is exact since (3.6) is a free resolution of $K_*(B)$. The vertical Künneth maps $\alpha(A, C)$ and $\alpha(A, F)$ are isomorphisms by (2.14) since $A \in \mathfrak{N}$ and $K_*(C)$ and $K_*(F)$ are free abelian groups. Thus

$$\mathrm{Coker}((1 \otimes \nu)_*) \cong \mathrm{Coker}(1 \otimes \nu_*) = K_*(A) \otimes K_*(B)$$

and

$$\mathrm{Ker}((1 \otimes \nu)_*) \cong \mathrm{Ker}(1 \otimes \nu_*) = \mathrm{Tor}(K_*(A), K_*(B)) \ .$$

One checks that $\hat{\alpha}$ in (4.4) corresponds to $\alpha(A, B)$ under these identifications. This proves the Künneth Theorem under the assumption that B is unital. The final step of the proof is contained in the

following lemma.

LEMMA 4.6. *Suppose that the Künneth Theorem holds for the pairs* (A, B^+), $(A, \mathbb{C})$ *with* A *nuclear. Then it holds for* (A, B).

Proof. Contemplate the following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & K_*(A)\otimes K_*(B^+) & \xrightarrow{\alpha} & K_*(A\otimes B^+) & \xrightarrow{\beta} & \mathrm{Tor}(K_*(A), K_*(B^+)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\tau_*} & & \downarrow{\scriptstyle (1\otimes\tau)_*} & & & & \\
0 & \longrightarrow & K_*(A)\otimes K_*(\mathbb{C}) & \xrightarrow{\alpha} & K_*(A\otimes\mathbb{C}) & \longrightarrow & 0 & & \\
 & & \downarrow{\scriptstyle 1\otimes\partial_*} & & \downarrow{\scriptstyle \partial_*} & & & & \\
(4.7) & & K_*(A)\otimes K_*(B) & \xrightarrow{\alpha} & K_*(A\otimes B) & & & & \\
 & & \downarrow{\scriptstyle 1\otimes\sigma_*} & & \downarrow{\scriptstyle (1\otimes\sigma)_*} & & & & \\
0 & \longrightarrow & K_*(A)\otimes K_*(B^+) & \xrightarrow{\alpha} & K_*(A\otimes B^+) & \xrightarrow{\beta} & \mathrm{Tor}(K_*(A), K_*(B^+)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\tau_*} & & \downarrow{\scriptstyle (1\otimes\tau)_*} & & & & \\
0 & \longrightarrow & K_*(A)\otimes K_*(\mathbb{C}) & \xrightarrow{\alpha} & K_*(A\otimes\mathbb{C}) & \longrightarrow & 0 & &
\end{array}
$$

where the vertical maps are induced by the natural maps

$$0 \longrightarrow B \xrightarrow{\sigma} B^+ \xrightarrow{\tau} \mathbb{C} \longrightarrow 0$$

and

$$0 \longrightarrow A\otimes B \xrightarrow{1\otimes\sigma} A\otimes B^+ \xrightarrow{1\otimes\tau} A\otimes\mathbb{C} \longrightarrow 0\,.$$

The boundary map $\partial\colon K_*(\mathbb{C}) \to K_*(B)$ is the zero map and the resulting short exact sequences in K-theory are split, so the left column is indeed exact. Furthermore, $1\otimes\tau_*$ is surjective, which implies that $(1\otimes\tau)_*$ is surjective and that $\partial_*\colon K_*(A\otimes\mathbb{C}) \to K_*(A\otimes B)$ is injective.

Unsplicing the middle column yields the short exact sequence

$$0 \longrightarrow \mathrm{Cok}((1\otimes\tau)_*) \longrightarrow K_*(A\otimes B) \longrightarrow \mathrm{Ker}((1\otimes\tau)_*) \longrightarrow 0$$

but $(1\otimes\tau)_*$ is surjective, so we have

$$K_*(A\otimes B) \cong \mathrm{Ker}((1\otimes\tau)_*)\,.$$

Rewrite (4.7) in light of the above information and one obtains the following commutative diagram with exact rows and columns:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & K_*(A)\otimes K_*(B) & \xrightarrow{\alpha_B} & K_*(A\otimes B) & & \mathrm{Tor}(K_*(A),\, K_*(B)) & & \\
(4.8) & & \downarrow{\scriptstyle 1\otimes\sigma_*} & & \downarrow{\scriptstyle (1\otimes\sigma)_*} & & \downarrow & & \\
0 & \longrightarrow & K_*(A)\otimes K_*(B^+) & \xrightarrow{\alpha} & K_*(A\otimes B^+) & \longrightarrow & \mathrm{Tor}(K_*(A),\, K_*(B^+)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\tau_*} & & \downarrow{\scriptstyle (1\otimes\tau)_*} & & \downarrow & & \\
0 & \longrightarrow & K_*(A)\otimes K_*(\mathbb{C}) & \xrightarrow{\alpha} & K_*(A\otimes\mathbb{C}) & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

The Serpent Lemma applied to (4.8) yields an exact sequence

$$0 \longrightarrow K_*(A)\otimes K_*(B) \xrightarrow{\alpha_B} K_*(A\otimes B) \xrightarrow{\beta} \mathrm{Tor}(K_*(A),\, K_*(B)) \longrightarrow 0$$

which is, of course, the Künneth formula for the pair (A, B). This completes the proof of Lemma 4.6 and the Künneth Theorem. □

REMARK 4.9. It would be very satisfying to drop the assumption that $A \in \mathfrak{N}$ (replacing it by the assumption that A is nuclear and perhaps separable). The assumption is used in an essential way only at (4.5). At that point one needs to know that if $K_*(B)$ is free and if $f\colon A' \to A''$ induces an isomorphism

$$f_*\colon K_*(A') \longrightarrow K_*(A'')$$

then

$$(f\otimes 1)_*\colon K_*(A'\otimes B) \longrightarrow K_*(A''\otimes B)$$

is an isomorphism. A direct proof of this fact would yield the Künneth Theorem in general. The difficulty is equivalent to the following conjecture, about which we are very mildly optimistic:

Conjecture. Suppose that A and B are C^*-algebras (separable, nuclear if that is necessary) with $K_*(A) = 0$ and $K_*(B)$ free. Then $K_*(A\otimes B) = 0$.

REMARK 4.10. We conjecture that the Künneth formula splits (unnaturally), at least if A and B are separable so that $\mathrm{Tor}(K_*(A), K_*(B))$ is countable. Some generalization of Bödigheimer's technique [3] should suffice. The question is related to the introduction of coefficients into K-theory-discussed below.

REMARK 4.11. Suppose that $N \in \mathfrak{N}$ is some fixed C^*-algebra with $K_0(N) = G$, $K_1(N) = 0$. As in [16], define

$$K_q(B; G) \cong K_q(B \otimes N) .$$

The Künneth Theorem implies that for any C^*-algebra B there is a natural short exact sequence

$$0 \longrightarrow K_q(B) \otimes G \longrightarrow K_q(B; G) \longrightarrow \operatorname{Tor}(K_{q-1}(B), G) \longrightarrow 0 \tag{4.12}$$

which shows that, at least up to group extension (see 4.11), the groups $K_q(B; G)$ are independent of choice of N. (The sequence 4.12 was shown to be exact by J. Cuntz [5] when N is a Cuntz algebra). Suppose further that $N \otimes N \otimes \mathscr{K} \cong N \otimes \mathscr{K}$ (so that $G \otimes G \cong G$). Then there is a Künneth formula for the theory $K_*(-; G)$ of the form

$$\begin{aligned} 0 \longrightarrow \sum_p K_p(A; G) \otimes K_{q-p}(B; G) &\longrightarrow K_q(A \otimes B; G) \\ \longrightarrow \sum_p \operatorname{Tor}(K_p(A; G), K_{q-p-1}(B; G)) &\longrightarrow 0 \end{aligned} \tag{4.13}$$

for $p, q \in \boldsymbol{Z}/2$. The Cuntz algebras $\mathscr{O}_{n+1}$ (for n prime) yield coefficients in the group $\boldsymbol{Z}/n$ and the UHF algebra associated to Q yields rational coefficients, for example.

There is joint work in progress by the author and J. Cuntz on the uniqueness of $K_*(-; \boldsymbol{Z}/n)$.

Added in proof. Pimsher and Voiculescu have extended their results of [12] to $K_*(A \times_\alpha F_n)$ ("K-groups of reduced crossed products by free groups," preprint INCREST). Proposition 2.7 extends and hence the Kunneth Theorem holds for the category obtained by also forcing $\mathfrak{N}$ to be closed under crossed products by free groups.

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